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On Fuzzy Connectedness

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ABSTRACT

If there exists a fuzzy continuous function from a fuzzy topological space (X,T) onto the discrete fuzzy topological space with two elements, then the space (X,T) is fuzzy disconnected. However, the converse is not true at all. We introduce an example for this and suggest a sufficient condition for which the converse holds.

1. Introduction

A fuzzy set is a class which admits partial membership in it. Chang generated a natural framework for constructing fuzzy topologies which are generalizations of general topologies in a sense by paying attention to basic concepts such as open sets, closed sets, neighborhoods, interior set, fuzzy continuity, fuzzy compactness, etc.[1]. The behavior of fuzzy connectedness is known as a little bit unexpected, compared with connectedness in general topology. Let 2 be the discrete fuzzy topological space with two elements. If there exists a fuzzy continuous function from a fuzzy topological space $(X, \mathcal{T})$ onto 2, then the fuzzy topological space $(X, \mathcal{T})$ is disconnected in the sense of fuzzy. However, the converse does not hold contrary to general topology. In this paper, we introduce an example which shows this and suggest a sufficient condition for which the converse holds.

2. Preliminaries

In this chapter, we introduce basic definitions and well-known results in fuzzy set and fuzzy topology theory, most of which come from [1], [2] and [3]. By a fuzzy set on a set X we mean a function from a fuzzy topological space X into the unit interval [0,1]. I^X denotes the collection of all fuzzy sets on X. Let A be a fuzzy set on X. The set $\{x \in X | A(x)>0\}$ is called the support of A, denoted by supp(A). Let $\{A_i | i \in I\}$ be a family of fuzzy sets on X. Then, we have that

$\bigcup_{i \in I} supp(A_i) = supp(\bigcup_{i \in I} A_i)$, however

$\bigcap_{i \in I} supp(A_i) \neq supp(\bigcap_{i \in I} A_i)$.

A fuzzy point on X is a fuzzy set whose support is a singleton of X. The fuzzy point P such that supp(P) $=\{a\}$ and $P(a)=\alpha$ is denoted by P(a, α).

Definition 2.1. By a fuzzy topology for X we mean a collection of fuzzy sets on X including ϕ and X which is closed under finite intersection and arbitrary union. A pair $(X, \mathcal{T})$ of a set X and a fuzzy topology $\mathcal{T}$ on X is called a fuzzy topological space.

Each member of $\mathcal{T}$ is called an open fuzzy set and the complement of an open fuzzy is called a closed fuzzy set in X. A fuzzy set U is called a neighborhood of $A \in I^X$ if there is an open fuzzy set $O \in \mathcal{T}$ such that $A \subset O \subset U$. Each open fuzzy set is a neighborhood of its subset.

Definition 2.2. Let (X,T) be a general topological space. The collection

$$\tilde{T} = \{G \mid G \in I^X \text{ and } supp(G) \in T\}$$

is called the fuzzy topology induced by T. The pair $(X, \tilde{T})$ is called the fuzzy topological space induced by (X, T)

Definition 2.3. Let X and Y be ordinary sets and $f: X \rightarrow Y$ a function.

(1) For $A \in I^X$, the image of A, written as $f[A]$, is a fuzzy set on Y defined by for each $y \in Y$,

$$f[A](y) = \begin{cases} lub_{f(x)=y}\{A(x)\} & \text{if } f^{-1}(y) \neq \phi, \\ 0 & otherwise \end{cases}$$

(2) For $B \in I^Y$, the inverse of B, written as $f^{-1}[B]$, is a fuzzy set on X defined by

$$f^{-1}[B](x) = B(f(x)) \text{ for all } x \in X$$

Lemma 2.4. [1]. Let $f:X \to Y$ be a function.

(1) $f^{-1}[B^c] = f^{-1}[B]^c$ for any fuzzy set B on Y.

(2) $f[A^c] \supset f[A]\}^c$ for any fuzzy set A on X, if f is surjective.

(3) $B_1 \subset B_2$ implies $f^{-1}[B_1] \subset f^{-1}[B_2]$ where B_1 and B_2 are fuzzy sets on Y.

(4) $A_1 \subset A_2$ implies $f[A_1] \subset f[A_2]$, where A_1 and A_2 are fuzzy sets on X.

(5) $B \supset f[f^{-1}[B]]$ for any fuzzy set B on Y.

(6) $A \subset f^{-1}[f[A]]$ for any fuzzy set A on X.

(7) Let $f:X \to Y$ and $g:Y \to Z$ be functions. Then $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for any fuzzy set C *on* Z, where $g \circ f$ is the composition of f and g.

Remark 2.5. We can find an incorrect observation in[1]. The condition that f is surjective is necessary in Lemma 2.4. (2). For if $f^{-1}(y) = \phi$, $f[A^c](y) = 0$ and $\{f[A]^c\}(y) = 1$.

From the definition 2.3, we easily observe

(1) $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$

(2) $f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2]$,

for any fuzzy subsets B^1 and B_2 on Y.

Definition 2.6. Let $f:(X, \mathcal{T}) \to (Y, \mathcal{U})$ be a function between two fuzzy topological spaces. f is said to be fuzzy continuous if

$$f^{-1}(B) \in \mathcal{T} \text{ for each } B \in \mathcal{U}.$$

Various types of fuzzy continuity were studied by Chang [1].

Proposition 2.7. [1]. If f be a function from a fuzzy topological space X into a fuzzy topological space Y, then the conditions below are related as follows: (FC1) and (FC2) are equivalent; (FC3) and (FC4) are equivalent; (FC1) implies (FC3), and (FC4) implies (FC5).

(FC1) f is fuzzy continuous.

(FC2) The inverse of every closed fuzzy set is closed.

(FC3) For each fuzzy set A on X, the inverse of every neighborhood of $f[A]$ is a neighborhood of A.

(FC4) For each fuzzy set A on X and each neighborhood V of $f[A]$, there is a neighborhood W of A such that $f[W] \subset V$.

(FC5) For each sequence (A_n) of fuzzy sets converging to a fuzzy set A in X, the sequence $(f[A_n])$ converges to $f(A)$ in Y.

Remark 2.8. (1) Actually, (FC1)-(FC4) are all equivalent. (2) In general, (FC5) does not imply (FC 1). However, we can obtain the following fact:

Proposition 2.9. Let $(X, \mathcal{T})$ and $(X, \mathcal{U})$ be two fuzzy topological spaces and $f:X \to Y$ be a function. Assume that the neighborhood system of each fuzzy in Y is countable. If for each sequence (A_n) of fuzzy sets on X which converges to A in I^X, the sequence $(f[A_n])$ converges to $f[A]$ in Y, then f is fuzzy continuous.

Proof. Let $B \in \mathcal{U}$. To show that $f^{-1}[B]$ is open, by Theorem 3.1 (a) in [1], it suffices to show that each sequence (A_n) in I^X which converges to a fuzzy set A^* with $A^* \subset f^{-1}[B]$ is eventually in $f^{-1}[B]$. Thus, let (A_n) be a sequence in I^X which converges to a fuzzy set A^* with $A^* \subset f^{-1}[B]$. By the hypothesis, $f[A_n]$ converges to the fuzzy set $f[A^*]$. By Lemma 2.4 (4) and (5), we have $f[A^*] \subset f[f^{-1}[B]] \subset B$. Hence, $(f[A_n])$ is eventually in B, since B is a neighborhood of $f[A^*]$. That is, there is an $m \in N$ such that for all $n > m$, $f[A_n] \subset B$. By Lemma 2.4 (3) and (6),

$$A_n \subset f^{-1}[f[A_n]] \subset f^{-1}[B] \text{ for all } n > m.$$

Therefore, (A_n) is eventually in $f^{-1}[B]$. □

3. Fuzzy Connectedness

Let 2 be the discrete fuzzy topological space with two elements, say 0,1. Each open set in 2 is of the form $sP(0,1) \cup tP(1,1)$, $s,t \in [0,1]$. If there exists a surjective fuzzy continuous function $f:(X, \mathcal{T}) \to 2$, then $(X, \mathcal{T})$ is not fuzzy connected. However, we claim that the converse does not hold. For this, we introduce an example to support our claim. And also, we suggest a sufficient condition for which the converse is satisfied.

Definition 3.1 Let A and B be fuzzy sets in a fuzzy topological space $(X, \mathcal{T})$.

(1) A and B are said to be Q-separated if there exist closed fuzzy sets F and H in $(X, \mathcal{T})$ such that $A \subset F$, $B \subset H$, $F \cap B = \phi$.

(2) A and B are said to be separated if there exist open fuzzy sets U and V in $(X, \mathcal{T})$ such that $A \subset U$, $B \subset V$, $U \cap B = \phi$, and $V \cap A = \phi$.

Definition 3.2. Let D be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$.

(1) D is called disconnected if there are non-empty fuzzy sets in the subspace $(supp(D), \mathcal{T}_{supp(D)})$ such that A and B are Q-separated and $A \cup B = D$. D is connected if it is not disconnected.

(2) D is called O-disconnected if there are non-empty fuzzy set A and B in the subspace $(supp(D), \mathcal{T}_{supp(A)})$ such that A and B are separated and $A \cup B = D$. D is O-connected if it is not O-disconnected.

Lemma 3.3. Let A and B be fuzzy sets in a fuzzy topological space $(X, \mathcal{T})$. A and B are Q-separated in X if and only if $cl(A) \cap B = \phi = A \cap cl(B)$.

Proof. The proof is straightforward from the definition.

Proposition 3.4. [3]. In a fuzzy topological space induced by a general topological space, Two types of fuzzy connectedness in Definition 3.2 are equivalent.

It is known that if there is a surjective fuzzy continuous function $f:(X, \mathcal{T}) \to 2$, then $(X, \mathcal{T})$ is both O-disconnected and disconnected. However, the converse is not true at all. The following example shows this:

Example 3.5. Let R and Q be the set of all real numbers and rational numbers, respectively. Let $\mathcal{T} = \{\chi_Q, \chi_{Q^c}, R, \phi\}$, where χ_X *denotes the characteristic function of* X in R. Then, $(R, \mathcal{T})$ is a fuzzy topological space which is disconnected by Lemma 3. 3., since $(\chi_Q)^c = \chi_{Q^c}$. We want to show that there is no surjective fuzzy continuous function from $(R, \mathcal{T})$ onto 2. Let $f:(X, \mathcal{T}) \to 2$ be a surjective function. If $f^{-1}[P(0,1)] = \phi$, then we get

$$f^{-1}[P(0.1)](x) = 0 \text{ for all } x \in R$$
$$\Rightarrow P(0,1)(f(x)) = 0 \text{ for all } x \in R$$
$$\Rightarrow f(x) = 1 \text{ for all } x \in R.$$

This is impossible since f is surjective. Hence, $f^{-1}[P(0,1)] \neq \phi$. In the same way, we can see that $f^{-1}[P(1,1)] \neq \phi$. From Lemma 2.4 and Remark 2.5, we have

$$\phi^{-1}[P(0.1)] \cap f^{-1}[P(1,1)] = \phi$$

and

$$\phi^{-1}[P(0,1)] \cup f^{-1}[P(1,1)] = R.$$

This enables us assume, without loss of generality, that

$$f^{-1}[P(0,1)] = \chi_Q \text{ and } f^{-1}[P(1,1)] = \chi_{Q^c}.$$

This means that f is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in Q \\ 1 & \text{if } x \in Q^c. \end{cases}$$

Let us consider the inverse image of an open set $sP(0,1) \cup tP(1,1)$ in 2 under f.

If $0 < s, t < 1$, then we can observe that $f^{1}[sP(0,1) \cup tP(1,1)] \notin \mathcal{T}$. That is, f can not be fuzzy continuous. □

Now, it is natural to ask under what conditions the converse holds. We formulate a proposition as one of solutions to this question.

Proposition 3.6. Let (X, T) be a disconnected general topological space. If $(X, \tilde{T})$ is the fuzzy topological space induced by (X, T), then there exists a surjective fuzzy continuous function $f:(X, \tilde{T}) \to 2$.

Proof. Since (X,T) is disconnected, there exists a surjective continuous function $f:(X, T) \to \{0,1\}$, where $\{0,1\}$ is the discrete space with two elements. Consider the underlying function $f:(X, \tilde{T}) \to 2$. It suffices to show that f is fuzzy continuous. Each open fuzzy set in 2 is of the form $sP(0,1) \cup tP(1,1)$, $0 \leq s, t \leq 1$. Since

$$f^{1}[sP(0,1) \cup tP(1,1)]$$
$$= f^{1}[sP(0,1)] \cup f^{1}[tP(1,1)],$$

it suffices to show that $f^{1}[sP(0,1)]$ and $f^{1}[tP(1,1)]$ are open in $(X, \tilde{T})$ for $0 < s, t \leq 1$.

Observe that

$$f^{1}[sP(0,1)](x)$$
$$= sP(0,1)(f(x))$$
$$= \begin{cases} s \text{ if } f(x = 0 \\ 0 \text{ if } f(x = 1 . \end{cases}$$

Hence, we can see that

$$f^{1}(0) = supp(f^{-1}[sP(0,1)]).$$

Since f is continuous, $supp(f^{-1}[sP(0,1)])$ is open in (X, T). Since $(X, \tilde{T})$ is the fuzzy topological space induced by (X, T), $f^{-1}[sP(0,1)]$ is open in $(X, \tilde{T})$. In the same way, we can see that $f^{-1}[tP(1,1)]$ is open in $(X, \tilde{T})$. □

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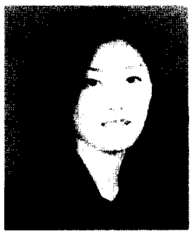

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