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# Bayes Estimation in a Hierarchical Linear Model †

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**ABSTRACT**

In the problem of estimating a vector of unknown regression coefficients under the sum of squared error losses in a hierarchical linear model, we propose the hierarchical Bayes estimator of a vector of unknown regression coefficients in a hierarchical linear model, and then prove the admissibility of this estimator using Blyth's (1951) method.

**Key Words** : Hierarchical Bayes Estimator; Admissibility; Hierarchical Linear Model.

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## 1. INTRODUCTION

Consider the linear model

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{y}$ is a $n \times 1$ vector of observations $y_i$, $X$ is a $n \times p$ known design matrix with rank($X$)$= q(q \leq p)$, $\boldsymbol{\beta}$ is a $p \times 1$ vector of parameters, and $\boldsymbol{e}$ is a $n \times 1$ vector of random errors. The hierarchical Bayes (HB) procedure which is a particular modeling of the prior information decompose the prior distribution into several conditional levels of distribution between structural and subjective items of information. A crucial point in the historical development of Bayesian methods was the recognition that the statistician's knowledge about the parameters in a model could also be subject to modeling. Prior knowledge can be roughly classified as structural relationships and parametric assessments, the latter being potentially more controversial than the former. The hierarchical linear model has proven to be successful in modeling structural knowledge. Good (1965) in his important work on the estimation of proportions, introduced the idea of specifying the prior distribution in stages. Lindley and Smith (1972) in their fundamental paper on the Bayesian linear model gave definition meaning to the term "hierarchical prior specification" and provided the basis for what has proven to be most fruitful area of development within the Bayesian paradigm.

In particular, hierarchical linear models were first introduced by Lindley and Smith (1972). Smith (1973a, 1973b) extended the use of general Bayesian linear model to estimation of parameters in the third stage of the hierarchy, as well as second stage. Strawderman (1971) proposed the HB procedure in estimating the mean vector under the squared error loss. Albert (1988) considered computation methods of Bayesian hierarchical generalized linear model. Pericchi and Nazaret (1988) characterized the consequences of being imprecise in the higher levels of the hierarchy, conditional on the data. Also, Berger and Robert (1990) proposed subjective hierarchical Bayes estimator of a multivariate normal mean and proved minimaxity of HB estimator. Recently, O'Hagan (1994) proposed hierarchical linear model. Datta and Ghosh (1995) proposed a class of hierarchical Bayes estimators which overcomes the Neyman-Scott problem in estimating the error variance in one-way analysis of variance (ANOVA) models and verified the minimaxity of HB estimator. Blyth (1951) proposed a sufficient admissibility condition, relating admissibility of an estimator with the existence of sequence of prior distributions approximating this estimator.

In this paper, we consider the problem of estimating $\boldsymbol{\beta}$ under the sum of squared error losses $L_0(\boldsymbol{\beta}, d) = |\boldsymbol{\beta} - d|^2$. In Section 2, we provide the hierarchical Bayes estimator of a vector of unknown regression coefficient in a hierarchical linear model. In Section 3, we propose some lemmas playing an important role in subsequence analysis and show that the admissibility of a HB estimator $\widehat{\beta}_{HB}$ using Blyth's (1951) method.

## 2. HIERARCHICAL BAYES ESTIMATOR IN A HIERARCHICAL LINEAR MODEL

In this section, we derive a hierarchical Bayes estimator in the following model ;

(i) conditional on $\boldsymbol{\beta}$ and $\gamma$, $\boldsymbol{y} \sim N_n(X\boldsymbol{\beta},\ \alpha^{-1}I_n)$, where $X$ is a $n \times p$ $(p < n)$ design matrix with rank($X$)$= q \le p$.

(ii) conditional on $\gamma$, $\boldsymbol{\beta} \sim N_p(X_1\gamma,\ \alpha^{-1}V)$, where $X_1$ is a $p \times q(q \le p)$ design matrix with rank($X_1$)$= q$, $\alpha$ is a positive known constant, and $V$ is a $p \times p$ known positive definite matrix.

(iii) $\gamma$ is uniform over $R^q$.

**Theorem 2.1.** For the hierarchical model defined in the above, the hierarchical Bayes estimator of $\boldsymbol{\beta}$ is

$$\widehat{\beta}_{HB} = [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y}. \tag{2.1}$$

**Proof.** The joint (improper) density of $\boldsymbol{y}$, $\boldsymbol{\beta}$ and $\gamma$ under the above model is given by

$$\begin{aligned} f(\boldsymbol{y},\boldsymbol{\beta},\gamma) &= (2\pi)^{-\frac{1}{2}(n+p)}\alpha^{\frac{1}{2}(n+p)}|V|^{-\frac{1}{2}} \\ &\quad \cdot exp\Big\{-\frac{\alpha}{2}(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})\Big\} \\ &\quad \cdot exp\Big\{-\frac{\alpha}{2}(\boldsymbol{\beta}-X_1\gamma)'V^{-1}(\boldsymbol{\beta}-X_1\gamma)\Big\} \\ &\propto exp\Big\{-\frac{\alpha}{2}[\boldsymbol{y}'\boldsymbol{y} - 2\boldsymbol{y}'X\boldsymbol{\beta} + \boldsymbol{\beta}'X'X\boldsymbol{\beta} + \boldsymbol{\beta}'V^{-1}\boldsymbol{\beta} \\ &\qquad -\boldsymbol{\beta}'V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\boldsymbol{\beta}]\Big\} \\ &\quad \cdot exp\Big\{-\frac{\alpha}{2}[\gamma-(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\boldsymbol{\beta}]'[X_1'V^{-1}X_1] \\ &\qquad [\gamma-(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\boldsymbol{\beta}]\Big\}. \end{aligned} \tag{2.2}$$

Next, by integrating (2.2) with respect to $\gamma$, we get the joint (improper) density of $\boldsymbol{y}$ and $\boldsymbol{\beta}$ which is given by

$$\begin{aligned} f(\boldsymbol{y},\beta) \ \propto \ & exp\Big\{ -\frac{\alpha}{2}[\boldsymbol{y}'\boldsymbol{y} - 2\boldsymbol{y}'X\boldsymbol{\beta} + \boldsymbol{\beta}'X'X\boldsymbol{\beta} + \boldsymbol{\beta}'V^{-1}\boldsymbol{\beta} \\ & -\boldsymbol{\beta}'V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\boldsymbol{\beta}]\Big\}. \end{aligned}$$

Since $f(\beta|\boldsymbol{y}) \propto f(\boldsymbol{y},\boldsymbol{\beta})$, the conditional density of $\boldsymbol{\beta}$, given $\boldsymbol{y}$, becomes

$$\begin{aligned} & f(\boldsymbol{\beta}|\boldsymbol{y}) \\ & \propto \ exp\Big\{ -\frac{\alpha}{2}\big[\boldsymbol{\beta} - [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y}\big]' \\ & \qquad \cdot\big[X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\big] \\ & \qquad \cdot\big[\boldsymbol{\beta} - [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y}\big]\Big\}. \qquad (2.3) \end{aligned}$$

From (2.3), the conditional expectation of $\boldsymbol{\beta}$, given $\boldsymbol{y}$, is

$$\begin{aligned} E(\boldsymbol{\beta}|\boldsymbol{y}) \ &= \ \frac{\int \boldsymbol{\beta} f(\boldsymbol{\beta},\boldsymbol{y})d\beta}{\int f(\boldsymbol{\beta},\boldsymbol{y})d\beta} \\ &= \ \frac{\frac{1}{G}\int \beta \cdot G \cdot exp\Big\{ -\frac{\alpha}{2}[\boldsymbol{\beta} - DX'\boldsymbol{y}]'D^{-1}[\boldsymbol{\beta} - DX'\boldsymbol{y}]\Big\}d\boldsymbol{\beta}}{\frac{1}{G}\int G \cdot exp\Big\{ -\frac{\alpha}{2}[\boldsymbol{\beta} - DX'\boldsymbol{y}]'D^{-1}[\boldsymbol{\beta} - DX'\boldsymbol{y}]\Big\}d\boldsymbol{\beta}}, \\ & \qquad \text{where } D = [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1} \\ & \qquad \text{and } G = (2\pi)^{-\frac{p}{2}}\alpha^{\frac{p}{2}}|D|^{-\frac{1}{2}} \\ &= \ [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y}. \end{aligned}$$

Therefore, the hierarchical Bayes estimator of $\beta$ is

$$\begin{aligned} \widehat{\beta}_{HB} \ &= \ E(\boldsymbol{\beta}|\boldsymbol{y}) \\ &= \ [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y}. \qquad (2.4) \end{aligned}$$

**Remark 2.1.** If $V \to \infty$ in (2.4), the hierarchical Bayes estimator of $\boldsymbol{\beta}$ is identical to the least squares estimator $\widehat{\beta}_{LS} = (X'X)^{-1}X'\boldsymbol{y}$ when rank($X$)$= p$.

## 3. ADMISSIBILITY OF HIERARCHICAL BAYES ESTIMATOR

In this section, we give two matrix lemmas and a property of expectation in quadratic form, playing an important role in subsequence analysis.

**Lemma 3.1.** Let $A(n)$ and $B(n)$ be respectively $k \times l$, $l \times m$ matrices with elements depending on $n$. Then

$$\lim_{n\to\infty} A_{k\times l}(n) \cdot B_{l\times m}(n) = [\lim_{n\to\infty} A_{k\times l}(n)] \cdot [\lim_{n\to\infty} B_{l\times m}(n)].$$

**Proof.** Let

$$A_{k\times l}(n) = \begin{pmatrix} a_{11}(n) & a_{12}(n) & \cdots & a_{1l}(n) \\ \vdots & & \ddots & \vdots \\ a_{k1}(n) & a_{k2}(n) & \cdots & a_{kl}(n) \end{pmatrix},$$

and

$$B_{l\times m}(n) = \begin{pmatrix} b_{11}(n) & b_{12}(n) & \cdots & b_{1m}(n) \\ \vdots & & \ddots & \vdots \\ b_{l1}(n) & b_{l2}(n) & \cdots & b_{lm}(n) \end{pmatrix}.$$

Then

$$A_{k\times l}B_{l\times m}(n) = \begin{pmatrix} \sum_{j=1}^{l} a_{1j}(n)b_{j1}(n) & \cdots & \sum_{j=1}^{l} a_{1j}(n)b_{jm}(n) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{l} a_{kj}(n)b_{j1}(n) & \cdots & \sum_{j=1}^{l} a_{kj}(n)b_{jm}(n) \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} &\lim_{n\to\infty} A_{k\times l}B_{l\times m}(n) \\ &= \begin{pmatrix} \sum_{j=1}^{l} \lim_{n\to\infty} a_{1j}(n) \lim_{n\to\infty} b_{j1}(n) & \cdots & \sum_{j=1}^{l} \lim_{n\to\infty} a_{1j}(n) \lim_{n\to\infty} b_{jm}(n) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{l} \lim_{n\to\infty} a_{kj}(n) \lim_{n\to\infty} b_{j1}(n) & \cdots & \sum_{j=1}^{l} \lim_{n\to\infty} a_{kj}(n) \lim_{n\to\infty} b_{jm}(n) \end{pmatrix} \\ &= [\lim_{n\to\infty} A_{k\times l}(n)] \cdot [\lim_{n\to\infty} B_{l\times m}(n)]. \end{aligned}$$

**Lemma 3.2.** Let $A_k(n)$ be a $k \times k$ nonsingular matrix with elements depending on $n$, then

$$\lim_{n\to\infty} A_k^{-1}(n) = [\lim_{n\to\infty} A_k(n)]^{-1}.$$

**Proof.** We have

$$I = A_k^{-1}(n) \cdot A_k(n),$$

where $A_k^{-1}(n)$ is the inverse of $A_k(n)$.
And

$$I = \lim_{n\to\infty} I = [\lim_{n\to\infty} A_k^{-1}(n)][\lim_{n\to\infty} A_k(n)].$$

Therefore $\lim_{n\to\infty} A_k^{-1}(n) = [\lim_{n\to\infty} A_k(n)]^{-1}$.

**Lemma 3.3.** When $\boldsymbol{x}$ is $N(\boldsymbol{\mu}, V)$

$$E(\boldsymbol{x}'A\boldsymbol{x}) = \boldsymbol{\mu}'A\boldsymbol{\mu} + tr(AV),$$

and when $\boldsymbol{\mu} = \mathbf{0}$

$$E(\boldsymbol{x}'A\boldsymbol{x}) = tr(AV).$$

**Proof.** See Searle (1970, p54).

In the following theorem we prove admissibility of hierarchical Bayes estimator $\widehat{\beta}_{HB}$ in (2.1) under the sum of squared error losses $L_0(\boldsymbol{\beta}, d) = |\boldsymbol{\beta} - d|^2$.

**Theorem 3.1.** The hierarchical Bayes estimator

$$\begin{aligned}\widehat{\beta}_{HB} &= [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}X'\boldsymbol{y} \\ &= DX'\boldsymbol{y},\end{aligned}$$

is admissible under the sum of squared error losses, where $D = [X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}$.

**Proof.** Consider the sequence of prior $\{\pi_n\}$ for $\boldsymbol{\beta}$ where $\pi_n$ is $N_p(\mathbf{0},\ \lambda^{-1}B_n)$ with

$$B_n^{-1} = (V^{-1} + \frac{1}{n}I_p) - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}.$$

And the joint density of $\boldsymbol{y}$ and $\boldsymbol{\beta}$ with respect to prior $\pi_n$ is given by

$$f(\boldsymbol{y}, \boldsymbol{\beta}) \;=\; (2\pi)^{-\frac{1}{2}(n+p)}\lambda^{\frac{1}{2}(n+p)}$$

$$
\begin{aligned}
&\cdot exp\Big\{-\frac{\lambda}{2}(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})\Big\} \\
&\cdot exp\Big\{-\frac{\lambda}{2}\boldsymbol{\beta}'B_n^{-1}\boldsymbol{\beta}\Big\} \\
\propto\ & exp\Big\{-\frac{\lambda}{2}[\boldsymbol{y}'\boldsymbol{y}-\boldsymbol{y}'X(X'X+B_n^{-1})^{-1}X'\boldsymbol{y}]\Big\} \\
&\cdot exp\Big\{-\frac{\lambda}{2}[\boldsymbol{\beta}-(X'X+B_n^{-1})^{-1}X'\boldsymbol{y}]' \\
&\qquad\cdot[X'X+B_n^{-1}][\boldsymbol{\beta}-(X'X+B_n^{-1})^{-1}X'\boldsymbol{y}]\Big\}. \qquad (3.1)
\end{aligned}
$$

Then

$$
\begin{aligned}
f(\beta|\boldsymbol{y}) &\propto exp\Big\{-\frac{\lambda}{2}[\boldsymbol{\beta}-(X'X+B_n^{-1})^{-1}X'\boldsymbol{y}]' \\
&\qquad [X'X+B_n^{-1}][\boldsymbol{\beta}-(X'X+B_n^{-1})^{-1}X'\boldsymbol{y}]\Big\} \\
&= exp\Big\{-\frac{\lambda}{2}[\boldsymbol{\beta}-D_nX'\boldsymbol{y}]'D_n^{-1}[\boldsymbol{\beta}-D_nX'\boldsymbol{y}]\Big\}, \\
&\qquad \text{where } D_n=(X'X+B_n^{-1})^{-1}.
\end{aligned}
$$

Hence the posterior distribution of $\boldsymbol{\beta}$ given $\boldsymbol{y}$ is $N_p(D_nX'\boldsymbol{y},\ \lambda^{-1}D_n)$, where

$$
\begin{aligned}
D_n^{-1} &= X'X+B_n^{-1} \\
&= [X'X+V^{-1}-V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]+\frac{1}{n}I_p \\
&= D^{-1}+\frac{1}{n}I_p,
\end{aligned}
$$

where $D=[X'X+V^{-1}-V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}]^{-1}$.
Next, integrating with respect to $\boldsymbol{\beta}$ in (3.1), it follows that the marginal density of $\boldsymbol{y}$ is

$$
f(\boldsymbol{y}) \propto exp\Big\{-\frac{\lambda}{2}\boldsymbol{y}'[I_n-X(X'X+B_n^{-1})^{-1}X']\boldsymbol{y}\Big\}. \qquad (3.2)
$$

Therefore, from (3.2) the marginal distribution of $\boldsymbol{y}$ with respect to $\pi_n$ is

$$
N_n\Big(\mathbf{0},\ \lambda^{-1}[I_n-X(X'X+B_n^{-1})^{-1}X']^{-1}\Big).
$$

Let $\Delta_n=[B(\pi_n,DX'\boldsymbol{y})-B(\pi_n,D_nX'\boldsymbol{y})]$, where, $B(\pi,\delta)$ is the Bayes risk of an estimator $\delta$ of $\boldsymbol{\beta}$ with respect to a prior $\pi$.

Hence,

$$\begin{aligned}
\Delta_n &= (2\pi)^{-\frac{1}{2}(n+p)} \int |\lambda^{-1} D_n|^{-\frac{1}{2}} (\boldsymbol{\beta} - DX'\boldsymbol{y})'(\boldsymbol{\beta} - DX'\boldsymbol{y}) \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}(\boldsymbol{\beta} - D_n X'\boldsymbol{y})' D_n^{-1} (\boldsymbol{\beta} - D_n X'\boldsymbol{y})\Big\} \\
&\qquad \cdot |\lambda^{-1}[I_n - X(X'X + B_n^{-1})^{-1} X']^{-1}|^{-\frac{1}{2}} \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}\boldsymbol{y}'[I_n - X(X'X + B_n^{-1})^{-1}X']\boldsymbol{y}\Big\} d\boldsymbol{\beta} d\boldsymbol{y} \\
&\quad -(2\pi)^{-\frac{1}{2}(n+p)} \int |\lambda^{-1} D_n|^{-\frac{1}{2}} (\boldsymbol{\beta} - D_n X'\boldsymbol{y})'(\boldsymbol{\beta} - D_n X'\boldsymbol{y}) \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}(\boldsymbol{\beta} - D_n X'\boldsymbol{y})' D_n^{-1} (\boldsymbol{\beta} - D_n X'\boldsymbol{y})\Big\} \\
&\qquad \cdot |\lambda^{-1}[I_n - X(X'X + B_n^{-1})^{-1} X']^{-1}|^{-\frac{1}{2}} \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}\boldsymbol{y}'[I_n - X(X'X + B_n^{-1})^{-1}X']\boldsymbol{y}\Big\} d\boldsymbol{\beta} d\boldsymbol{y}. \quad (3.3)
\end{aligned}$$

By transformations, $\boldsymbol{x} = \beta - DX'y$, $\boldsymbol{z} = \beta - D_n X'y$, (3.2) becomes

$$\begin{aligned}
&(2\pi)^{-\frac{1}{2}(n+p)} \lambda^{\frac{1}{2}(n+p)} \int |D_n|^{-\frac{1}{2}} (\boldsymbol{x}'\boldsymbol{x}) \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{x} + DX'\boldsymbol{y} - D_n X'\boldsymbol{y}]' D_n^{-1} [\boldsymbol{x} + DX'\boldsymbol{y} - D_n X'\boldsymbol{y}]\Big\} \\
&\qquad \cdot |I_n - XD_n X'|^{\frac{1}{2}} exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{y}'(I_n - XD_n X')\boldsymbol{y}]\Big\} d\boldsymbol{x} d\boldsymbol{y} \\
&-(2\pi)^{-\frac{1}{2}(n+p)} \lambda^{\frac{1}{2}(n+p)} \int |D_n|^{-\frac{1}{2}} \boldsymbol{z}'\boldsymbol{z} \cdot exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{z}' D_n^{-1} \boldsymbol{z}]\Big\} \\
&\qquad \cdot |I_n - XD_n X'|^{\frac{1}{2}} exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{y}'(I_n - XD_n X')\boldsymbol{y}]\Big\} d\boldsymbol{z} d\boldsymbol{y} \\
= \; &(2\pi)^{-\frac{1}{2}(n+p)} \lambda^{\frac{1}{2}(n+p)} |D_n|^{-\frac{1}{2}} |I_n - XD_n X'|^{\frac{1}{2}} \\
&\cdot \int \boldsymbol{x}'\boldsymbol{x} \Big[ exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{x} + DX'\boldsymbol{y} - D_n X'\boldsymbol{y}]' D_n^{-1} [\boldsymbol{x} + DX'\boldsymbol{y} - D_n X'\boldsymbol{y}]\Big\} \\
&- exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{x}' D_n^{-1} \boldsymbol{x}]\Big\}\Big] exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{y}'(I_n - XD_n X')\boldsymbol{y}]\Big\} d\boldsymbol{x} d\boldsymbol{y}.
\end{aligned}$$

And using Lemma 3.3, one gets

$$\begin{aligned}
\Delta_n &= (2\pi)^{-\frac{1}{2}(n+p)} \lambda^{\frac{1}{2}(n+p)} |D_n|^{-\frac{1}{2}} |I_n - XD_n X'|^{\frac{1}{2}} \\
&\quad \cdot \int \Big[ (D_n X'\boldsymbol{y} - DX'\boldsymbol{y})'(D_n X'\boldsymbol{y} - DX'\boldsymbol{y}) + \mathrm{tr}(\lambda^{-1} D_n) - \mathrm{tr}(\lambda^{-1} D_n)\Big] \\
&\qquad \cdot exp\Big\{ -\frac{\lambda}{2}[\boldsymbol{y}'(I_n - XD_n)\boldsymbol{y}]\Big\} d\boldsymbol{y} \\
&= (2\pi)^{-\frac{1}{2}(n+p)} \lambda^{\frac{1}{2}(n+p)} |D_n|^{-\frac{1}{2}} |I_n - XD_n X'|^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
& \cdot \int \boldsymbol{y}'X(D_n - D)'(D_n - D)X'\boldsymbol{y} \\
& \quad \cdot exp\Big\{ - \frac{\lambda}{2}[\boldsymbol{y}'(I_n - XD_nX')\boldsymbol{y}]\Big\}d\boldsymbol{y} \\
= & \ \mathrm{tr}\Big[\lambda^{-1}(I_n - XD_nX')^{-1}X(D_n - D)'(D_n - D)X'\Big].
\end{aligned}
$$

Next, by Lemma 3.2

$$
\lim_{n\to\infty} B_n^{-1} = V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}, \tag{3.4}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} D_n &= \Big[X'X + V^{-1} - V^{-1}X_1(X_1'V^{-1}X_1)^{-1}X_1'V^{-1}\Big]^{-1} \\
&= D.
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} D_n - D = O.
$$

Hence, using (3.4) and Lemma 3.1, (3.3) becomes

$$
\begin{aligned}
& \lim_{n\to\infty} tr\Big[\lambda^{-1}(I_n - XD_nX')^{-1}X(D_n - D)'(D_n - D)X'\Big] \\
= & \ \lambda^{-1}tr\Big[\lim_{n\to\infty}(I_n - XD_nX')^{-1}X(D_n - D)'(D_n - D)X'\Big] \\
= & \ 0,
\end{aligned}
$$

that is, $\Delta_n = [B(\pi_n, DX'\boldsymbol{y}) - B(\pi_n, D_nX'\boldsymbol{y})] \to 0$ as $n \to \infty$. By Blyth's (1951) method (see Berger 1985, p547) $\widehat{\beta}_{HB}$ is admissible.

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