한국퍼지 및 지능 시스템학회 논문지 1998, Vol. 8, No. 4. (1998) 78~81

Products of Fuzzy Finite State Machines

Sung-Jin Cho*, Jae-Gyeom Kim and Soong-Hee Lee*****

**Department of Applied Mathematics, Pukyong National University*
***Department of Mathematics, Kyungsung University*
****Department of Information and Communication Engineering, Inje University*

ABSTRACT

We introduce the concepts of coverings, direct products, cascade products and wreath products of fuzzy finite state machines and investigate their algebraic structures.

1. Introduction

Since Wee[7] in 1967 introduced the concept of fuzzy automata following Zadeh[8], fuzzy automata theory has been developed by many researchers. Recently Malik *et al.*[4-6] introduced the concepts of fuzzy finite state machines and fuzzy transformation semigroups based on Wee's concept[7] of fuzzy automata and related concepts and applied algebraic technique. Cho *et al.*[2,3] introduced the notion of a T-fuzzy finite state machine that is an extension of a fuzzy finite state machine. Even if $T=\wedge$, our notion is different from the notion of Malik *et al.*[5]. In this paper, we introduce the concepts of coverings, restricted direct products, full direct products, cascade products and wreath products of fuzzy finite state machines (in the sense of Malik *et al.*[5]) that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures.

For the terminology in (crisp) algebraic automata theory, we refer to [1].

2. Coverings

Definition 2.1 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. If $\xi: X_1 \to X_2$ is a function and $\eta: Q_2 \to Q_1$ is a surjective partial function such that $\tau_1^+(\eta(p), x, \eta(q)) \leq \tau_2^+(p, \xi(x), q)$ for all p, q in the domain of η and $x \in X_1^+$, then we say that (η, ξ) is a covering of M_1 by M_2 and that M_2 covers M_1 and denote by $M_1 \leq M_2$. Moreover, if the inequality turns out equality whenever the left hand side of the inequality is not zero [resp. the inequality always turns out equality], then we say that (η, ξ) is a strong covering [resp. a complete covering] of M_1 by M_2 and that M_2 strongly covers [resp. completely covers] M_1 and denote by $M_1 \leq_s M_2$ [resp. $M_1 \leq_c M_2$].

In Definition 2.1, we abused the function ξ. We will write the natural semigroup homomorphism from X_1^+ to X_2^+ induced by ξ by ξ also for convenience sake. We give an example that is elementary and important.

Example 2.2 Let $M=(Q, X, \tau)$ be a fuzzy finite state machine. Define an equivalence relation ~ on X by $a \sim b$ if and only if $\tau(p, a, q)=\tau(p, b, q)$ for all $p,q \in Q$. Construct a fuzzy finite state machine $M_1=(Q, X/\sim, \tau^\sim)$ by defining $\tau^\sim(p, [a], q)=\tau(p, a, q)$. Now define $\xi: X \to X/\sim$ by $\xi(a)=[a]$ and $\eta=1_Q$. Then (η, ξ) is a complete covering of M by M_1 clearly.

Proposition 2.3 Let M_1, M_2 and M_3 be fuzzy finite state machines. If $M_1 \leq M_2$ [resp. $M_1 \leq_s M_2$, $M_1 \leq_c M_2$] and $M_2 \leq M_3$ [resp. $M_2 \leq_s M_3$, $M_2 \leq_c M_3$], then $M_1 \leq M_3$ [resp. $M_1 \leq_s M_3$, $M_1 \leq_c M_3$].

Proof. It is straightforward.

3. Direct Products

In this section, we consider restricted direct products and full direct products of fuzzy finite state machines.

Definition 3.1 Let $M_1=(Q_1, X, \tau_1)$ and $M_2=(Q_2, X, \tau_2)$ be fuzzy finite state machines. The restricted

(*),(**) : Partially supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1996.

direct product $M_1 \wedge M_2$ of M_1 and M_2 is the fuzzy finite state machine $(Q_1 \times Q_2, X, \tau_1 \wedge \tau_2)$ with

$$(\tau_1 \wedge \tau_2)((p_1, p_2), a, (q_1, q_2)) = \wedge(\tau_1(p_1, a, q_1), \tau_2(p_2, a, q_2)).$$

Theorem 3.2 Let $M_1=(Q_1, X, \tau_1)$ and $M_2=(Q_2, X, \tau_2)$ be fuzzy finite state machines. Then $(\tau_1 \wedge \tau_2)^+((p_1, p_2), x, (q_1, q_2)) = \wedge(\tau_1^+(p_1, x, q_1), \tau_2^+(p_2, x, q_2))$ for all $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$ and $x \in X^+$.

Proof. Let $a_1, \cdots, a_n \in X$. Then we have

$$\begin{aligned}
&(\tau_1 \wedge \tau_2)^+((p_1, p_2), a_1 \cdots a_n, (q_1, q_2)) \\
&= \vee\{\wedge((\tau_1 \wedge \tau_2)((p_1, p_2), a_1, (r_{11}, r_{12})), \\
&\quad (\tau_1 \wedge \tau_2)((r_{11}, r_{12}), a_2, (r_{21}, r_{22})), \\
&\quad \cdots, (\tau_1 \wedge \tau_2)((r_{(n-1)1}, r_{(n-1)2}), \\
&\quad a_n, (q_1, q_2))) \mid (r_{i1}, r_{i2}) \in Q_1 \times Q_2\} \\
&= \vee\{\wedge(\wedge(\tau_1(p_1, a_1, r_{11}), \tau_2(p_2, a_1, r_{12})), \\
&\quad \wedge(\tau_1(r_{11}, a_2, r_{21}), \tau_2(r_{12}, a_2, r_{22})), \\
&\quad \cdots, \wedge(\tau_1(r_{(n-1)1}, a_n, q_1), \\
&\quad \tau_2(r_{(n-1)2}, a_n, q_2)) \mid r_{i1} \in Q_1, r_{i2} \in Q_2\} \\
&= \wedge(\vee\{\wedge(\tau_1(p_1, a_1, r_{11}), \tau_1(r_{11}, a_2, r_{21}), \\
&\quad \cdots, \tau_1(r_{(n-1)1}, a_n, q_1)) \mid r_{i1} \in Q_1\}, \\
&\quad \vee\{\wedge(\tau_2(p_2, a_1, r_{12}), \tau_1(r_{12}, a_2, r_{22}), \\
&\quad \cdots, \tau_2(r_{(n-1)2}, a_n, q_2)) \mid r_{i2} \in Q_1\}) \\
&= \wedge(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, a_1 \cdots a_n, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Definition 3.3 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The full direct product $M_1 \times M_2$ of M_1 and M_2 is the fuzzy finite state machine $(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times \tau_2)$ with $(\tau_1 \times \tau_2)((p_1, p_2), (a, b), (q_1, q_2)) = \wedge(\tau_1(p_1, a, q_1), \tau_2(p_2, b, q_2))$.

Theorem 3.4 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. Then

$$\begin{aligned}
&(\tau_1 \times \tau_2)^+((p_1, p_2), (a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) \\
&= \wedge(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2))
\end{aligned}$$

for all $a_1, \cdots, a_n \in X_1$, $b_1, \cdots, b_n \in X_2$, $p_1, q_1 \in Q_1$ and

Proof. Let $a_1, \cdots, a_n \in X_1$ and $b_1, \cdots, b_n \in X_2$. Then we have

$$\begin{aligned}
&(\tau_1 \times \tau_2)^+((p_1, p_2), (a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) \\
&= \vee\{\wedge((\tau_1 \times \tau_2)((p_1, p_2), (a_1, b_1), (r_{11}, r_{12})), \\
&\quad (\tau_1 \times \tau_2)((r_{11}, r_{12}), (a_2, b_2), (r_{21}, r_{22})), \\
&\quad \cdots, (\tau_1 \times \tau_2)((r_{(n-1)1}, r_{(n-1)2}), (a_n, b_n), \\
&\quad (q_1, q_2))) \mid (r_{i1}, r_{i2}) \in Q_1 \times Q_2\} \\
&= \vee\{\wedge(\wedge(\tau_1(p_1, a_1, r_{11}), \tau_2(p_2, b_1, r_{12})), \\
&\quad \wedge(\tau_1(r_{11}, a_2, r_{21}), \tau_2(r_{12}, b_2, r_{22})), \\
&\quad \cdots, \wedge(\tau_1(r_{(n-1)1}, a_n, q_1), \\
&\quad \tau_2(r_{(n-1)2}, b_n, q_2)) \mid r_{i1} \in Q_1, r_{i2} \in Q_2\} \\
&= \wedge(\vee\{\wedge(\tau_1(p_1, a_1, r_{11}), \tau_1(r_{11}, a_2, r_{21}), \\
&\quad \cdots, \tau_1(r_{(n-1)1}, a_n, q_1)) \mid r_{i1} \in Q_1\}, \\
&\quad \vee\{\wedge(\tau_2(p_2, b_1, r_{12}), \tau_2(r_{12}, b_2, r_{22}), \\
&\quad \cdots, \tau_2(r_{(n-1)2}, b_n, q_2)) \mid r_{i2} \in Q_2\} \\
&= \wedge(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Proposition 3.5 Let $M_1=(Q_1, X, \tau_1)$ and $M_2=(Q_2, X, \tau_2)$ be fuzzy finite state machines. Then $M_1 \wedge M_2 \leq_c M_1 \times M_2$.

Proof. Let $\eta = 1_{Q_1 \times Q_2}$ and define $\xi: X \to X \times X$ by $\xi(a) = (a, a)$. Then (η, ξ) is a complete covering of $M_1 \wedge M_2$ by $M_1 \times M_2$ clearly.

The following proposition is a direct consequence of the associativity of $\wedge$.

Proposition 3.6 Let M_1, M_2 and M_3 be fuzzy finite state machines. Then the following are hold:

(i) $(M_1 \wedge M_2) \wedge M_3 = M_1 \wedge (M_2 \wedge M_3)$.

(ii) $(M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3)$.

4. Cascade Products and Wreath Products

In this section, we consider cascade products and wreath products of fuzzy finite state machines.

Definition 4.1 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The cascade product $M_1 \omega M_2$ of M_1 and M_2 with respect to $\omega: Q_2 \times X_2 \to X_1$ is the fuzzy finite state machine $(Q_1 \times Q_2, X_2, \tau_1 \omega \tau_2)$ with

$$\begin{aligned}
&(\tau_1 \omega \tau_2)((p_1, p_2), b, (q_1, q_2)) \\
&= \wedge(\tau_1(p_1, \omega(p_2, b), q_1), \tau_2(p_2, b, q_2))
\end{aligned}$$

Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines and $\omega: Q_2 \times X_2 \to X_1$. Define ω^+:

$Q_2 \times X_2^+ \to X_1^+$ by

$$\omega^+(p_2, b_1 b_2 \cdots b_n) = \omega(p_2, b_1)\omega(u_1, b_2) \cdots \omega(u_{n-1}, b_n),$$

where $p_2, u_1, u_2, \cdots, u_{n-1} \in Q_2$ and $b_1, \cdots, b_n \in X_2$ such that

$$\begin{aligned}&\tau_1^+(p_1, \omega(p_2, b_1)\omega(u_1, b_2) \cdots \omega(u_{n-1}, b_n), q_1)\\ &\quad = \vee \{\tau_1^+(p_1, \omega(p_2, b_1)\omega(r_1, b_2) \cdots \\ &\qquad \omega(r_{n-1}, b_n), q_1) \mid r_1, r_2, \cdots, r_{n-1} \in Q_2\}\end{aligned}$$

where $p_1, q_1 \in Q_1$ and $b \in X_2$.

Theorem 4.2 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. Then

$$\begin{aligned}&(\tau_1 \omega \tau_2)^+((p_1, p_2), x, (q_1, q_2))\\ &\quad = \wedge (\tau_1^+(p_1, \omega^+(p_2, x), q_1), \tau_2^+(p_2, x, q_2))\end{aligned}$$

where $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$ and $x \in X_2^+$.

Proof. Let $b_1, \cdots, b_n \in X_2$. Then we have

$$\begin{aligned}&(\tau_1 \omega \tau_2)^+((p_1, p_2), b_1 \cdots b_n, (q_1, q_2))\\ &= \vee \{\wedge ((\tau_1 \omega \tau_2)((p_1, p_2), b_1, (r_{11}, r_{12})),\\ &\quad (\tau_1 \omega \tau_2)((r_{11}, r_{12}), b_2, (r_{21}, r_{22})),\\ &\quad \cdots, (\tau_1 \omega \tau_2)((r_{(n-1)1}, r_{(n-1)2}),\\ &\quad b_n, (q_1, q_2))) \mid (r_{i1}, r_{i2}) \in Q_1 \times Q_2\}\\ &= \vee \{\wedge (\wedge (t_1(p_1, \omega(p_2, b_1), r_{11}), \tau_2(p_2, b_1, r_{12})),\\ &\quad \wedge (\tau_1(r_{11}, \omega(r_{12}, b_2), r_{21}), \tau_2(r_{12}, b_2, r_{22})),\\ &\quad \cdots, \wedge (\tau_1(r_{(n-1)1}, \omega(r_{(n-1)2}, b_n), q_1),\\ &\quad \tau_2(r_{(n-1)2}, b_n, q_2))) \mid r_{i1} \in Q_1, r_{i2} \in Q_2\}\\ &= \wedge (\vee \{\vee \{\wedge (\tau_1(p_1, \omega(p_2, b_1), r_{11}),\\ &\quad \tau_1(r_{11}, \omega(r_{12}, b_2), r_{21}) \cdots,\\ &\quad \tau_1(r_{(n-1)1}, \omega(r_{(n-1)2}, b_n), q_1)) \mid r_{i1} \in Q_1\} \mid r_{i2} \in Q_2\},\\ &\quad \vee \{\wedge (\tau_2(p_2, b_1, r_{12}), \tau_2(r_{12}, b_2, r_{22}),\\ &\quad \cdots, \tau_2(r_{(n-1)2}, b_n, q_2)) \mid r_{i2} \in Q_2\})\\ &= \wedge (\vee \{\tau_1^+(p_1, \omega(p_2, b_1)\omega(r_{12}, b_2)\\ &\quad \cdots \omega(r_{(n-1)2}, b_n), q_1) \mid r_{i2} \in Q_2\},\\ &\quad \tau_2^+(p_2, b_1 \cdots b_n, q_2))\\ &= \wedge (\tau_1^+(p_1, \omega^+(p_2, b_1 \cdots b_n), q_1),\\ &\quad \tau_2^+(p_2, b_1 \cdots b_n, q_2))\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $_2, q_2 \in Q_2$.

Definition 4.3 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ be fuzzy finite state machines. The wreath product $M_1 \circ M_2$ of M_1 and M_2 is the fuzzy finite state machin $(Q_1 \times Q_2, X_1^{Q_2} \times X_2, \tau_1 \circ \tau_2)$ with

$$\begin{aligned}&(\tau_1 \circ \tau_2)((p_1, p_2), (f, b), (q_1, q_2))\\ &\quad = \wedge (\tau_1(p_1, f(p_2), q_1), \tau_2(p_2, b, q_2))\end{aligned}$$

Theorem 4.4 Let $M_1=(Q_1, X_1, \tau_1)$ and $M_2=(Q_2, X_2, \tau_2)$ are fuzzy finite state machines. Then

$$M_1 \omega M_2 \leq_c M_1 \circ M_2$$

Proof. Let ξ: $X_2 \to X_1^{Q_2} \times X_2$ be a function such that $\xi(x_2)=(\xi_1(x_2), \xi_2(x_2))$ where $x_2 \in X_2$, $\xi_1(x_2)$: $Q_2 \to X_1$ is a function defined by $\xi_1(x_2)(p_2)=\omega(p_2, \xi_2(x_2))$ and $\xi_2=I_{X_2}$. And let $\eta=1_{Q_1 \times Q_2}$. Then for each (p_1, p_2), $(q_1, q_2) \in Q_1 \times Q_2$ and $x_2 \in X_2$, we have

$$\begin{aligned}&(\tau_1 \omega t_2)(\eta((p_1, p_2)), x_2, \eta((q_1, q_2))\\ &\quad = (\tau_1 \omega t_2)((p_1, p_2), \xi_2(x_2), (q_1, q_2))\\ &\quad = \wedge (\tau_1(p_1, \omega(p_2, \xi_2(x_2)), q_1), \tau_2(p_2, \xi_2(x_2), q_2))\\ &\quad = \wedge (\tau_1(p_1, \xi_1(x_2)(p_2), q_1), \tau_2(p_2, \xi_2(x_2), q_2))\\ &\quad = (\tau_1 \circ \tau_2)((p_1, p_2), (\xi_1(x_2), \xi_2(x_2)), (q_1, q_2))\\ &\quad = (\tau_1 \circ \tau_2)((p_1, p_2), \xi(x_2), (q_1, q_2))\end{aligned}$$

Hence

$$M_1 \omega M_2 \leq M_1 \circ M_2$$

Corollary 4.5 Let $M_1=(Q_1, X_1, \tau_1)$, $M_2=(Q_2, X_2, t_2)$ and $M=(Q, X, \tau)$ are fuzzy finite state machines. If $M \leq M_1 \omega M_2$, then $M \leq M_1 \circ M_2$.

Proof. It is clear from Theorem 4.4 and Proposition 2.3.

Remark. Corollary 4.5 can be proved directly.

5.Conclusion

In this paper, we introduce the concepts of coverings, restricted direct products, full directed products, cascade products and wreath products of fuzzy finite state machines that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures.

References

[1] W. M. L. Holcombe, *Algebraic automata theory*, (Cambridge University Press, 1982).

[2] S. J. Cho, J. G. Kim and S. T. Kim, On T-gen-

eralized subsystems of *T*-generalized state machines, *Far East J. Math. Sci.*, **5**(1), pp. 131-151, 1997.
[3] Y. H. Kim, J. G. Kim and S. J. Cho, Products of *T*-generalized state machines and *T*-generalized transformation semigroups, *Fuzzy Sets and Systems*, **93**, pp. 87-97, 1998.
[4] D. S. Malik, J. N. Mordeson and M. K. Sen, On subsystems of a fuzzy finite state machine, *Fuzzy Sets and Systems*, **68**, pp. 83-92, 1994.
[5] D. S. Malik, J. N. Mordeson and M. K. Sen, Semigroups of fuzzy finite state machines, in: P. P. Wang, ed., *Advances in Fuzzy Theory and Technology*, **II**, pp. 87-98, 1994.
[6] D. S. Malik, J. N. Mordeson and M. K. Sen, Submachines of fuzzy finite state machines, *J. Fuzzy Math.* **4**, pp. 781-792, 1994.
[7] W. G. Wee, On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph.D. Thesis, Purdue Univ., 1967.
[8] L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8**, pp. 338-353, 1965.

김 재 겸(Jae-Gyeom Kim) 정회원
1981년 : 고려대학교 수학과(이학사)
1983년 : 고려대학교 수학과 대학원(이학사)
1987년 : 고려대학교 수학과 대학원(이학박사)
1989년~현재 : 경성대학교 수학과 재직(부교수)
주관심분야 : Automata 이론, 암호이론, ATM 망 제어

조 성 진(Sung-Jin Cho) 정회원
1979년 : 강원대학교 수학교육과(이학사)
1981년 : 고려대학교 수학과 대학원(이학석사)
1988년 : 고려대학교 수학과 대학원(이학박사)
1988년~현재 : 부경대학교 자연과학대학 수리과학부 재직(부교수)
주관심분야 : Automata 이론

이 숭 희(Soong Hee Lee)
1987년 : 경북대학교 전자공학과(공학사)
1990년 : 경북대학교 전자공학과(공학석사)
1995년 : 경북대학교 전자공학과(공학박사)
1987년~1997년 : 한국전자통신연구원 선임연구원
1997년~현재 : 인제대학교 정보통신공학과 재직(전임강사)
주관심분야 : ATM/B-ISDN, 통신시스템