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Another Normalization of Fuzzy Ideals in Near-rings

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ABSTRACT

In this paper we give another normalization of fuzzy ideals in near-rings.

1. Introduction

W. Liu[17] has studied fuzzy ideals of a ring, and many researchers[5,12,13,22] are engaged in extending the concepts. The notion of fuzzy ideals and its properties were applied to various areas: semigroups [14,15,16,19,21], distributive lattices[2], artinian rings [18], BCK-algebras[20]. S. Abou-Zaid[1] studied fuzzy ideals in near-rings, and many followers[8,9,10, 11] discussed further properties of fuzzy ideals in near-rings. The concept of normalization of fuzzy ideals was applied to *BCK*-algebras[7], Gamma-rings[6], and near-rings[11]. In this paper we give another normalization of fuzzy ideals in near-rings and discuss some properties of fuzzy ideals.

A non-empty set R with two binary operations "+" and "$\cdot$" is called a *near-ring*[3] if (1) $(R, +)$ is a group, (2) $(R, \cdot)$ is a semigroup, (3) $x\cdot(y+z)=x\cdot y+x\cdot z$ for all $x,y,z\in R$. We will use the word 'near-ring' to mean 'left near-ring'. We denote xy instead of $x\cdot y$. Note that $x0=0$ and $x(-y)=-xy$ but in general $0x\neq 0$ for some $x\in R$. An *ideal* I of a near-ring R is a subset of R such that (4) $(I, +)$ is a normal subgroup of $(R, +)$, (5) $RI\subseteq I$, (6) $(r+i)s-rs\in I$ for any $i\in I$ and any $r,s\in R$. Note that I is a *left ideal* of R if I satisfies (4) and (5), and I is a *right ideal* of R if I satisfies (4) and (6).

2. Normalizaton of Fuzzy Ideals

Let R be a near-ring and μ be a fuzzy subset of R. We say μ a *fuzzy subnear-ring* of R if (7) $\mu(x-y)\geq\min\{\mu(x), \mu(y)\}$, (8) $\mu(xy)\geq\min\{\mu(x), \mu(y)\}$, for all $x,y\in R$. μ is called a *fuzzy ideal* of R if μ is a fuzzy subnear-ring of R and (9) $\mu(x)=\mu(y+x-y)$, (10) $\mu(xy)\geq\mu(y)$, (11) $\mu((x+i)y-xy\geq\mu(i)$, for any $x,y,i\in R$. Note that μ is a fuzzy left ideal of R if it satisfies (7), (9) and (10), and μ is a fuzzy right ideal of R if it satisfies (7), (8), (9) and (11) (see[1]).

We give a example of fuzzy ideals of near-rings which was discussed in[12].

Example 2.1 ([12]) Let $R:=\{a, b, c, d\}$ be a set with two binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then we can easily see that $(R; +, \cdot)$ is a (left) near-ring. Define a fuzzy subset μ: $R\rightarrow[0,1]$ by $\mu(c)=\mu(d)<\mu(b)<\mu(a)$. Then μ is a fuzzy ideal of R.

Lemma 2.2 ([12]) If a fuzzy subset μ of R satisfies the property (7) then

(i) $\mu(0_R)\geq\mu(x)$,

(ii) $\mu(-x)=\mu(x)$, *for all* $x\in R$.

For a given fuzzy left (right resp.) ideal of R, we note that $\mu(0)$ is the largest value of μ. It is often convenient to have $\mu(0)=1$. A fuzzy left (right resp.) ideal μ of R is said to be *normal* if $\mu(0)=1$.

Theorem 2.3 Let μ be a fuzzy left (right resp.) ideal of R and let μ be a fuzzy subset in R defined by

$$\hat{\mu}(x):=\frac{1}{\mu(0)}\mu(x)$$

for all $x\in R$. Then $\hat{\mu}$ is a normal fuzzy left (resp.

right) ideal of R containing μ.

Proof. Let μ be a fuzzy left ideal of R. Then

$$\begin{aligned}\min\{\hat{\mu}(x),\hat{\mu}(y)\} &= \min\{\frac{1}{\mu(0)}\mu(x),\ \frac{1}{\mu(0)}\mu(y)\}\\ &= \frac{1}{\mu(0)}\min\{\mu(x),\mu(y)\}\\ &\le \frac{1}{\mu(0)}\mu(x-y)\\ &= \hat{\mu}(x-y),\end{aligned}$$

$$\begin{aligned}\hat{\mu}(xy) &= \frac{1}{\mu(0)}\mu(xy)\\ &\ge \frac{1}{\mu(0)}\mu(y)\\ &= \hat{\mu}(y)\end{aligned}$$

and

$$\begin{aligned}\hat{\mu}(x) &= \frac{1}{\mu(0)}\mu(x)\\ &= \frac{1}{\mu(0)}\mu(y+x-y)\\ &= \hat{\mu}(y+x-y),\end{aligned}$$

for all $x,y\in R$. Hence $\hat{\mu}$ is a fuzzy left ideal of R. Suppose that μ is a fuzzy right ideal of R. Then

$$\begin{aligned}\hat{\mu}(xy) &= \frac{1}{\mu(0)}\mu(xy)\\ &\ge \frac{1}{\mu(0)}\min\{\mu(x),\mu(y)\}\\ &= \min\{\hat{\mu}(x),\hat{\mu}(y)\}\end{aligned}$$

and

$$\begin{aligned}\hat{\mu}((x+i)y-xy) &= \frac{1}{\mu(0)}\mu((x+i)y-xy)\\ &\ge \frac{1}{\mu(0)}\mu(i)\\ &= \hat{\mu}(i)\end{aligned}$$

for all $x,y,i\in R$. Hence $\hat{\mu}$ is a right ideal of R. Clearly $\hat{\mu}(0)=1$ and $\mu\subseteq\hat{\mu}$. This completes the proof.

Noticing that $\mu\subseteq\hat{\mu}$, we have the following corollary.

Corollary 2.4 If μ is a fuzzy left (right resp.) ideal of R satisfying $\hat{\mu}(x)=0$ for some $x\in R$, then $\mu(x)=0$ also.

Lemma 2.5 ([1]) Let μ be a fuzzy left (right resp.) ideal of R. Then $R_\mu:=\{x\in R|\ \mu(x)=\mu(0)\}$ is a left (right resp.) ideal of R.

Using the Lemma 2.5 we prove the following theorem.

Theorem 2.6 Let μ and ν be fuzzy left (right resp.) ideals of R. If $\mu\subseteq\nu$ and $\mu(0)\subseteq\nu(0)$, then $R_\mu\subseteq R_\nu$.

Proof. Assume that $\mu\subseteq\nu$ and $\mu(0)=\nu(0)$. If $x\in R_\mu$ then $\nu(x)\ge\mu(x)=\mu(0)=\nu(0)$. Noticing that $\nu(x)\le\nu(0)$ for all $x\in R$, we have $\nu(x)=\nu(0)$, that is, $x\in R_\nu$. This completes the proof.

Corollary 2.7 If μ and ν are normal fuzzy left (right resp.) ideals of R satisfying $\mu\subseteq\nu$, then $R_\mu\subseteq R_\nu$.

Proof. Since μ and ν are normal, $\mu(0)=\nu(0)=1$. It follows from Theorem 2.6 that $R_\mu\subseteq R_\nu$.

Proposition 2.8 ([1]) Let χ_I be a characteristic function of a non-empty subset I of R. Then χ_I is a fuzzy left (right resp.) ideal if and only if I is a left (right resp.) ideal of R.

Theorem 2.9 For any left (right resp.) ideal I of R, the characteristic function χ_I of I is a normal fuzzy left (right resp.) ideal of R and $R_{\chi_I}=I$.

Proof. It follows from Proposition 2.8 that χ_I is a fuzzy left (right resp.) ideal of R. Since I is a left (right resp.) ideal of R, $0\in I$ and hence $\chi_I(0)=1$, i.e., χ_I is a normal fuzzy left (right resp.) ideal of R. Moreover,

$$\begin{aligned}R_{\chi_I} &= \{x\in R\,|\,\chi_I(x)=\chi_I(0)\}\\ &= \{x\in R\,|\,\chi_I(x)=1\}\\ &= I,\end{aligned}$$

completing the proof.

Theorem 2.10 A fuzzy left (right resp.) ideal μ of R is normal if and only if $\hat{\mu}=\mu$.

Proof. Assume that μ is a normal fuzzy left (right resp.) ideal of R and let $x\in$R. Then $\hat{\mu}(x)=\frac{1}{\mu(0)}\mu(x)=\mu(x)$, and hence $\hat{\mu}=\mu$.

Theorem 2.11 If μ is a fuzzy left (right resp.) ideal of R, then $\hat{\hat{\mu}}=\hat{\mu}$.

Proof. For any $x\in R$ we have $\hat{\hat{\mu}}(x)=\frac{1}{\hat{\mu}(0)}\hat{\mu}(x)=\hat{\mu}(0)$, since $\hat{\mu}(0)=\frac{1}{\mu(0)}\mu(0)=1$, completing the proof.

Since μ is normal, $\hat{\mu}=\mu$. By applying Theorem 2.11 we obtain:

Corollary 2.12 If μ is a normal fuzzy left (right resp.) ideal of R, then $\hat{\hat{\mu}}=\mu$.

Theorem 2.13 Let μ be a fuzzy left (right resp.) ideal of R. If there exists a fuzzy left (right resp.) ideal ν of R satisfying $\hat{\nu}\subseteq\mu$, then μ is normal.

Proof. Suppose there exists a fuzzy left (right resp.) ideal ν of R such that $\hat{\nu}\subseteq\mu$. Then $1=\hat{\nu}(0)\leq\mu(0)$, whence $\mu(0)=1$. The proof is complete.

By using Theorem 2.10, we have the following corollary.

Corollary 2.14 Let μ be a fuzzy left (right resp.) ideal of R. If there exists a fuzzy left (resp. right) ideal ν of R satisfying $\hat{\nu}\subseteq\mu$, then $\hat{\mu}=\mu$.

Theorem 2.15 Let μ be a non-constant normal fuzzy left (resp. right) ideal of R, which is maximal in the poset of normal fuzzy left (right resp.) ideals under set inclusion. Then μ takes only the values 0 and 1.

Proof. Note that $\mu(0)=1$. Let $x\in R$ be such that $\mu(x)\neq 1$. It is enough to show that $\mu(x)=0$. Assume that there exists $a\in R$ such that $0<\mu(a)<1$. Define a fuzzy subset $\nu: R\rightarrow[0,1]$ by $\nu(x):=\frac{1}{2}\{\mu(x)+\mu(a)\}$ for all $x\in R$. Then clearly ν is well-defined. Assume that μ is a normal fuzzy left ideal of R. Let $x,y\in R$. Then

$$\begin{aligned}\nu(x-y)&=\frac{1}{2}\{\mu(x-y)+\mu(a)\}\\&\geq\frac{1}{2}\{\min\{\mu(x),\mu(y)\}+\mu(a)\}\\&=\min\{\frac{1}{2}\{\mu(x)+\mu(a)\},\frac{1}{2}\{\mu(y)+\mu(a)\}\}\\&=\min\{\nu(x),\nu(y)\},\end{aligned}$$

$$\begin{aligned}\nu(x)&=\frac{1}{2}\{\mu(x)+\mu(a)\}\\&=\frac{1}{2}\{\mu(y+x-y)+\mu(a)\}\\&=\nu(y+x-y)\end{aligned}$$

and

$$\begin{aligned}\nu(xy)&=\frac{1}{2}\{\mu(xy)+\mu(a)\}\\&\geq\frac{1}{2}\{\mu(y)+\mu(a)\}\\&=\nu(y).\end{aligned}$$

Hence ν is a fuzzy left ideal of R. Suppose that μ is a fuzzy right ideal of R. Then

$$\begin{aligned}\nu(xy)&=\frac{1}{2}\{\mu(xy)+\mu(a)\}\\&\geq\frac{1}{2}\{\min\{\mu(x),\mu(y)\}+\mu(a)\}\\&=\min\{\frac{1}{2}\{\mu(x)+\mu(a)\},\frac{1}{2}\{\mu(y)+\mu(a)\}\}\\&=\min\{\nu(x),\nu(y)\},\end{aligned}$$

and

$$\begin{aligned}\nu((x+i)y-xy)&=\frac{1}{2}\{\mu((x+i)y-xy)+\mu(a)\}\\&\geq\frac{1}{2}\{\mu(i)+\mu(a)\}\\&=\nu(i)\end{aligned}$$

for all $x,y,i\in R$. Hence ν is a fuzzy right ideal of R.

Since $\hat{\nu}(0)=1$, $\hat{\nu}$ is a normal fuzzy left (right resp.) ideal of R. Noticing that

$$\hat{\nu}(0)=1>\hat{\nu}(a)=\frac{2\mu(a)}{\mu(0)+\mu(a)}>\mu(a),$$

we know that $\hat{\nu}$ is non-constant. It follows from $\hat{\nu}(a)>\mu(a)$ that μ is not maximal. This proves the theorem.

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