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Initial Smooth Fuzzy Topological Spaces

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ABSTRACT

We will define a base of a smooth fuzzy topological space and investigate some properties of bases. We will prove the existences of initial smooth fuzzy topological spaces. From this fact, we can define subspaces and products of smooth fuzzy topological spaces.

1. Introduction

R. N. Hazra *et al.*[4] introduced the fuzziness in the concept of openness of a fuzzy subset as a generalization of Chang's fuzzy topology[1]. It has been developed in many directions[2, 3, 5, 6]. Moreover, R. Srivastava[6] introduced the concept of a base for a smooth fuzzy topological space in view of the definition of R. N. Hazra *et al.*[4].

In this paper, we will define a base of a smooth fuzzy topological space in view of the definition of K.C. Chattopadhyay *et al.*[2] as a generalization of R. Srivastava [6] and investigate some properties of bases. We will prove the existences of initial smooth fuzzy topological spaces. From this fact, we can define subspaces and products of smooth fuzzy topological spaces.

In this paper, all the notations and the definitions are standard in fuzzy set theory.

2. Preliminaries

Definition 2.1 [2]. Let X be a nonempty set. A function $\tau : I^X \to I$ is called a *gradation of openness* on X if it satisfies the following conditions:

(01) $\tau(\tilde{0}) = \tau(\tilde{1})=1$, where $\tilde{0}(x)=0$, $\tilde{1}(x)=1$, for all $x\in X$,

(02) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1, \mu_2\in I^X$,

(03) $\tau(\bigvee_{i\in\Lambda}\mu_i)\geq\bigwedge_{i\in\Lambda}\tau(\mu_i)$, for any family $\{\mu_i|i\in\Lambda\}\subseteq I^X$.

The pair (X, τ) is called a *smooth fuzzy topological space.*

Let τ be a gradation of openness on X and $F : I^X\to I$ be defined by $F(\mu)=\tau(\mu^c)$. Then F is called a *gradation of closedness* on X.

Let (X, τ) be a smooth fuzzy topological space, then for each $r\in I$, $\tau^r=\{\mu\in I^X|\tau(\mu)\geq r\}$ is a Chang's fuzzy topology on X.

If $f: X\to Y$ is a function, then for any $\mu\in X$, the *direct image* of μ is defined as

$$f(\mu)(y)=\begin{cases} \sup\mu(x)_{x\in f^{-1}(y)} & \text{if } f^{-1}(y)\neq\emptyset, \\ 0 & \text{otherwise}\end{cases}$$

and for $\nu\in I^Y$, the preimage of ν is defined as $f^{-1}(\nu)=\nu^{\circ f}$.

Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. A function $f:(X, \tau_1)\to(Y, \tau_2)$ is called a *gradation preserving map* (gp-map) if $\tau_2(\mu)\leq\tau_1(f^{-1}(\mu))$ for all $\mu\in I^Y$.

Let τ_1 and τ_2 be gradations of openness on X. We say τ_1 is *finer* than τ_2 (τ_2 is *coarser* than τ_1), denoted by $\tau_2\leq\tau_1$, if $\tau_2(\mu)\leq\tau_1(\mu)$ for all $\mu\in I^X$.

We can easily prove the following lemma.

Lemma 2.2. If $f: X\to Y$ is a function, then we have the following properties for direct and inverse images of fuzzy sets under mappings: for μ, $\mu_i\in I^Y$ and ν, $\nu_i\in I^X$,

(1) $\mu\geq f(f^{-1}(\mu))$ with equality if f is suvjective,

(2) $\nu\leq f^{-1}(f(\nu))$ with equality if f is injective,

(3) $f^{-1}(\mu^c)=f^{-1}(\mu)^c$,

(4) $f^{-1}(\bigvee_{i\in I}\mu_i)=\bigvee_{i\in I}f^{-1}(\mu_i)$,

(5) $f^{-1}(\bigwedge_{i\in I}\mu_i)=\bigwedge_{i\in I}f^{-1}(\mu_i)$,

(6) $f(\bigvee_{i\in I}\nu_i)=\bigvee_{i\in I}f(\nu_i)$,

(7) $f(\bigwedge_{i\in I} \nu_i) = \bigwedge_{i\in I} f(\nu_i)$, with equality if f is injective.

3. Initial smooth fuzzy topological spaces

Definition 3.1. Let X be a nonempty set and $\tilde{0}\notin\Theta\subset I^X$. A function $\beta:\Theta\to I$ is called a *base* on X if it satisfies the following conditions:

(B1) $\beta(\tilde{1})=1$,
(B2) $\beta(\mu_1\wedge\mu_2)\geq\beta(\mu_1)\wedge\beta(\mu_2)$, for all $\mu_1, \mu_2\in\Theta$.

Remark 1. (1) Let (X, τ) be a smooth fuzzy topological space. Let

$$\tilde{0}\notin\Theta=\{\tilde{1},\mu,\nu,\mu\wedge\nu\in I^X \mid \tau(\mu\wedge\nu)\geq\tau(\mu)\wedge\tau(\nu)\}$$

be given. Define the function $\beta:\Theta\to I$ on X by, for all $\mu\in\Theta$,

$$\beta(\mu)=\tau(\mu).$$

Then β is a base on X. In this case, β is called a *base* for τ on X.

(2) Let β be a base on X. Then for each $r\in I$, $\beta^r=\{\mu\in I^X|\beta(\mu)\geq r\}$ is a base for a Chang's fuzzy topology on X.

A base β always *generates* a gradation of openness on X in the following sense:

Theorem 3.2. Let β be a base on X. Define the function $\tau_\beta:I^X\to I$ as follows: for each $\mu\in I^X$,

$$\tau_\beta(\mu)=\begin{cases}\beta(\mu) & \text{if } \mu\in\Theta\\ sup\{\bigwedge_{j\in\Lambda}\beta(\mu_j)\} \text{ if } \mu=\bigvee_{j\in\Lambda}\mu_j, & \text{for any } \{\mu_j\}_{j\in\Lambda}\subseteq\Theta,\\ 1 & \text{if } \mu=\tilde{0},\\ 0 & \text{otherwise.}\end{cases}$$

Then (X, τ_β) is a smooth fuzzy topological space.

The proof of Theorem 3.2 is trivial from the definition of a gradation of openness.

From Theorem 3.2, we can define the following definition.

Definition 3.3. If β is a base on X, then τ_β is called the *gradation of openness generated by* β. (X, τ_β) is called a *smooth fuzzy topological space generated by* a baseβ on X.

Remark 2. If β is a base for τ on X, in general, τ_β is coarser than τ.

Example 1. Let X be a nonempty set and two distinct fuzzy sets λ, $\mu\in I^X$ such that $\lambda\wedge\mu\neq\tilde{0}$. Define the function $\tau:I^X\to I$ as follows:

$$\tau(\tilde{1})=1,\ \tau(\tilde{0})=1,\ \tau(\lambda)=\frac{1}{2},\ \tau(\mu)=\frac{1}{4},$$
$$\tau(\lambda\vee\mu)=\frac{2}{3},\ \tau(\lambda\wedge\mu)=\frac{1}{3},\ \tau(\nu)=0 \text{ otherwise.}$$

Put $\Theta=\{\tilde{1}, \lambda, \mu, \lambda\wedge\mu\}$. Define $\beta:\Theta\to I$ by

$$\beta(\tilde{1})=1,\ \beta(\lambda)=\frac{1}{2},\ \beta(\mu)=\frac{1}{4},\ \beta(\lambda\wedge\mu)=\frac{1}{3}.$$

Then β is a base on X. From Theorem 3.2, τ_β generated by β is as follows:

$$\tau_\beta(\tilde{1})=1,\ \tau_\beta(\lambda)=\frac{1}{2},\ \tau_\beta(\mu)=\frac{1}{4},\ \tau_\beta(\lambda\wedge\mu)=\frac{1}{3}.$$
$$\tau_\beta(\tilde{1})=1,\ \tau_\beta(\lambda\vee\mu)=\frac{1}{4},\ \tau_\beta(\nu)=0 \text{ otherwise.}$$

Hence τ_β is coarser than τ.

Theorem 3.4 [2]. Let $(X, \tau_i)_{i\in\Gamma}$ be smooth fuzzy topological spaces. Define the function $\tau:I^X\to I$ on X by, for every $\mu\in I^X$,

$$\tau(\mu)=\bigwedge_{i\in\Gamma}\tau_i(\mu).$$

Then τ is a gradation of openness on X.

Theorem 3.5. Let $(X, \tau_i)_{i\in\Gamma}$ be smooth fuzzy topological spaces. Let

$$\Theta=\{\tilde{0}\neq\mu=\bigwedge_{i=1}^{n}\mu_{k_i} \mid \tau_{k_i}(\mu_{k_i})>0 \text{ for all } k_i\in K\}$$

for every finite index set $K=\{k_1, ..., k_n\}\subset\Gamma$. Define the function $\beta:\Theta\to I$ on X by

$$\beta(\mu)=sup\{\bigwedge_{i=1}^{n}\tau_{k_i}(\mu_{k_i}) \mid \mu=\bigwedge_{i=1}^{n}\mu_{k_i}\}$$

for every finite index set $K=\{k_1, ..., k_n\}\subset\Gamma$. Then:

(1) β is a base on X.

(2) The gradation of opennesss τ_β generated by β is the coarsest gradation of opennesss on X finer than τ_i, for all $i\in\Gamma$.

Proof : (1) (B1) It is trivial from the definition of β.
(B2) Suppose that there exist μ, $\nu\in\Theta$, $c\in(0, 1)$ such that

$$\beta(\mu\wedge\nu)<c<\beta(\mu)\wedge\beta(\nu).$$

Then there exist finite index sets $K=\{k_1, ..., k_n\}$, $L=\{l_1, ..., l_m\}$ such that $\mu=\bigwedge_{i=1}^{n}\mu_{k_i}$, $\nu=\bigwedge_{j=1}^{m}\nu_{l_j}$ and

$$\bigwedge_{i=1}^{n}\tau_{k_i}(\mu_{k_i})>c,\ \bigwedge_{j=1}^{m}\tau_{l_j}(\nu_{l_j})>c.$$

On the other hand, since $\mu\wedge\nu=(\bigwedge_{i=1}^{n}\mu_{k_i})\wedge(\bigwedge_{j=1}^{m}\nu_{l_j})$, we have

$$\beta(\mu\wedge\nu)\geq(\bigwedge_{i=1}^{n}\tau_{k_i}(\mu_{k_i}))\wedge(\bigwedge_{j=1}^{m}\tau_{l_j}(\nu_{l_j}))>c.$$

It is a contradiction.

(2) First, we will show that for all $\mu\in I^X$, $i\in\Gamma$

$$\tau_i(\mu)\leq\tau_\beta(\mu).$$

If $\mu\in\Theta$, by the definition of β, it is trivial. If $\mu\notin\Theta$ and $\mu\neq\tilde{0}$, we have $\tau_i(\mu)=0$, for all $i\in\Gamma$. Hence it is trivial. If $\mu=\tilde{0}$, we have $\tau_i(\mu)=\tau_\beta(\mu)=1$.

Second, if $\tau_i(\mu)\leq\tau^*(\mu)$, for all $\mu\in I^X$, $i\in\Gamma$ we will show that

$$\tau_\beta(\mu)\leq\tau^*(\mu),\ \text{for all } \mu\in I^X.$$

If $\mu\neq\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, it is trivial from Theorem 3.2.

Suppose there exists $\mu\in I^X$ such that $\tau_i(\mu)>\tau^*(\mu)$. If $\mu\in\Theta$, by the definition of β, then there exists a finite index set $K=\{k_1, ..., k_n\}\subset\Gamma$ such that $\mu=\bigwedge_{i=1}^{n}\mu_{k_i}$, and

$$\tau_\beta(\mu)\geq\bigwedge_{i=1}^{n}\tau_{k_i}(\mu_{k_i})>\tau^*(\mu).$$

On the other hand, since $\tau_{k_i}(\mu_{k_i})\leq\tau^*(\mu_{k_i})$, we have

$$\bigwedge_{i=1}^{n}\tau_{k_i}(\mu_{k_i})\leq\bigwedge_{i=1}^{n}\tau^*(\mu_{k_i})\leq\tau^*(\mu).$$

Hence, by the definition of τ_β, we have $\tau_\beta(\mu)\leq\tau^*(\mu)$ It is a contradiction.

If $\mu=\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, then we have

$$\bigwedge_{k\in\Lambda}\tau_\beta(\mu_k)\leq\bigwedge_{k\in\Lambda}\tau^*(\mu_k)\leq\tau^*(\bigvee_{k\in\Lambda}\mu_k)=\tau(\mu).$$

From Theorem 3.2, we have $\tau_\beta(\mu)\leq\tau^*(\mu)$. It is a contradiction.

Example 2. Let X be a set and two distinct fuzzy sets λ, $\mu\in I^X$ such that $\lambda\wedge\mu\neq\tilde{0}$.

Define the functions τ_1, $\tau_2: I^X\to I$ as follows:

$$\tau_1(\tilde{1})=1, \tau_1(\tilde{0})=1, \tau_1(\lambda)=\frac{1}{2}, \tau_1(\nu)=0 \text{ otherwise,}$$

and

$$\tau_2(\tilde{1})=1, \tau_2(\tilde{0})=1, \tau_2(\mu)=\frac{1}{3}, \tau_2(\nu)=0 \text{ otherwise.}$$

Then $\Theta=\{\tilde{1}, \lambda, \mu, \lambda\wedge\mu\}$ and it is defined $\beta:\Theta\to I$ by

$$\beta(\tilde{1})=1, \beta(\lambda)=\frac{1}{2}, \beta(\mu)=\frac{1}{3}, \beta(\lambda\wedge\mu)=\frac{1}{3}.$$

From Theorem 3.5, β is a base on X. From Theorem 3.2, τ_β generated by β is as follows:

$$\tau_\beta(\tilde{1})=1, \tau_\beta(\lambda)=\frac{1}{2}, \tau_\beta(\mu)=\frac{1}{3}, \tau_\beta(\lambda\wedge\mu)=\frac{1}{3},$$

and

$$\tau_\beta(\tilde{0})=1, \tau_\beta(\lambda\vee\mu)=\frac{1}{3}, \ \tau_\beta(\nu)=0 \text{ otherwise.}$$

The gradation of openness τ_β is the coarsest gradation of opennes on X finer than τ_1 and τ_2.

Theorem 3.6. Let X be a nonempty set and $\Delta(X)$ be the set of all gradations of openness on X. Then the partially ordered set $(\Delta(X), \leq)$ is a complete lattice.

Proof: Define τ_0, $\tau_1: I^X\to I$ by, for all $\mu\in I^X$,

$$\tau_0(\mu)=\begin{cases}1 & \text{if } \mu=\tilde{0} \text{ or } \tilde{1}, \\ 0 & \text{otherwise}\end{cases}$$

and for all $\mu\in I^X$, $\tau_1(\mu)=1$, respectively. Then $\Delta(X)$ have the least element τ_0 and the greatest element τ_1, respectively. Hence it is clear from Theorem 3.4 and Theorem 3.5.

Definition 3.7. Let $(X_i, \tau_i)_{i\in\Gamma}$ be a family of smooth fuzzy topological spaces, X a set and $f_i: X\to X_i$ a function for each $i\in\Gamma$. The *initial structure* τ is the coarsest gradation of openness on X with respect to which for each $i\in\Gamma$, f_i is a gp-map.

Theorem 3.8. Let X be a set and (Y, τ) a smooth fuzzy topological space. Let $f: X\to Y$ be a function and $Y_\mu=\{\nu\in I^Y | f^{-1}(\nu)=\mu\}$. Define, for every $\mu\in I^X$,

$$f^{-1}(\tau)(\mu)=\begin{cases}\sup\{\tau(\nu)\mid \nu\in Y_\mu\} & \text{if } Y_\mu\neq\phi,\\ 0 & \text{otherwise}.\end{cases}$$

Then $f^{-1}(\tau)$ is the initial gradation of openness on X with respect to which f is a gp-map.

Proof: First, we will show that $f^{-1}(\tau)$ is a gradation of openness on X.
(O1) It is trivial from the definition of $f^{-1}(\tau)$.
(O2) We will show that for each $\mu_1, \mu_2\in I^X$,

$$f^{-1}(\tau)(\mu_1\wedge\mu_2)\geq f^{-1}(\tau)(\mu_1)\wedge f^{-1}(\tau)(\mu_2).$$

If $f^{-1}(\tau)(\mu_1)=0$ or $f^{-1}(\tau)(\mu_2)=0$, it is trivial.
If $f^{-1}(\tau)(\mu_1)\neq0$ and $f^{-1}(\tau)(\mu_2)\neq0$, for every ε such that

$$f^{-1}(\tau)(\mu_1)\wedge f^{-1}(\tau)(\mu_2)>\varepsilon>0,$$

then there exist $\nu_1, \nu_2\in I^Y$ such that

$$\tau(\nu_1)\geq f^{-1}(\tau)(\mu_1)-\varepsilon,\ f^{-1}(\nu_1)=\mu_1$$
and
$$\tau(\nu_2)\geq f^{-1}(\tau)(\mu_2)-\varepsilon,\ f^{-1}(\nu_2)=\mu_2,$$

respectively. Since $f^{-1}(\nu_1\wedge\nu_2)=\mu_1\wedge\mu_2$ from Lemma 2. 2 (5), we have

$$\begin{aligned}f^{-1}(\tau)(\mu_1\wedge\mu_2)&\geq\tau(\nu_1\wedge\nu_2)\\&\geq\tau(\nu_1)\wedge\tau(\nu_2)\\&\geq f^{-1}(\tau)(\mu_1)\wedge f^{-1}(\tau)(\mu_2)-\varepsilon.\end{aligned}$$

(O3) We will show that for any family $\{\mu_j\mid j\in\Lambda\}I^X$,

$$f^{-1}(\tau)(\bigvee_{j\in\Lambda}\mu_j)\geq\bigwedge_{j\in\Lambda}f^{-1}(\tau)(\mu_j).$$

If $f^{-1}(\tau)(\mu_j)=0$ for some $j\in\Lambda$, it is trivial. For $f^{-1}(\tau)(\mu_j)\neq0$ for all $j\in\Lambda$, suppose that there exist $r\in(0,1)$ and a family $\{\mu_j\}_{j\in\Lambda}$ such that

$$f^{-1}(\tau)(\bigvee_{j\in\Lambda}\mu_j)<r<\bigwedge_{j\in\Lambda}f^{-1}(\tau)(\mu_j).$$

Since $f^{-1}(\tau)(\mu_j)>r$ for all $j\in\Lambda$, by the definition of $f^{-1}(\tau)$, there exists a family $\{\nu_j\}_{j\in\Lambda}$ such that $f^{-1}(\nu_j)=\mu_j$ and $\tau(\nu_j)\geq r$, respectively.

On the other hand, since $f^{-1}(\bigvee_{j\in\Lambda}\nu_j)=\bigvee_{j\in\Lambda}\mu_j$ from Lemma 2.2 (4), we have

$$f^{-1}(\tau)(\bigvee_{j\in\Lambda}\mu_j)\geq\tau(\bigvee_{j\in\Lambda}\nu_j)\geq\bigwedge_{j\in\Lambda}\tau(\nu_j)\geq r.$$

It is a contradiction.

Second, for each $\nu\in I^Y$, by the definition of $f^{-1}(\tau)$ we have

$$f^{-1}(\tau)(f^{-1}(\nu))\geq\tau(\nu)$$

Hence f is a gp-map.

Finally, if $f:(X,\tau^*)\to(Y,\tau)$ is a gp-map, we will show that for all $\mu\in I^X$,

$$f^{-1}(\tau)(\mu)\leq\tau^*(\mu).$$

Suppose that there exists $\mu\in I^X$ such that $f^{-1}(\tau)(\mu)>\tau^*(\mu)$. From the definition of $f^{-1}(\tau)$, there exists a $\nu\in I^Y$ such that $f^{-1}(\nu)=\mu$ and

$$f^{-1}(\tau)(\mu)\geq\tau(\nu)>\tau^*(\mu).$$

On the other hand, since $f:(X,\tau^*)\to(Y,\tau)$ is a gp-map, then

$$\tau(\nu)\leq\tau^*(f^{-1}(\nu))=\tau^*(\mu).$$

It is a contradiction.

From Theorem 3.8, we can define a subspace.

Definition 3.9. Let (X, τ) be a smooth fuzzy topological space and A be a subset of X. The pair $(A, i^{-1}(\tau))$ is said to be a *subspace* of (X, τ) if $i^{-1}(\tau)$ is endowed with the initial gradation of openness on A with respect to which the inclusion map i is a gp-map.

Remark 3. The above definition of a subspace coincides with that defined by K.C. Chattopadhyay *et al.*[2].

Example 3. Let $A=\{x, y\}$ and $X=\{x, y, z\}$ be sets. Let $\nu_1, \nu_2: X\to I$ be fuzzy sets as follows:

$$\nu_1(x)=\frac{1}{2},\ \nu_1(y)=\frac{1}{3},\ \nu_1(z)=\frac{1}{4},$$

and

$$\nu_2(x)=\frac{1}{2},\ \nu_2(y)=\frac{1}{3},\ \nu_2(z)=\frac{3}{4}.$$

Define the gradation of openness $\tau: I^X\to I$ as follows:

$$\tau(\tilde{0})=\tau(\tilde{1})=1,\ \tau(\nu_1)=\frac{2}{3},$$

$$\tau(\nu_2)=\frac{3}{4},\ \tau(\nu)=0 \text{ otherwise}.$$

Let $i: A\to X$ be an inclusion map and $\mu\in I^A$ such that

$$\mu(x)=\frac{1}{2},\ \mu(y)=\frac{1}{3}.$$

From Theorem 3.8 we have

$$i^{-1}(\tau)(\tilde{0})=i^{-1}(\tau)(\tilde{1})=1,$$
$$i^{-1}(\tau)(\mu)=\frac{3}{4},\ i^{-1}(\tau)(\nu)=0 \text{ otherwise.}$$

Therefore $(A,\ i^{-1}(\tau))$ is a subspace of $(X,\ \tau)$.

Theorem 3.10 (*Existence of initial structures*). Let $(X_i,\ \tau_i)_{i\in\Gamma}$ be smooth fuzzy topological spaces, X a set and $f_i : X\rightarrow X_i$ a function, for each $i\in\Gamma$. Let

$$\Theta=\{\tilde{0}\neq\mu=\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\mid \tau_{k_j}(\nu_{k_j})>0 \text{ for all } k_j\in K\}$$

for every finite index set $K=\{k_1,\ ...,\ k_n\}\subset\Gamma$. Define the functions $\beta,\ \beta^* : \Theta\rightarrow I$ on X by

$$\beta(\mu)=\sup\{\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})\mid\mu=\bigwedge_{j=1}^{n}\mu_{k_j}\}$$

and

$$\beta^*(\mu)=\sup\{\bigwedge_{j=1}^{n}\tau_{k_j}(\nu_{k_j})\mid\mu=\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\}$$

where $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ for every finite index $K=\{k_1,\ ...,\ k_n\}\subset\Gamma$. Then:

(1) For every $\mu\in\Theta$, $\beta(\mu)=\beta^*(\mu)$.
(2) β is a base on X.
(3) The gradation of openness τ_β generated by β is the initial gradation of openness on X with respect to which for each $i\in\Gamma$, f_i is a gp-map.
(4) A map $f: (Z,\ \tau_Z)\rightarrow(X,\ \tau_\beta)$ is a gp-map iff for each $i\in\Gamma$, $f_i\circ f$ is a gp-map.

Proof: (1) For $\mu\in\Theta$ such that $\mu=\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})$, let $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ be given. By the definition of $f_{k_j}^{-1}(\tau_{k_j})$, we have

$$\tau_{k_j}(\nu_{k_j})\leq f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j}).$$

Hence $\beta(\mu)\geq\beta^*(\mu)$.

Suppose there exists $\mu\in\Theta$ such that $\beta(\mu)>\beta^*(\mu)$. By the definition of β, then there exists a finite index $K=\{k_1,\ ...,\ k_n\}\subset\Gamma$ such that

$$\beta(\mu)\geq\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})>\beta^*(\mu)$$

with $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ and $\mu=\bigwedge_{j=1}^{n}\mu_{k_j}$. For all $k_j\in K$, by the definition of $f_{k_j}^{-1}(\tau_{k_j})$, there exists a ρ_{k_j} such that

$$f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})\geq\tau_{k_j}(\rho_{k_j})>\beta^*(\mu)$$

with $\mu_{k_j}=f_{k_j}^{-1}(\rho_{k_j})$. It follows that

$$\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})\geq\bigwedge_{j=1}^{n}\tau_{k_j}(\rho_{k_j})>\beta^*(\mu).$$

On the other hand, since $\mu=\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\rho_{k_j})$, by the definition of β^*, we have

$$\bigwedge_{j=1}^{n}\tau_{k_j}(\rho_{k_j})\leq\beta^*(\mu).$$

It is a contradiction.

(2) First, let

$$\Theta_1=\{\tilde{0}\neq\mu=\bigwedge_{i=1}^{m}\mu_{l_i}\mid f_{l_i}^{-1}(\tau_{l_i})(\mu_{l_i})>0 \text{ for all } l_i\in L\}$$

for every finite index set $L=\{l_1,\ ...,\ l_n\}\subset\Gamma$. We will show that $\Theta=\Theta_1$.

Let $\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\in\Theta$. From the definition of $f_{kj}^{-1}(\tau_{kj})$, we have

$$f_{k_j}^{-1}(\tau_{k_j})(f_{k_j}^{-1}(\nu_{k_j}))\geq\tau_{k_j}(\nu_{k_j})>0.$$

Hence, $\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j})\in\Theta_1$, Therefore $\Theta\subseteq\Theta_1$.

Let $\bigwedge_{i=1}^{m}\mu_{l_i}\in\Theta_1$. Since $f_{l_i}^{-1}(\tau_{l_i})(\mu_{l_i})>0$, for all $i=1,\ ...m.$, by the definition of $f_{l_i}^{-1}(\tau_{l_i})$, there exists ν_{l_i} such that $f^{-1}\ (\tau_{l_i})=\mu_{l_i}$ and

$$f_{l_i}^{-1}(\tau_{l_i})(\mu_{l_i})\geq\tau_{l_i}(\nu_{l_i})>0.$$

Hence $\bigwedge_{i=1}^{m}\mu_{l_i}\in\Theta$. Therefore $\Theta_1\subseteq\Theta$.

Finally, from Theorem 3.8, since for each $i\in\Gamma$, $f_i^{-1}\ (\tau_i)$ is the initial gradation of openness on X with respect to which f_i is a gp-map, by Theorem 3.5, β is a base on X.

(3) For every $\nu_i\in I^{X_i}$, by the definition of $f_i^{-1}(\tau_i)$, we have

$$\tau_i(\nu_i)\leq f_i^{-1}(\tau_i)(f_i^{-1}(\nu_i))\leq\tau_\beta(f_i^{-1}(\nu_i)).$$

Hence for each $i\in\Gamma$, f_i is a gp-map.

If $f_i : (X,\ \tau^*)\rightarrow(X_i,\ \tau_i)$ is a gp-map, for each $i\in\Gamma$, we have

$$\tau_i(\nu_i)\leq\tau^*(f_i^{-1}(\nu_i)),\ \text{for all}\ \nu_i\in I^{X_i}.$$

We will show that $\tau_\beta(\mu)\leq\tau^*(\mu)$, for all $\mu\in I^X$.

If $\mu\neq\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, it is trivial from

Theorem 3.2.

Suppose there exists $\mu\in I^X$ such that $\tau_\beta(\mu)>\tau^*(\mu)$ If $\mu\in\Theta$, by the definition of β, then there exists a finite index $K=\{k_1, ..., k_n\}\subset\Gamma$ such that

$$\tau_\beta(\mu)\geq\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})>\tau^*(\mu)$$

with $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ and $\mu=\bigwedge_{j=1}^{n}\mu_{k_j}$.

On the other hand, since $\tau_k(\nu_k)\leq\tau^*(f_k^{-1}(\nu_k))$, for all $\nu_k\in I^{X_k}$, by the definition of $f_k^{-1}(\tau_k)$ we have

$$\begin{aligned}\tau_k(\nu_k)&\leq f_k^{-1}(\tau_k)(f_k^{-1}(\nu_k))\\ &\leq\tau^*(f_k^{-1}(\nu_k)).\ \text{(by Theorem 3.8)}\end{aligned}$$

It follows that

$$\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})\leq\bigwedge_{j=1}^{n}\tau^*(\mu_{k_j})\leq\tau^*(\mu),$$

with $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ and $\mu=\bigwedge_{j=1}^{n}\mu_{k_j}$. Hence, by the definition of τ_β, we have $\tau_\beta(\mu)\leq\tau^*(\mu)$. It is a contradiction.

If $\mu=\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, then we have

$$\bigwedge_{k\in\Lambda}\tau_\beta(\mu_k)\leq\bigwedge_{k\in\Lambda}\tau^*(\mu_k)\leq\tau^*(\mu).$$

Hence, by the definition of τ_β, we have $\tau_\beta(\mu)\leq\tau^*(\mu)$. It is a contradiction.

(4) Necessity of the composition condition is clear since the composition of gp-maps is a gp-map.

Conversely, suppose that $f:(Z,\tau_Z)\to(X,\tau_\beta)$ is not a gp-map. There exists $\mu\in I^X$ such that

$$\tau_z(f^{-1}(\mu))<\tau_\beta(\mu).$$

If $\mu\neq\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, it is a contradiction from Theorem 3.2.

If $\mu\in\Theta$, by the definition of β, then there exists a finite index $K=\{k_1, ..., k_n\}\subset\Gamma$ such that

$$\tau_\beta(\mu)\geq\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})>\tau_Z(f^{-1}(\mu))$$

with $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ and $\mu=\bigwedge_{j=1}^{n}\mu_{k_j}$.

On the other hand, for each $i\in\Gamma$, f_i, *of* is a gp-map, we have for all $\nu_i\in I^{X_i}$,

$$\begin{aligned}\tau_i(\nu_i)&\leq\tau_Z((f_i of)^{-1}(\nu_i))\\ &=\tau_Z(f^{-1}(f_i^{-1}(\nu_i))).\end{aligned}$$

By the definition of $f_i^{-1}(\tau_i)$, we have

$$\begin{aligned}\tau_i(\nu_i)&\leq f_i^{-1}(\tau_i)(f_i^{-1}(\nu_i))\\ &\leq\tau_Z(f^{-1}(f_i^{-1}(\nu_i))).\ \text{(by Theorem 3.8)}\end{aligned}$$

It follows that

$$\begin{aligned}\bigwedge_{j=1}^{n} f_{k_j}^{-1}(\tau_{k_j})(\mu_{k_j})&\leq\bigwedge_{j=1}^{n}\tau_Z(f^{-1}(\mu_{k_j}))\\ &\leq\tau_Z(\bigwedge_{j=1}^{n}f^{-1}(\mu_{k_j}))\\ &=\tau_Z(f^{-1}(\mu)),\end{aligned}$$

with $\mu_{k_j}=f_{k_j}^{-1}(\nu_{k_j})$ and $\mu=\bigwedge_{j=1}^{n}\mu_{k_j}$. Hence, by the definition of τ_β, we have $\tau_\beta(\mu)\leq\tau_Z(f^{-1}(\mu))$. It is a contradiction.

If $\mu=\bigvee_{k\in\Lambda}\mu_k$, for any $\{\mu_k\}_{k\in\Lambda}\subseteq\Theta$, then we have

$$\tau_Z(f^{-1}(\mu_k))\geq\tau_\beta(\mu_k),$$

It follows that

$$\begin{aligned}\bigwedge_{k\in\Lambda}\tau_\beta(\mu_k)&\leq\bigwedge_{k\in\Lambda}\tau_Z(f^{-1}(\mu_k))\\ &\leq\tau_Z(\bigvee_{k\in\Lambda}f^{-1}(\mu_k))\\ &\leq\tau_Z(f^{-1}(\mu)).\end{aligned}$$

Hence, we have $\tau_\beta(\mu)\leq\mu_Z(f^{-1}(\mu))$. It is a contradiction.

From Theorem 3.10, we can define a product in the obvious way.

Definition 3.11. Let X be the product $\Pi_{k\in\Gamma}X_k$ of the family $\{(X_k,\tau_k)\,|\,k\in\Gamma\}$ of smooth fuzzy topological spaces. The initial gradation of openness $\tau=\otimes\tau_k$ on X with respect to which all the projections $\pi_k:X\to X_k$ are gp-maps is called the *product gradation of openness* of $\{\tau_k|k\in\Gamma\}$, and $(X,\otimes\tau_k)$ is called the *product smooth fuzzy topological space*.

Corollary 3.12. Let τ_i be a gradation of openness on X_i, for each $i\in\Gamma$ and $X=\Pi_{i\in\Gamma}X_i$ a set and for each $k\in\Gamma$, $\pi_i:X\to X_i$ a projection mapping. Let

$$\Theta=\{\tilde{0}\neq\mu=\bigwedge_{j=1}^{n}\pi_{k_j}^{-1}(\nu_{k_j})\,|\,\tau_{k_j}(\nu_{k_j})>0\text{ for all }k_j\in K\}$$

for every finite index set $K=\{k_1, ..., k_n\}\subset\Gamma$. Define the function $\beta:\Theta\to I$ on X by

$$\beta(\mu)=\sup\{\bigwedge_{j=1}^{n}\tau_{k_j}(\nu_{k_j})\,|\,\mu=\bigwedge_{j=1}^{n}\pi_{k_j}^{-1}(\nu_{k_j})\}$$

for every finite index set $K=\{k_1, ..., k_n\}\subset\Gamma$. Then:

(1) β is a base on X.

(2) The gradation of openness τ_β generated by β is the product gradation of openness on X with respect to which for each $i \in \Gamma$, π_i is a gp-map.

(3) A map $f : (Z, \tau_Z) \to (X, \tau)$ is a gp-map iff for each $i \in \Gamma$, $\pi_i \circ f$ is a gp-map.

Example 4. Let X and Y be nonempty sets.
Let $\tau_1 : I^X \to I$ be defined by

$$\tau_1(\tilde{0}) = \tau_1(\tilde{1}) = 1, \ \tau_1(\nu) = \frac{1}{2},$$
$$\tau_1(\rho) = 0 \quad \text{otherwise},$$

and $\tau_2 : I^Y \to I$ defined by

$$\tau_2(\tilde{0}) = \tau_2(\tilde{1}) = 1, \ \tau_2(\mu) = \frac{1}{3},$$
$$\tau_2(\rho) = 0 \quad \text{otherwise}.$$

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projection maps.
If $\pi_1^{-1}(\nu) \neq \pi_2^{-1}(\mu)$, by Theorem 3.10, we have

$$\Theta = \{\tilde{1}, \ \pi_1^{-1}(\nu), \ \pi_2^{-1}(\mu), \ \pi_1^{-1}(\nu) \wedge \pi_2^{-1}(\mu)\}.$$

Hence it is defined $\beta : \Theta \to I$ by

$$\beta(\pi_1^{-1}(\nu)) = \tau_1(\nu) = \frac{1}{2}, \ \beta(\pi_2^{-1}(\mu)) = \tau_2(\mu) = \frac{1}{3},$$
$$\beta(\pi_1^{-1}(\nu) \wedge \pi_2^{-1}(\mu)) = \frac{1}{3}, \ \beta(\tilde{1}) = 1.$$

From Theorem 3.2, τ_β generated by β is as follows:

$$\tau_\beta(\tilde{1}) = 1, \ \tau_\beta(\tilde{0}) = 1,$$
$$\tau_\beta(\pi_1^{-1}(\nu)) = \frac{1}{2}, \ \tau_\beta(\pi_2^{-1}(\mu)) = \frac{1}{3},$$
$$\tau_\beta(\pi_1^{-1}(\nu) \wedge \pi_2^{-1}(\nu)) = \frac{1}{3},$$
$$\tau_\beta(\pi_1^{-1}(\nu) \vee \pi_2^{-1}(\nu)) = \frac{1}{3},$$
$$\tau_\beta(\rho) = 0 \quad \text{otherwise}.$$

Then $(X \times Y, \ \tau_\beta)$ is the product smooth fuzzy topological space.

If $\pi_1^{-1}(\nu) = \pi_2^{-1}(\mu)$, by Theorem 3.10, we have

$$\Theta = \{\tilde{1}, \ \pi_1^{-1}(\nu)\}.$$

Hence it is defined $\beta : \Theta \to I$ by

$$\beta(\pi_1^{-1}(\nu)) = \tau_1(\nu) \vee \tau_2(\mu) = \frac{1}{2},$$
$$\beta(\tilde{1}) = 1.$$

From Theorem 3.2, τ_β generated by β is as follows:

$$\tau_\beta(\tilde{1}) = 1, \ \tau_\beta(\tilde{0}) = 1,$$
$$\tau_\beta(\pi_1^{-1}(\nu)) = \frac{1}{2},$$
$$\tau_\beta(\rho) = 0 \quad \text{otherwise}.$$

Then $(X \times Y, \ \tau_\beta)$ is the product smooth fuzzy topological space.

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