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Model Misspecification in Nonstationary Seasonal Time Series †

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ABSTRACT

In this paper we analytically study model misspecification that arises in regression analysis of nonstationary seasonal time series. We assume the underlying data generating process is a seasonally or a regularly and seasonally integrated process. We first study consequences of totally misspecified cases where seasonal indicator variables, a linear time trend, or another statistically independent seasonally integrated process are used as predictor variables in order to model the nonstationary seasonal behavior of the dependent variable. Then we study consequences of partially misspecified cases where the dependent variable and a predictor variable are cointegrated at some, but not all of the frequencies corresponding to the nonstationary roots.

Key Words : Total misspecification; partial misspecification; seasonal cointegration; partially cointegrated; spurious regression
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1. INTRODUCTION

Nonstationary time series arise frequently in business and economics, and modeling nonstationary time series plays an important role in empirical studies. In general, for a certain class of time series, the nonstationary part of a time series can be modeled by a deterministic function or a stochastic function. Because the characteristics of and the forecasts from these two types of functions are quite different it has drawn a great deal of attention to decide which type of the functions is appropriate for describing the nonstationary behavior. For example, the issue of the trend stationary versus the difference stationary, that is, the linear time trend versus the random walk, has long been studied and resulted in great body of literature on statistical inference of a unit root. Especially, some of the likely consequences of model misspecification discussed in Granger and Newbold (1974) brought greater attention to the issue, and Phillips (1986) and Durlauf and Phillips (1988) provided analytical studies of the findings in Granger and Newbold (1974).

Seasonal time series also arise frequently. It is also important to determine if the seasonal behavior can be properly modeled by a deterministic function such as seasonal indicator variables and other periodic functions or by a stochastic function such as a seasonally integrated process. The issue has been studied, almost in parallel with unit root tests, as seasonal unit root tests. Abeysinghe (1991) offers, through a Monte Carlo simulation study, some likely consequences when a nonstationary seasonal process is regressed on another statistically independent nonstationary seasonal process and seasonal indicator variables. And Abeysinghe (1994) analytically studied effects on the sample autocovariance by inappropriate use of seasonal indicator variables.

In this paper we study some consequences of two types of model misspecification in regression of a nonstationary seasonal time series which is seasonally integrated or regularly and seasonally integrated. The first type is misspecification of predictors by deterministic functions or by another statistically independent seasonally integrated process. The second type is partial misspecification where the predictor variable and the dependent variable are cointegrated at some, but not all of the seasonal frequencies.

2. PRELIMINARY RESULTS

For a seasonally integrated process Y_t and a process X_t, which can be seasonally integrated or deterministic, one may consider a regression model

$$Y_t = \alpha X_t + N_t. \tag{2.1}$$

In this paper we are mainly concerned with two cases: a totally misspecified case where $\alpha = 0$ so that $Y_t = N_t$; and a partially misspecified case where $\alpha \neq 0$ and N_t is nonstationary, that is, the predictor variable cannot "capture fully" the nonstationary behavior of the dependent variable. If X_t and Y_t are independent in (2.1), it is an example of a totally misspecified model. If X_t and Y_t are quarterly seasonal time series cointegrated at frequency zero only, it is an example of a partially misspecified model. To be specific, for such X_t and Y_t there exists non-zero α such that $Y_t - \alpha X_t$ does not have a unit root in its autoregressive (AR) representation, that is,

$$(1+B)(1+B^2)Y_t = \alpha(1+B)(1+B^2)X_t + u_t,$$

where u_t is a stationary process and B is the usual back shift operator such that $BY_t = Y_{t-1}$. Then,

$$Y_t = \alpha X_t + N_t,$$

where $\alpha \neq 0$ and nonstationary N_t satisfies $(1+B)(1+B^2)N_t = u_t$, and thus the model in (2.1) is partially misspecified. We note that Shin and Sarkar (1995) considered partially misspecified cases where the order of X_t is strictly higher than that of N_t. However, for seasonal nonstationary time series, partially misspecified cases can occur where the orders of X_t and N_t are the same. In the above example X_t and N_t are all $O_p(t^{1/2})$.

For X_t and N_t in (2.1) we assume that there exist real numbers a_n and b_n such that as $n \to \infty$ $(a_n^{-1}X_{[nr]}, b_n^{-1}N_{[nu]})$ converges in distribution to $(f(r), g(u))$ for some continuous functions f and g, which are possibly random, on $[0,1]$, where $[x]$ denote the integer part of x. We note for a totally misspecified model $Y_t = O_p(b_n)$ and for a partially misspecified model $Y_t = O_p(max\{a_n, b_n\})$. Then we have the following asymptotic results for least squares regression of (2.1).

Lemma 1. For the model in (2.1), based on n pairs of observations (X_t, Y_t), $t = 1, \ldots, n$, let $\hat{\alpha}$ be the least squares estimator of α, $t_{\alpha=\alpha_0}$ the regression t statistic for testing $\alpha = \alpha_0$, $\hat{\sigma}^2$ the mean squared error, $\hat{N}_t$ the residual, and R^2 the coefficient of determination. Then, under the assumption stated above,

$$\frac{a_n}{b_n}(\hat{\alpha}-\alpha) \to^D \frac{\int_0^1 f(r)g(r)dr}{\int_0^1 f(r)^2 dr} \equiv \xi,$$

$$b_n^{-1}\hat{N}_{[nr]} \xrightarrow{D} g(r) - \xi f(r),$$

$$b_n^{-2}\hat{\sigma}^2 \xrightarrow{D} \int_0^1 \{g(r)-\xi f(r)\}^2 dr \equiv \zeta,$$

$$n^{-1/2} t_{\alpha=\alpha_0} \xrightarrow{D} \xi\{\frac{\int_0^1 f(r)^2 dr}{\zeta}\}^{1/2}.$$

Further, if $\alpha = 0$ or if $\alpha \neq 0$ and $a_n/b_n \to 0$, then

$$R^2 \xrightarrow{D} \frac{\int_0^1 f(r)^2 dr}{\int_0^1 g(r)^2 dr}\xi^2.$$

If $\alpha \neq 0$ and $a_n/b_n \to c$, then

$$R^2 \xrightarrow{D} (\xi + \alpha c)^2 \frac{\int_0^1 f(r)^2 dr}{\int_0^1 g(r)^2 dr}.$$

Finally, if $\alpha \neq 0$ and $a_n/b_n \to \infty$, then $R^2 \to^P 1$.

A detailed proof of the lemma is in the technical appendix. The asymptotic behavior of $\hat{\alpha} - \alpha$ depends on the orders of N_t and X_t. Especially, if X_t is of higher order than N_t, then $\hat{\alpha}$ converges to α and is consistent, while the regression t statistic diverges. Therefore, although $\hat{\alpha}$ is consistent in a totally misspecified model, the null hypothesis of $\alpha = 0$ will be rejected and a spurious relationship occurs because of the divergent t statistic. If X_t and N_t are of the same order, then $\hat{\alpha} - \alpha$ converges to a non-degenerate limiting distribution and $\hat{\alpha}$ is inconsistent. If X_t is of lower order than N_t, then $\hat{\alpha} - \alpha$ diverges. Regardless of the order of N_t and X_t, in a misspecified regression model as in (2.1), the order of the residual $\hat{N}_t$ is the same as the order of N_t. For partially misspecified models, if the order of N_t is higher than or equal to that of X_t, then R^2 converges to a non-degenerate limiting distribution; if the order of N_t is lower, then R^2 converges to 1 in probability. However, in totally misspecified models, R^2 converges to a non-degenerate limiting distribution regardless of the orders of N_t and X_t.

In a simulation study of Abeysinghe (1991), X_t and Y_t are generated independently from various data generating processes which are all $O_p(t^{1/2})$.

Therefore, the higher rejection rate of $\alpha = 0$ in the simulation study can be explained by the divergent t statistic in Lemma 1.

Often, the dependent variable Y_t and the predictor variable X_t are "adjusted" for another variable Z_t. Examples of Z_t include a time variable for detrending and periodic functions for seasonal adjustment. In order to study the effect of data adjustment by a variable Z_t which is statistically independent of both X_t and Y_t and also of N_t, one may consider

$$Y_t = \alpha X_t + \gamma Z_t + N_t \tag{2.2}$$

with $\gamma = 0$. For Z_t we also assume that there exists a real number c_n such that as $n \to \infty$ $c_n^{-1} Z_{[nr]}$ converges in distribution to a continuous function $h(r)$, which is possibly random on [0, 1].

We can rewrite (2.2) as

$$Y_t = \alpha \hat{r}_t^x + \gamma^* Z_t + N_t,$$

where $\hat{r}_t^x = X_t - (\sum Z_t^2)^{-1}(\sum X_t Z_t) Z_t$ and $\gamma^* = \gamma + (\sum Z_t^2)^{-1}(\sum X_t Z_t)\alpha$. Then, because $\hat{r}_t^x$ and Z_t are orthogonal the least squares estimator of α in (2.2) and its statistical properties involve Y_t, $\hat{r}_t^x$, and N_t .

Because the $\hat{r}_t^x$ are the residuals from a (totally) misspecified regression model of X_t on Z_t, their stochastic order is the same as the orders of X_t according to Lemma 1. Therefore, asymptotic properties of $\hat{\alpha}$, the least squares estimator of α in (2.2) depends on the orders of X_t and N_t. By similar arguments, asymptotic properties of $\hat{\gamma}$ depends of those of Z_t and N_t, and we have the following results.

Lemma 2. Let $\hat{\alpha}$ be the least squares estimator of α, $t_{\alpha=\alpha_0}$ the regression t statistic for testing $\alpha = \alpha_0$, $\hat{\gamma}$ the least squares estimator of γ, and $t_{\gamma=0}$ the regression t statistic for testing $\gamma = 0$ in (2.2). Then, $\hat{\alpha} = O_p(b_n/a_n)$, $t_{\alpha=\alpha_0} = O_p(n^{1/2})$, $\hat{\gamma} = O_p(c_n/a_n)$, and $t_{\gamma=0} = O_p(n^{1/2})$.

The stochastic order of the least squares estimator of α in (2.2) is not affected by the presence of Z_t nor by the stochastic order of it, although the asymptotic distribution of $\hat{\alpha}$ is affected by the presence of Z_t. Because of the divergent t statistics, spurious relation occurs between Y_t and Z_t, and between Y_t and X_t if $\alpha = 0$. The results in Lemma 2 can be easily extended to cases where two or more variables which are independent of the X_t and N_t are included as "adjustment" variables.

3. SEASONALLY INTEGRATED PROCESSES

We consider the following seasonally integrated process with period s.

$$(1 - B^s)Y_t = \varepsilon_t, \tag{3.1}$$

where the ε_t are independent random variables with $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = \sigma^2$, and $sup_t E(|\varepsilon_t|^{2+\delta}) < \infty$. If starting values of Y_t contain a strong seasonality, a realization may be consistently repetitive. In such cases one may use seasonal indicator variables or another seasonal time series X_t to account for the seasonal behavior. Here we assume X_t is generated from

$$X_t = X_{t-s} + e_t \tag{3.2}$$

and the e_t are independent and satisfy the same moment conditions as the ε_t. Also the e_t and the ε_t are independent, and thus X_t and Y_t are independent. Therefore, for Y_t, $t = 1, \ldots, n$ (and $n = sm$ for brevity), one may consider the following models:

$$Y_t = \sum_{j=1}^{s} \beta_j \delta_{jt} + N_t, \tag{3.3}$$

$$Y_t = \sum_{j=1}^{s} \beta_j \delta_{jt} + \gamma t + N_t \tag{3.4}$$

$$Y_t = \alpha X_t + N_t, \tag{3.5}$$

$$Y_t = \sum_{j=1}^{s} \beta_j \delta_{jt} + \alpha X_t + N_t, \tag{3.6}$$

$$Y_t = \sum_{j=1}^{s} \beta_j \delta_{jt} + \gamma t + \alpha X_t + N_t, \tag{3.7}$$

where $\delta_{jt} = 1$ if $j \equiv t \ (mod\ s)$ or 0 otherwise. All the models are totally misspecified for Y_t. Model (3.4) applies to a case where Y_t is detrended compared with model (3.3). Compared with model (3.5), model (3.6) applies to a case where Y_t and X_t are deseasonalized and model (3.7) to a case where Y_t and X_t are detrended and deseasonalized.

For these totally misspecified regression models we summarize asymptotic properties of the estimators and other related regression statistics frequently used in the following theorem.

Theorem 1. If the regression models in (3.3) through (3.7) are fitted by the least squares method for Y_t generated from (3.1), then as $n \to \infty$, $\hat{\beta}_j$ diverges for $j = 1, \ldots, s$, the regression t statistic for testing $\beta_j = 0$ diverges, $\hat{\gamma}$ converges to zero in probability, the regression t statistic for testing $\gamma = 0$ diverges, $\hat{\alpha}$ converges to a non-degenerate limiting distribution, the regression t statistic for testing $\alpha = 0$ diverges, the mean squared error $\hat{\sigma}^2$ diverges, and the R^2 converges to a non-degenerate limiting distribution.

Proofs of this theorem is based on Lemmas 1 and 2, and can be easily established by noting the orders of the dependent variable Y_t and the predictor variables. Thus details are omitted. The explicit asymptotic results of this theorem can be found in the technical appendix. Although the ϵ_t and the e_t are assumed to be independent, the results of this and the following theorems can be easily extended to a case where the ϵ_t and the e_t are weakly stationary processes. In all models a spurious relationship between Y_t and each of the predictor variables occurs mainly because of the divergent regression t statistics. The $\hat{\beta}_j$ are inconsistent, and because of the divergent $\hat{\beta}_j$'s and the corresponding t statistic, one may incorrectly conclude that the process has a deterministic seasonal component. For models (3.4) and (3.7) $\hat{\gamma}$ is the only consistent estimator. However, because of the divergent t statistic one may reject the null hypothesis of $\gamma = 0$, and incorrectly conclude that there is a linear trend in the process. Because of the divergent t statistics for testing $\alpha = 0$ one incorrectly conclude that X_t is related to Y_t.

As in the technical appendix, these t-statistics, when properly standardized, have limiting distributions. For example, the regression t-statistics for testing $\alpha = 0$ need to be multiplied by $n^{-1/2}$, and then compared with percentiles of the corresponding limiting distribution. Because these limiting distributions are non-standard in the sense that they are functional of stochastic integrals of Brownian motions, percentiles are usually generated by a Monte Carlo simulation.

When models (3.5), (3.6), and (3.7) are compared, deseasonalization or detrending does not correct the spurious regression relation between Y_t and X_t. When models (3.3), (3.4), (3.6), and (3.7) are compared, introducing a statistically independent seasonal process as a predictor does not correct the spurious regression relation between Y_t and deterministic predictors such a seasonal indicator variables and a linear time trend. For $s = 1$, that is, for (regularly) integrated processes, some of the results in this theorem are identical to some of the results in Theorem 1 of Phillips (1986) and in Theorems 2.1 and 2.2 of Durlauf and Phillips (1988).

4. REGULARLY AND SEASONALLY INTEGRATED PROCESSES

Time series that are both regularly and seasonally integrated are found to be useful to model variety of seasonal time series. Examples include the airline data of Series G in Box, Jenkins, and Reinsel (1994) and the employment data of Series W9 of Wei (1989). Therefore, we consider the following regularly and seasonally integrated process with period s.

$$(1-B)(1-B^s)Y_t = \varepsilon_t \tag{4.1}$$

For Y_t in (4.1) one may consider regression models in (3.3) through (3.7). Here we assume X_t is also a regularly and seasonally integrated process generated from

$$(1-B)(1-B^s)X_t = e_t. \tag{4.2}$$

These ε_t and e_t satisfy the conditions stated in (3.1) and (3.2). Then, by noting $Y_t = O_p(t^{3/2})$, we have the following results.

Theorem 2. If the regression models in (3.3) through (3.7) are fitted by the least squares method for Y_t generated from (4.1) and X_t from (4.2), then as $n \to \infty$, the results stated in Theorem 1 hold except $\hat{\gamma}$ now diverges. If, instead, X_t generated from (3.2) is used, then $\hat{\alpha}$ diverges.

Proofs of this theorem is also based on Lemmas 1 and 2, and can be easily established by noting the orders of the dependent variable Y_t and the predictor variables. Thus details are omitted. The explicit asymptotic results of this theorem can be found in the technical appendix. Unlike the results in Theorem 1, $\hat{\gamma}$ is no longer consistent. This is because the dependent variable Y_t in (4.1) is $O_p(t^{3/2})$ while the predictor variable t is $O(t)$. In all models considered, spurious regression relations occur. It is noted that $\hat{\alpha}$ diverges if a seasonally integrated process X_t generated from (3.2) is used instead of (4.2). This is because X_t in (3.2) is of lower order than Y_t in (4.1).

5. PARTIALLY COINTEGRATED SEASONAL PROCESSES

For two seasonally integrated processes

$$X_t = X_{t-s} + e_t \quad \text{and} \quad Y_t = Y_{t-s} + \varepsilon_t,$$

we consider cases where X_t and Y_t are cointegrated at some of but not all of the frequencies corresponding to the roots of $1 - B^s = 0$. For such cases we call X_t and Y_t are partially cointegrated seasonal processes. For partially cointegrated seasonal processes one may also consider least squares regression of Y_t on X_t as in (3.5), (3.6), or (3.7). Unlike the previously considered case where e_t and ε_t are independent, for partially cointegrated X_t and Y_t models in (3.5), (3.6), and (3.7) are not totally misspecified because, as will be shown later in detail, $\alpha \neq 0$ and N_t is nonstationary. Therefore, models (3.5), (3.6) and (3.7) are partially misspecified. To be specific, in much of the following discussions emphasis will be placed on the situation of quarterly seasonal time series so that $s = 4$. Then the possible nonstationary roots are 1, -1, and $\pm i$ which correspond to frequencies zero, 1/2, and 1/4, respectively. For more about seasonal cointegration refer to Hylleberg et al.(1990), Lee (1992), and Ahn and Reinsel (1994).

When X_t and Y_t are cointegrated at frequency zero, then according to the definition of Hylleberg et al. (1990) there exist a constant $\alpha \neq 0$ such that $Y_t - \alpha X_t$ dose not have a root of one, which is called a unit root, in its autoregressive (AR) representation. That is,

$$(1+B)(1+B^2)Y_t = \alpha(1+B)(1+B^2)X_t + u_t, \tag{5.1}$$

where u_t is a stationary processes and the partial sum process $S_t = \sum_0^t u_j$ is assumed to satisfy a functional central limit theorem of the type, for example, discussed and applied in Phillips (1987). Then (5.1) can be rewritten as in (3.5) with N_t satisfying $(1 + B)(1 + B^2)N_t = u_t$. Since $\alpha \neq 0$ and N_t is nonstationary, model (3.5) is partially misspecified for X_t and Y_t in (5.1).

When X_t and Y_t are cointegrated at frequency 1/2, there exist a constant α such that $Y_t - \alpha X_t$ dose not have a root of -1 in its AR representation. That is,

$$(1-B)(1+B^2)Y_t = \alpha(1-B)(1+B^2)X_t + u_t.$$

Then, this can be rewritten as in (3.5) with N_t satisfying $(1-B)(1+B^2)N_t = u_t$. Model (3.5) is also partially misspecified for X_t and Y_t cointegrated at frequency 1/2.

When X_t and Y_t are cointegrated at frequency 1/4, there are two possible cases: in one case a cointegrating combination is a contemporaneous linear combination and in the other case a cointegrating combination includes a lagged variable. In the first case there exists a constant α such that $Y_t - \alpha X_t$ does not have roots of $\pm i$ in its AR representation. That is,

$$(1-B^2)Y_t = \alpha(1-B^2)X_t + u_t.$$

Then, this can be rewritten as in (3.5) with N_t satisfying $(1 - B^2)N_t = u_t$, and model (3.5) is partially misspecified.

In the other case, there exist α and α_1 such that $Y_t - \alpha X_t - \alpha_1 X_{t-1}$ does not have roots of $\pm i$ in its AR representation, that is,

$$(1 - B^2)Y_t = (1 - B^2)(\alpha X_t + \alpha_1 X_{t-1}) + u_t. \tag{5.2}$$

This corresponds to the case where polynomial cointegrating vectors (PCIV's) exit, see Hylleberg et al. (1990). Then (5.2) can be rewritten as in (3.5) with N_t satisfying $(1 - B^2)N_t = (1 - B^2)\alpha_1 X_{t-1} + u_t$. Model (3.5) is misspecified for X_t and Y_t which are cointegrated at frequency 1/4 with a PCIV. We note that in this case a linear combination $Y_t + \alpha_1 X_{t+1} - \alpha X_t$ does not have roots of $\pm i$ in its AR representation, either; see pages 322 and 326 of Ahn and Reinsel (1994). This linear combination yields a partially misspecified model with N_t in (3.5) satisfying $(1 - B^2)N_t = -(1 - B^2)\alpha_1 X_{t+1} + u_t$. Therefore, in model (3.5) the "noise term" N_t is not well defined, although the parameter α is identifiable.

When X_t and Y_t are partially cointegrated at two of the frequencies corresponding to the nonstationary roots with one cointegrating vector, by similar arguments we can easily establish that regression model of Y_t on X_t is partially misspecified and the orders of X_t and N_t are the same. For example, when X_t and Y_t are cointegrated at frequencies zero and 1/4 with a PCIV, there exist a linear combination $Y_t - \alpha X_t - \alpha_1 X_{t-1}$ which does not have roots of 1 and $\pm i$ in its AR representation, that is,

$$(1 + B)(Y_t - \alpha X_t - \alpha_1 X_{t-1}) = u_t.$$

This can be rewritten as in (3.5) with N_t satisfying $(1 + B)N_t = (1 + B)\alpha_1 X_{t-1} + u_t$.

In all cases of partially cointegrated seasonal processes discussed above, X_t and N_t are all $O_p(t^{1/2})$. Therefore, Lemma 1 is directly applicable when least squares regression model (3.5) is considered, and Lemma 2 is applicable when (3.6) and (3.7) are considered because $\beta_j = 0$ for all j and $\gamma = 0$. We have the following results.

Theorem 3. If least squares regression models (3.5), (3.6) and (3.7) are fitted and X_t and Y_t are partially cointegrated, then as $n \to \infty$, $\hat{\alpha} - \alpha$ converges to a non-degenerate limiting distribution, the regression t statistic for testing α diverges, $\hat{\beta}_j$ diverges for $j = 1, \ldots, s$, the regression t statistic for testing $\beta_j = 0$ diverges, $\hat{\gamma}$ converges to zero in probability, the regression t statistic for testing $\gamma = 0$ diverges, the mean squared error $\hat{\sigma}^2$ diverges, and the R^2 converges to a non-degenerate limiting distribution.

The limiting distributions are different for different cointegrated cases and are explicitly given in the technical appendix. Similar to the case where X_t and Y_t are independent the least squares estimator $\hat{\alpha}$ is inconsistent for α and thus for the cointegrating vector $(1, -\alpha)'$. This is because whether X_t and Y_t are independent or partially cointegrated the orders of X_t and N_t are the same. If X_t and Y_t are cointegrated at two different frequencies with linearly independent cointegrating vectors, then the parameter α in (3.5) is not identifiable. For example if X_t and Y_t are cointegrated at frequency zero with a cointegrating vector $(1, \ -\alpha_0)$ and at frequencies 1/2 and 1/4 with a cointegrating vector $(1, \ -\alpha_1)$ with $\alpha_0 \neq \alpha_1$, then

$$Y_t = \alpha_0 X_t + N_t^0, \tag{5.3}$$

$$Y_t = \alpha_1 X_t + N_t^1, \tag{5.4}$$

where $(1+B)(1+B^2)N_t^0 = u_t^0$, $(1-B)N_t^1 = u_t^1$, u_t^0 is the u_t satisfying (5.1), and u_t^1 satisfies $(1-B)Y_t = \alpha_1(1-B)X_t + u_t^1$. In order to properly estimate such α_0 and α_1 one may use generalized least squares estimation with the filtered series. For example regress $(1+B)(1+B^2)Y_t$ on $(1+B)(1+B^2)X_t$ to estimate α_0 and regress $(1-B)Y_t$ on $(1-B)X_t$ to estimate α_1. However, with filtered series information on the long run relationship between the levels of X_t and Y_t is lost. As conitegration is concerned with the long run relationship among the levels of the series involved, estimation of cointegrating vectors with unfiltered series is desirable. For an extensive discussion of estimation of seasonal cointegrating vectors with unfiltered series, refer to Ahn and Reinsel (1994).

6. A NUMERICAL EXAMPLE

To illustrate some of the consequences of partial misspecification of nonstationary seasonal time series model, we consider quarterly United Kingdom data on the logarithm of consumption expenditure (Y_t) and the logarithm of personal disposable income (X_t) for the period 1955 through 1979. The data are extensively analyzed in Ahn and Reinsel (1994) for modeling procedures for seasonal nonstationary vector AR models. The scatter plot of Y_t and X_t

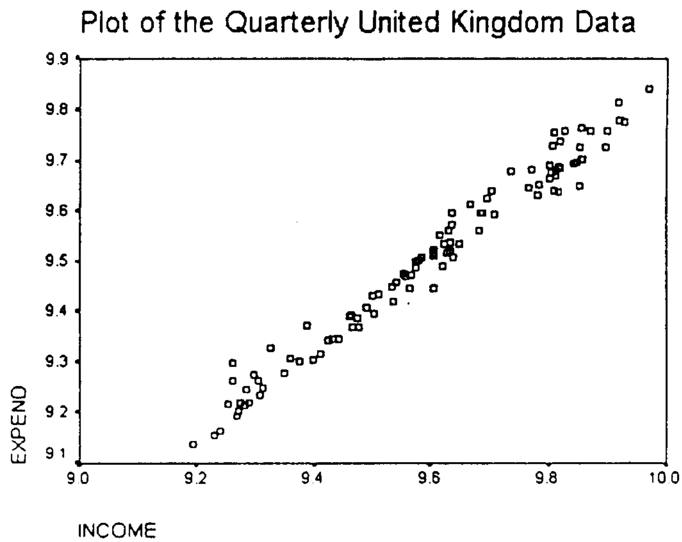

Figure 1 Scatter Plot of the Quarterly United Kingdom Data on the Logarithm of Consumption Expenditure (EXPEND) and the logarithm of Personal Disposable Income (INCOME) for the Period 1955 through 1979.

in Figure 1 shows strong linear relationship between the two variables, and one may consider a simplt linear regression model for the data. When the regression model (3.5) is fitted with a constant term, we obtain $\hat{\alpha} = 0.86090$ with "standard error" 0.01429 and $R^2 = 0.974$. But the residuals show a strong seasonality, which indicates the error term in the regression model is nonstationary and thus indicates a partially misspecified model. Ahn and Reinsel (1994) obtained that X_t and Y_t are cointegrated at frequency zero with a cointegrating vector $(1, \ -0.8649)$ and at frequencies 1/2 and 1/4 with a cointegrating vector $(1, \ -2.7310)$. That is, $Y_t - 0.8649X_t$ does not have a unit root and $Y_t - 2.7310X_t$ does not have nonstationary roots of -1 and $\pm i$. The estimates of the cointegrating vectors were obtained by Gaussian reduced rank (GRR) estimation. Therefore, for the data the regression model in (3.5) is misspecified. Further, the parameter α is not identifiable because X_t and Y_t are cointegrated at different frequencies with linearly independent cointegrating vectors.

Even though the parameter α is not identifiable, the least squares (LS) estimate of α is very close to the GRR estimate of the cointegrating vector at frequency zero. That is, in this example the LS estimator closely estimates the value of α corresponding to frequency zero, that is, α_0 in (5.3), but not the value corresponding to frequencies 1/2 and 1/4, that is, α_1 in (5.4). This is because the least squares estimate minimizes the residual sum of squares and for the data the sample variance of $Y_t - 0.8649X_t$ is smaller than that of $Y_t - 2.7310X_t$.

TECHNICAL APPENDIX

In this appendix we provide a proof of Lemma 1 and explicit forms of the limiting distributions of the various estimators and test statistics appeared in the text.

A1. Proof of Lemma 1:

Since $(a_n^{-1}X_{[nr]}, b_n^{-1}N_{[nu]})$ converges in distribution to $(f(r), g(u))$, $(n^{-1}a_n^{-1}b_n^{-1}\sum X_{[nr]}N_{[nr]}, n^{-1}a_n^{-2}\sum X_{[nr]}^2)$ converges in distribution to $(\int_0^1 f(r)g(r)dr, \int_0^1 f(r)^2 dr)$ by similar arguments used in (A.1) of Phillips (1986). Therefore,

$$\frac{a_n}{b_n}(\hat{\alpha} - \alpha) = \frac{a_n}{b_n}\frac{\sum X_t N_t}{\sum X_t^2} \xrightarrow{D} \frac{\int_0^1 f(r)g(r)dr}{\int_0^1 f(r)^2 dr} \equiv \xi.$$

From $\hat{N}_t = N_t - (\hat{\alpha} - \alpha)X_t$ it follows immediately

$$b_n^{-1}\hat{N}_{[nr]} \xrightarrow{D} g(r) - \xi f(r).$$

Since $\hat{\sigma}^2 = \sum \hat{N}_{[nr]}^2/(n-1)$,

$$b_n^{-2}\hat{\sigma}^2 \xrightarrow{D} \int_0^1 \{g(r) - \xi f(r)\}^2 dr \equiv \zeta.$$

Since $t_{\alpha=\alpha_0} = (\hat{\alpha} - \alpha_0)/(\hat{\sigma}^2/\sum X_t^2)^{1/2}$,

$$n^{-1/2}t_{\alpha=\alpha_0} = \frac{(a_n/b_n)(\hat{\alpha} - \alpha_0)}{(b_n^{-2}\hat{\sigma}^2/n^{-1}a_n^{-2}\sum X_t^2)^{1/2}} \xrightarrow{D} \xi\{\frac{\int_0^1 f(r)^2 dr}{\zeta}\}^{1/2}$$

under the null hypothesis $\alpha = \alpha_0$.

To prove the asymptotic properties of R^2, we first need the asymptotic properties of Y_t for each of the cases because

$$R^2 = 1 - \frac{\sum \hat{N}_t^2}{\sum Y_t^2}$$

involves Y_t. If $\alpha = 0$ or if $\alpha \neq 0$ and $a_n/b_n \to 0$, then $b_n^{-1}Y_{[nr]} \xrightarrow{D} g(r)$. If $\alpha \neq 0$ and $a_n/b_n \to c$, then $a_n^{-1}Y_{[nr]} \xrightarrow{D} \alpha f(r) + g(r)/c$. Finally if $\alpha \neq 0$ and $a_n/b_n \to \infty$, then $a_n^{-1}Y_{[nr]} \xrightarrow{D} \alpha f(r)$. Now, the results follow by simple arithmetic. For example, if $\alpha \neq 0$ and $a_n/b_n \to c$,

$$\begin{aligned} R^2 &= 1 - \frac{n^{-1}b_n^{-2}\sum \hat{N}_t^2 b_n^2}{n^{-1}a_n^{-2}\sum Y - t^2 a_n^2} \\ &\xrightarrow{D} 1 - \frac{\zeta}{\int_0^1 \{\alpha c f(r) + g(r)\}^2 dr}. \end{aligned}$$

A2. Explicit formulae for limiting distribution:

Most of the limiting distributions involve simple functionals of Brownian motions. They are obatined using an approach similar to that in Li (1991), Ahn and Cho (1993), and Park and Cho(1995) for seasonal models. The approach essentially makes use of invariance principles, also known as functional central limit theorem, and is explored in detail in Phillips (1986, 1987) for nonseasonal models. Basically, the statistics in the text are functions of quantities whose limiting behavior can be found in the aforementioned articles, and the limiting distributions of the statistics follow by the continuous mapping theorem.

A2.1. Formulae for Theorem 1:

A2.1.1. Formulae for model (3.3):

$$
\begin{aligned}
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}\int W_j(r)dr \equiv \xi_{11}^j \\
n^{-1/2}t_{\beta=0} &\xrightarrow{D} \xi_{11}^j/(s\zeta_{11})^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_{j=1}^{s}\{\int W_j(r)^2dr - (\int W_j(r)dr)^2\} \equiv \zeta_{11} \\
R^2 &\xrightarrow{D} 1 - \frac{\zeta_{11}}{\sum_j \int_0^1 W_j(r)^2 dr},
\end{aligned}
$$

where $W_j(r)$ is a Brownian motion as a limiting distribution of the partial sum process $\sum_k \varepsilon_{(k-1)s+j}$ corresponding to the j-th season and ε_t is as defined in (3.1).

A2.1.2. Formulae for model (3.4):

$$
\begin{aligned}
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}\int W_j(r)dr - 6s^{-5/2}\sum_j \int rW_j(r)dr \\
&\quad +3s^{-3/2}\sum_j \int W_j(r)dr \equiv \xi_{12}^j \\
n^{-1/2}t_{\beta=0} &\xrightarrow{D} \xi_{12}^j/(s\zeta_{12})^{1/2} \\
n^{1/2}\hat{\gamma} &\xrightarrow{D} 12s^{-5/2}\sum_j \int rW_j(r)dr - 6s^{-3/2}\sum_j \int W_j(r)dr \equiv \xi_{13} \\
n^{-1/2}t_{\gamma=0} &\xrightarrow{D} \xi_{13}/(12\zeta_{12})^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j \int W_j(r)^2dr - s^{-2}\sum_{j=1}^{s}(\int W_j(r)dr)^2 \\
&\quad -12\{s^{-5/2}\sum_j \int rW_j(r)dr - \frac{1}{2}s^{-3/2}\sum_j \int W_j(r)dr\}^2 \equiv \zeta_{12} \\
R^2 &\xrightarrow{D} 1 - \frac{\zeta_{12}}{s^{-2}\sum_j \int W_j(r)^2dr - s^{-3}(\sum_j \int W_j(r)dr)^2}
\end{aligned}
$$

A2.1.3. Formulae for model (3.5):

$$
\begin{aligned}
\hat{\alpha} &\xrightarrow{D} \frac{\sum_j \int_0^1 W_j(r)V_j(r)dr}{\sum_j \int_0^1 V_j(r)^2 dr} \equiv \xi_{14} \\
n^{-1/2} t_{\alpha=0} &\xrightarrow{D} s^{-1/2}\xi_{14}\{\frac{\sum_j \int_0^1 V_j(r)^2 dr}{\zeta_{13}}\}^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j \int W_j(r)^2 dr - \frac{(\sum_j \int_0^1 W_j(r)V_j(r)dr)^2}{\sum_j \int_0^1 V_j(r)^2 dr} \equiv \zeta_{13} \\
R^2 &\xrightarrow{D} 1 - \frac{\zeta_{13}}{\sum_j \int_0^1 W_j(r)^2 dr},
\end{aligned}
$$

where $V_j(r)$ is a Brownian motion as a limiting distribution of the partial sum process $\sum_k e_{(k-1)s+j}$ corresponding to the j-th season and e_t is as defined in (3.2).

A2.1.4. Formulae for model (3.6):

$$
\begin{aligned}
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}\int W_j(r)dr - \xi_{16}s^{-1/2}\int V_j(r)dr \equiv \xi_{15}^j \\
n^{-1/2} t_{\beta_j=0} &\xrightarrow{D} \xi_{15}^j/(s\zeta_{14})^{1/2} \\
\hat{\alpha} &\xrightarrow{D} \frac{\sum_j \int W_j(r)V_j(r)dr - \sum_j \int W_j(r)dr \int V_j(r)dr}{\sum_j \int V_j(r)^2 dr - \sum_j(\int V_j(r)dr)^2} \equiv \xi_{16} \\
n^{-1/2} t_{\alpha=0} &\xrightarrow{D} s^{-3/2}\xi_{16}\{\frac{\sum_j \int V_j(r)^2 dr}{\zeta_{14}}\}^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j \int W_j^2(r)dr - s^{-2}\sum_{j=1}^{s}(\int W_j(r)dr)^2 \\
&\quad - \frac{[\sum_j \int W_j(r)V_j(r) - \sum_j \int W_j(r)dr \int V_j(r)dr]^2}{\sum_j \int V_j^2(r)dr - \sum_j(\int V_j(r)dr)^2} \equiv \zeta_{14} \\
R^2 &\xrightarrow{D} 1 - \frac{\zeta_{14}}{s^{-2}\sum_j \int W_j^2(r)dr - s^{-3}(\sum_j \int W_j(r)dr)^2}
\end{aligned}
$$

A2.1.5. Formulae for model (3.7):

$$
\begin{aligned}
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}\int W_j(r)dr + \frac{1}{ad-b^2}\sum_{i=1}^{s}\{c_{ij}s^{-1/2}\int W_i(r)dr\} \equiv \xi_{17}^j \\
n^{-1/2} t_{\beta_j=0} &\xrightarrow{D} \xi_{17}^j/(s\zeta_{15})^{1/2} \\
\hat{\alpha} &\xrightarrow{D} (ad-b^2)^{-1}[-\sum_j g_j s^{-3/2}(\int W_j(r)dr) - b\int rW(r)dr \\
&\quad + a\int W(r)V(r)dr] \equiv \xi_{18}
\end{aligned}
$$

$$
\begin{aligned}
n^{-1/2}t_{\alpha=0} &\xrightarrow{D} s^{-1}\xi_{18}\{\frac{\sum_j \int V_j(r)^2 dr}{\zeta_{15}}\}^{1/2} \\
n^{1/2}\hat{\gamma} &\xrightarrow{D} (ad-b^2)^{-1}[\sum_j f_j s^{-3/2}(\int W_j(r)dr) + d\int rW(r)dr \\
&\quad - b\int W(r)V(r)dr] \equiv \xi_{19} \\
n^{-1/2}t_{\gamma=0} &\xrightarrow{D} \xi_{19}/(6\zeta_{15})^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j \int W_j(r)^2 dr - s^{-2}\sum_j(\int W_j(r)dr)^2 \\
&\quad - (ad-b^2)^{-1}\{\sum_i\sum_j c_{ij}s^{-3}\int W_i(r)dr\int W_j(r)dr \\
&\quad + 2\sum_j f_j s^{-3/2}\int W_j(r)dr\int rV(r)dr \\
&\quad + 2\sum_j g_j s^{-3/2}\int W_j(r)dr\int W(r)V(r)dr + d(\int rW(r)dr)^2 \\
&\quad - 2b(\int rW(r)dr)\int W(r)V(r)dr + a(\int W(r)V(r)dr)^2\} \\
&\equiv \zeta_{15} \\
R^2 &\xrightarrow{D} 1 - \frac{\zeta_{15}}{s^{-2}\sum_j \int W_j(r)^2 dr - s^{-3}(\sum_j \int W_j(r)dr)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
a &= 1/12 \\
b &= \int rV(r)dr - \frac{1}{2}s^{-1/2}\sum_j \int V_j(r)dr \\
d &= \int V(r)^2 dr - s^{-2}\sum_j(\int V_j(r)dr)^2 \\
f_j &= -\frac{1}{2}d + bs^{-1/2}\int V_j(r)dr \\
g_j &= \frac{1}{2}b - as^{-1/2}\int V_j(r)dr \\
c_{ij} &= -\frac{1}{4}d + b\frac{s^{-3/2}}{2}\int V_j(r)dr + as^{-2}\int V_i(r)dr\int V_j(r)dr
\end{aligned}
$$

A2.2. Formulae of Theorem 2:

A2.2.1 Formulae for model (3.3):

$$
\begin{aligned}
n^{-3/2}\hat{\beta}_j &\xrightarrow{D} s^{-3/2}\int_0^1 \sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \equiv \xi_{21}^j \\
n^{-1/2}t_{\beta_j=0} &\xrightarrow{D} \xi_{21}^j/(s\zeta_{21})^{1/2} \\
n^{-3}\hat{\sigma}^2 &\xrightarrow{D} s^{-4}[\sum_{j=1}^{s}\int_0^1 (\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1)^2 dr_2 \\
&\quad -\sum_{j=1}^{s}(\int_0^1 \sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2)^2] \equiv \zeta_{21}
\end{aligned}
$$

$$
R^2 \xrightarrow{D} 1 - \frac{\zeta_{21}}{s^{-4}\sum_{j=1}^{s}\int_0^1(\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1)^2dr_2 - s^{-5}(\sum_{j=1}^{s}\int_0^1\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2)^2}
$$

A2.2.2. Formulae for model (6):

$$
\begin{aligned}
n^{-3/2}\hat{\beta}_j &\xrightarrow{D} s^{-3/2}\int_0^1 \sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad -6s^{-5/2}\sum_{j=1}^{s}\int_0^1 r_2\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad +3s^{-5/2}\int_0^1 \sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \equiv \xi_{22}^j \\
n^{-1/2}t_{\beta_j=0} &\xrightarrow{D} \xi_{22}^j/(s\zeta_{22})^{1/2} \\
n^{-1/2}\hat{\gamma} &\xrightarrow{D} 12s^{-7/2}\sum_{j=1}^{s}\int_0^1 r_2\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad -6s^{-5/2}\int_0^1 \sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 \equiv \xi_{23} \\
n^{-1/2}t_{\gamma=0} &\xrightarrow{D} \xi_{23}/(12\zeta_{22})^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
n^{-3}\hat{\sigma}^2 &\xrightarrow{D} s^{-4}[\sum_{j=1}^{s}\int_0^1(\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1)^2dr_2 - \sum_{j=1}^{s}(\int_0^1\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2)^2] \\
&\quad -12[s^{-7/2}\sum_{j=1}^{s}\int_0^1 r_2\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2 - \frac{1}{2}s^{-5/2}\int_0^1\sum_{i=1}^{j}\int_0^{r_2} W_i(r_1)dr_1dr_2]^2 \\
&\equiv \zeta_{22}
\end{aligned}
$$

$$R^2 \xrightarrow{D} 1 - \frac{\zeta_{22}}{s^{-4}\sum_{j=1}^{s}\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1)^2dr_2 - s^{-5}(\sum_{j=1}^{s}\int_0^1\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1dr_2)^2}$$

A2.2.3. Formulae for model (3.5):

$$\begin{aligned}
\hat{\alpha} &\xrightarrow{D} \frac{\int_0^1(\sum_j\int_0^{r_2}V_j(r_1)dr_1)(\sum_j\int_0^{r_2}W_j(r_1)dr_1)dr_2}{\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1)^2dr_2} \equiv \xi_{24} \\
n^{-1/2}t_{\alpha=0} &\xrightarrow{D} s^{-2}\xi_{24}\{\frac{\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1)^2dr_2}{\zeta_{23}}\}^{1/2} \\
n^{-3}\hat{\sigma}^2 &\xrightarrow{D} s^{-4}\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1)^2dr_2 \\
&\quad -\frac{\{\int_0^1(\sum_j\int_0^{r_2}V_j(r_1)dr_1)(\sum_j\int_0^{r_2}W_j(r_1)dr_1)dr_2\}^2}{\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1)^2dr_2} \\
&\equiv \zeta_{23}
\end{aligned}$$

$$R^2 \xrightarrow{D} 1 - \frac{\zeta_{23}}{s^{-4}\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1)^2dr_2 - s^{-5}(\sum_j\int_0^1\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1dr_2)^2}$$

A2.2.4. Formulae for model (3.6):

$$\begin{aligned}
n^{-3/2}\hat{\beta}_j &\xrightarrow{D} s^{-3/2}\int_0^1\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1dr_2 - \xi_{26}s^{-3/2}\int_0^1\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1dr_2 \\
&\equiv \xi_{25}^{j} \\
n^{-1/2}t_{\beta_j=0} &\xrightarrow{D} \xi_{25}^{j}/(s\zeta_{24})^{1/2} \\
\hat{\alpha} &\xrightarrow{D} [\int_0^1(\sum_j\int_0^{r_2}V_j(r_1)dr_1)(\sum_j\int_0^{r_2}W_j(r_1)dr_1)dr_2 \\
&\quad -\sum_j(\int_0^1\sum_i\int_0^{r_2}V_i(r_1)dr_1dr_2)(\int_0^1\sum_i\int_0^{r_2}W_i(r_1)dr_1dr_2)] \\
&\quad [\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1)^2dr_2 - \sum_j(\int_0^1\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1dr_2)^2] \\
&\equiv \xi_{26} \\
n^{-1/2}t_{\alpha=0} &\xrightarrow{D} s^{-2}\xi_{26}\{\frac{\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}V_i(r_1)dr_1)^2dr_2}{\zeta_{24}}\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
n^{-3}\hat{\sigma}^2 &\xrightarrow{D} s^{-4}\sum_j\int_0^1(\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1)^2dr_2 - s^{-4}\sum_j(\int_0^1\sum_{i=1}^{j}\int_0^{r_2}W_i(r_1)dr_1dr_2)^2] \\
&\quad +[\int_0^1(\sum_j\int_0^{r_2}V_j(r_1)dr_1)(\sum_j\int_0^{r_2}W_j(r_1)dr_1)dr_2
\end{aligned}$$

$$-\sum_j(\int_0^1 \sum_i \int_0^{r_2} V_i(r_1)dr_1dr_2)(\int_0^1 \sum_i \int_0^{r_2} W_i(r_1)dr_1dr_2)]^2$$
$$[\sum_j \int_0^1 (\sum_{i=1}^j \int_0^{r_2} V_i(r_1)dr_1)^2 dr_2 - \sum_j(\int_0^1 \sum_{i=1}^j \int_0^{r_2} V_i(r_1)dr_1dr_2)^2]$$
$$\equiv \quad \zeta_{24}$$

$$R^2 \xrightarrow{D} 1 - \frac{\zeta_{24}}{s^{-4}\sum_j \int_0^1(\sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1)^2dr_2 - s^{-5}(\sum_j \int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2)^2}$$

A2.2.5. Formulae for model (3.7):

$$\begin{aligned}
n^{-3/2}\hat\beta_j \quad &\xrightarrow{D} \quad s^{-3/2}\int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad + \frac{1}{\tilde a\tilde d - \tilde b^2}\sum_{i=1}^s \{\tilde c_{ij} s^{-3/2}\int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2\} \\
&\equiv \quad \xi_{27}^j \\
n^{-1/2}t_{\beta_j=0} \quad &\xrightarrow{D} \quad \xi_{27}^j/(s\zeta_{15})^{1/2} \\
\hat\alpha \quad &\xrightarrow{D} \quad (\tilde a\tilde d - \tilde b^2)^{-1}[-\sum_j \tilde g_j s^{-5/2}\int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad -\tilde b\sum_j \int_0^1 \sum_{i=1}^j \int_0^{r_2} r_1 W_i(r_1)dr_1dr_2 \\
&\quad +\tilde a\int_0^1(\sum_j \int_0^{r_2} V_j(r_1)dr_1)(\sum_j \int_0^{r_2} W_j(r_1)dr_1)dr_2] \quad \equiv \quad \xi_{28} \\
n^{-1/2}t_{\alpha=0} \quad &\xrightarrow{D} \quad s^{-2}\xi_{28}\{\frac{\sum_j \int V_j^2(r)dr}{\zeta_{15}}\}^{1/2} \\
n^{-1/2}\hat\gamma \quad &\xrightarrow{D} \quad (\tilde a\tilde d - \tilde b^2)^{-1}[\sum_j \tilde f_j s^{-5/2}\int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2 \\
&\quad +\tilde d\sum_j \int_0^1 \sum_{i=1}^j \int_0^{r_2} r_1 W_i(r_1)dr_1dr_2 \\
&\quad -\tilde b\int_0^1(\sum_j \int_0^{r_2} V_j(r_1)dr_1)(\sum_j \int_0^{r_2} W_j(r_1)dr_1)dr_2] \quad \equiv \quad \xi_{29} \\
n^{-1/2}t_{\gamma=0} \quad &\xrightarrow{D} \quad \xi_{29}/(6\zeta_{25})^{1/2}
\end{aligned}$$

$$n^{-3}\hat\sigma^2 \quad \xrightarrow{D} \quad s^{-4}\sum_j \int_0^1 (\sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1)^2 dr_2 - s^{-4}\sum_j(\int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2)^2]$$
$$-(\tilde a\tilde d - \tilde b^2)^{-1}\{\sum_i \sum_j \tilde c_{ij} s^{-5}$$

$$
\begin{aligned}
& * \int_0^1 \sum_{k=1}^{i} \int_0^{r_2} W_k(r_1)dr_1dr_2 \int_0^1 \sum_{k=1}^{j} \int_0^{r_2} W_k(r_1)dr_1dr_2 \\
& +2\sum_j \tilde{f}_j s^{-5/2} \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} W_i(r_1)dr_1dr_2 (\sum_j \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} r_1 V_i(r_1)dr_1dr_2) \\
& +2\sum_j \tilde{g}_j s^{-5/2} \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} W_i(r_1)dr_1dr_2 (\int_0^1 (\sum_j \int_0^{r_2} V_j(r_1)dr_1) \\
& \quad * (\sum_j \int_0^{r_2} W_j(r_1)dr_1)dr_2) \\
& +\tilde{d}(\sum_j \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} r_1 W_i(r_1)dr_1dr_2)^2 \\
& -2\tilde{b}(\sum_j \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} r_1 W_i(r_1)dr_1dr_2) \\
& \quad * (\int_0^1 (\sum_j \int_0^{r_2} V_j(r_1)dr_1)(\sum_j \int_0^{r_2} W_j(r_1)dr_1)dr_2) \\
& +\tilde{a}(\int_0^1 (\sum_j \int_0^{r_2} V_j(r_1)dr_1)(\sum_j \int_0^{r_2} W_j(r_1)dr_1)dr_2)^2\} \\
\equiv\ & \zeta_{25}
\end{aligned}
$$

$$
R^2 \xrightarrow{D} 1 - \frac{\zeta_{25}}{s^{-4}\sum_j \int_0^1 (\sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1)^2 dr_2 - s^{-5}(\sum_j \int_0^1 \sum_{i=1}^j \int_0^{r_2} W_i(r_1)dr_1dr_2)^2}
$$

where

$$
\begin{aligned}
\tilde{a} &= 1/12 \\
\tilde{b} &= s^{-7/2}\sum_j \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} r_1 V_i(r_1)dr_1dr_2 - \frac{1}{2}s^{-3/2}\sum_j \int_0^1 \sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1dr_2 \\
\tilde{d} &= s^{-4}\sum_j \int_0^1 (\sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1)^2 dr_2 - s^{-3}\sum_j (\int_0^1 \sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1dr_2)^2 \\
\tilde{f}_j &= -\frac{1}{2}d + bs^{-3/2}\int_0^1 \sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1dr_2 \\
\tilde{g}_j &= \frac{1}{2}b - as^{-3/2}\int_0^1 \sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1dr_2 \\
\tilde{c}_{ij} &= -\frac{1}{4}d + b\frac{s^{-5/2}}{2}\int_0^1 \sum_{i=1}^{j} \int_0^{r_2} V_i(r_1)dr_1dr_2 \\
&+ as^{-4}\int_0^1 \sum_{k=1}^{i} \int_0^{r_2} V_k(r_1)dr_1dr_2 \int_0^1 \sum_{k=1}^{j} \int_0^{r_2} V_k(r_1)dr_1dr_2
\end{aligned}
$$

A2.3. Formulae of Theorem 3:

For each of the partially contegrated cases, we first need to express N_t in terms of u_t. For example, if X_t and Y_t are cointegrated at frequency zero, then

$$N_t = \sum_{k=1}^{[t/4]} u_{t-4k} - \sum_{k=1}^{[t/4]} u_{t-1-4k}.$$

Let $U_j(r)$ be a Brownian motion as a limiting distribution of the partial sum process $\sum_k u_{(k-1)s+j}$ corresponding to the j-th season and u_t is as defined in (12). Then,

$$n^{-1/2} N_{[nr]} \xrightarrow{D} s^{-1/2}(U_j(r) - U_{j-1}(r)).$$

As the formulae for Theorem 3 are algebraically involved, here we present the foremulae for the partially cointegrated case at frequency zero only. The formulae for the other cases can be obtained from the authors upon request.

A2.3.1. Formulae for model (3.5):

$$\hat{\alpha} - \alpha_0 \xrightarrow{D} \frac{\sum_j \int_0^1 V_j(r)(U_j(r) - U_{j-1}(r))dr}{\sum_j \int_0^1 V_j^2(r)dr} \equiv \xi_{31}$$

$$n^{-1/2} t_{\alpha=\alpha_0} \xrightarrow{D} \{\sum_j \int V_j(r)^2 dr\}^{1/2} \xi_{31}/(s^2 \zeta_{31})^{1/2}$$

$$n^{-1}\hat{\sigma}^2 \xrightarrow{D} s^{-2} \sum_j \int (U_j(r) - U_{j-1}(r))^2 dr - \frac{\{\sum_j \int_0^1 V_j(r)(U_j(r) - U_{j-1}(r))dr\}^2}{\sum_j \int_0^1 V_j^2(r)dr} \equiv \zeta_{31}$$

$$\begin{aligned} R^2 \xrightarrow{D}\ & 1 - \zeta_{31} \quad [\alpha_0^2 (s^{-2} \sum_j \int V_j^2(r)dr - s^{-3}(\sum_j \int V_j(r)dr)^2) \\ & + 2\alpha_0 (s^{-2} \sum_j \int_0^1 V_j(r)(U_j(r) - U_{j-1}(r))dr \\ & - s^{-3} \sum_j \int V_j(r)dr \sum_j \int (U_j(r) - U_{j-1}(r))dr) \\ & + s^{-2} \sum_j \int_0^1 (U_j(r) - U_{j-1}(r))^2 dr - s^{-3}(\sum_j \int_0^1 U_j(r) - U_{j-1}(r)dr)^2] \end{aligned}$$

A2.3.2. Formulae for model (3.6):

$$\hat{\alpha} - \alpha_0 \xrightarrow{D} \frac{\sum_j \int V_j(r)(U_j(r) - U_{j-1}(r))dr - \sum_j \int V_j(r)dr \int (U_j(r) - U_{j-1}(r))dr}{\sum_j \int V_j(r)^2 dr - \sum_j (\int V_j(r)dr)^2} \equiv \xi_{32}$$

$$n^{-1/2} t_{\alpha=\alpha_0} \xrightarrow{D} \{\sum_j \int V_j(r)\}^{1/2} \xi_{32}/(s^2 \zeta_{32})^{1/2}$$

$$
\begin{aligned}
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}(\int U_j(r)-U_{j-1}(r))dr \\
&\quad -\frac{\sum_j\int V_j(r)(U_j(r)-U_{j-1}(r))dr-\sum_j\int V_j(r)dr\int(U_j(r)-U_{j-1}(r))dr}{\sum_j\int V_j(r)^2dr-\sum_j(\int V_j(r)dr)^2} \\
&\quad s^{-1/2}\int V_j(r)dr \\
&\equiv \xi_{33} \\
n^{-1/2}t_{\beta_j=0} &\xrightarrow{D} \xi_{33}/(s\zeta_{32})^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j\int(U_j(r)-U_{j-1}(r))^2dr-s^{-2}\sum_{j=1}^{s}(\int U_j(r)-U_{j-1}(r)dr)^2 \\
&\quad -\frac{[\sum_j\int(U_j(r)-U_{j-1}(r))V_j(r)-\sum_j\int U_j(r)-U_{j-1}(r)dr\int V_j(r)dr]^2}{\sum_j\int V_j(r)^2dr-\sum_j(\int V_j(r)dr)^2} \\
&\equiv \zeta_{32} \\
R^2 &\xrightarrow{D} 1-\zeta_{32}\quad [\alpha_0^2(s^{-2}\sum_j\int V_j^2(r)dr-s^{-3}(\sum_j\int V_j(r)dr)^2) \\
&\quad +2\alpha_0(s^{-2}\sum_j\int_0^1 V_j(r)(U_j(r)-U_{j-1}(r))dr \\
&\quad -s^{-3}\sum_j\int V_j(r)dr\sum_j\int(U_j(r)-U_{j-1}(r))dr) \\
&\quad +s^{-2}\sum_j\int_0^1(U_j(r)-U_{j-1}(r))^2dr-s^{-3}(\sum_j\int_0^1 U_j(r)-U_{j-1}(r)dr)^2]
\end{aligned}
$$

A2.3.3. Formulae for model (3.7):

$$
\begin{aligned}
\hat{\alpha}-\alpha_0 &\xrightarrow{D} (ad-b^2)^{-1}[-\sum_j g_j s^{-3/2}\int(U_j(r)-U_{j-1}(r))dr \\
&\quad -b\sum_j\int r(U_(r)-U_{j-1}(r))dr+a\sum_j\int(U_j(r)-U_{j-1}(r))V_j(r)dr] \\
&\equiv \xi_{34} \\
n^{-1/2}t_{\alpha=\alpha_0} &\xrightarrow{D} \{\sum_j\int V_j(r)^2dr\}^{1/2}\xi_{34}/(s^2\zeta_{33})^{1/2} \\
n^{-1/2}\hat{\beta}_j &\xrightarrow{D} s^{-1/2}(\int U_j(r)-U_{j-1}(r))dr \\
&\quad +\frac{1}{ad-b^2}\sum_{i=1}^{s}\{c_{ij}s^{-1/2}(\int U_j(r)-U_{j-1}(r))dr \\
&\quad +\alpha_0[s^{-1/2}\int V_j(r)dr+\frac{1}{ad-b^2}\sum_{i=1}^{s}\{c_{ij}s^{-1/2}\int V_i(r)dr\}] \\
&\equiv \xi_{35} \\
n^{-1/2}t_{\beta_j=0} &\xrightarrow{D} \xi_{35}/(s\zeta_{33})^{1/2} \\
n^{-1}\hat{\sigma}^2 &\xrightarrow{D} s^{-2}\sum_j\int(U_j(r)-U_{j-1}(r))^2dr-s^{-2}\sum_j(\int_0^1 U_j(r)-U_{j-1}(r)dr)^2
\end{aligned}
$$

$$
\begin{aligned}
& -(ad-b^2)^{-1}\{\sum_i\sum_j c_{ij}s^{-3}(\int_0^1 U_i(r)-U_{i-1}(r)dr)(\int_0^1 U_j(r)-U_{j-1}(r)dr) \\
& +2\sum_j f_j s^{-3/2}(\int_0^1 U_j(r)-U_{j-1}(r)dr)\int rV(r)dr \\
& +2\sum_j g_j s^{-3/2}(\int_0^1 U_j(r)-U_{j-1}(r)dr)\sum_j\int_0^1 (U_j(r)-U_{j-1}(r))V_j(r)dr \\
& +d(\sum_j\int_0^1 U_j(r)-U_{j-1}(r)dr)^2 \\
& -2b[\sum_j\int_0^1 r(U_j(r)-U_{j-1}(r))dr][\sum_j\int_0^1 (U_j(r)-U_{j-1}(r))V_j(r)dr] \\
& +a[\sum_j\int_0^1 (U_j(r)-U_{j-1}(r))V_j(r)dr]^2 \\
\equiv\ & \zeta_{33} \\
R^2 \xrightarrow{D}\ & 1-\zeta_{33}\quad [\alpha_0^2(s^{-2}\sum_j\int V_j^2(r)dr - s^{-3}(\sum_j\int V_j(r)dr)^2) \\
& +2\alpha_0(s^{-2}\sum_j\int_0^1 V_j(r)(U_j(r)-U_{j-1}(r))dr \\
& -s^{-3}\sum_j\int V_j(r)dr\sum_j\int (U_j(r)-U_{j-1}(r))dr) \\
& +s^{-2}\sum_j\int_0^1 (U_j(r)-U_{j-1}(r))^2dr - s^{-3}(\sum_j\int_0^1 U_j(r)-U_{j-1}(r)dr)^2]
\end{aligned}
$$

where a,b,d,c_{ij},f_j and g_j are defined in *A2.1.5*.

REFERENCES

(1) Abeysinghe, T. (1991) Inappropriate use of seasonal dummies in regression, *Economics Letters,* 36, 175-179.

(2) Abeysinghe, T. (1994) Deterministic seasonal models and spurious regressions, *Journal of Econometrics,* 61, 259-272.

(3) Ahn, S.K. and Cho, S. (1993) Some tests for unit roots in seasonal time series with deterministic trends, *Statistics and Probability Letters,* 16, 85-95.

(4) Ahn, S.K. and Reinsel, G.C. (1994) Estimation of partially nonstationary vector autoregressive models with seasonal behavior, *Journal of Econometrics,* 62, 317-350.

(5) Box, G.E.P., Jenkins, G.M., and Reinsel, G.C. (1994) *Time series analysis: Forecasting and control, 3ed,* Englewood Cliffs, NJ: Prentice-Hall.

(6) Durlauf, S. and Phillips, P.C.B. (1988) Trends versus random walks in time series analysis, *Econometrica,* 56, 1333-1354.

(7) Granger, C.W.J. and Newbold, P. (1974) Spurious regression in economics, *Journal of Econometrics,* 2, 111-120.

(8) Hylleberg, S., Engle, R.F., Granger, C.W.J., and Yoo, B.S. (1990) Seasonal integration and cointegration, *Journal of Econometrics,* 44, 215-238.

(9) Lee, H.S. (1992) Maximum likelihood inference on cointegration and seasonal cointegration, *Journal of Econometrics,* 54, 1-47.

(10) Li, W.K. (1991) Some lagrange multiplier tests for seasonal differencing, *Biometrika,* 78, 49-56.

(11) Park, Y. J. and Cho, S. (1995) Lagrange multiplier test for both regular and seasonal unit roots, *The Korean Communications in Statistics,* 2, 101-114.

(12) Phillips P.C.B. (1986) Understanding spurious regressions in economics, *Journal of Econometrics,* 33, 311-340.

(13) Phillips P.C.B.(1987) Time series regression with a unit root, *Econometrica,* 55, 277-301.

(14) Shin, D.W. and Sarkar, S. (1995) Model misspecification problems for nonstationary time series, *Technical Report Dept. of Statistics Ewha Womans University.*

(15) Wei, W.W.S. (1989) *Time series analysis*, Addison-Wesley.