Convergence Behavior of the Filtered-x LMS Algorithm for Active Noise Canceller

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Abstract

Application of the Filtered-X LMS adaptive filter to active noise cancellation requires to estimate the transfer characteristics between the output and the error signal of the adaptive canceler. In this paper, we derive an adaptive cancellation algorithm and analyze its convergence behavior when the acoustic noise is assumed to consist of multiple sinusoids. The results of the convergence analysis of the Filtered-X LMS algorithm indicate that the effects of parameter estimation inaccuracy on the convergence behavior of the algorithm are characterize by two distinct components:Phase estimation error and estimated magnitude. In particular, the convergence of the Filtered-X LMS algorithm is shown to be strongly affected by the accuracy of the phase response estimate. Simulation results of the algorithm are presented which support the theoretical convergence analysis.

I. Introduction

Adaptive approaches have widely been used in active noise cancellation applications in which the unwanted noise sound is adaptively synthesize with the equal amplitude but opposite phase, resulting in the cancellation of the acoustic noise as shown in Fig. 1.

In Fig. 1, the input microphone can be replaced by other non-acoustical sensors such as tachometers or accelerometers in which case the possibility of the speaker output feedback to the input microphone is removed [1]. For instance, periodic noises to be cancelled can be generated using its fundamental sinusoid. The adaptive filter output derives the loudspeaker in such a way that the acoustic noise and the loudspeaker output can be summed to null at the error microphone.

Although any adaptive algorithm can be used in Fig. 1 is not appropriate. The reason is that the acoustic path between the filter output and summation point of the error signal is frequency sensitive, which acts to distort the phase and magnitude of the error signal. In turn, the distortion of the phase and magnitude in the error path can degrade the convergence performance of the LMS algorithm. As a result the convergence rate is lowered, the residual error is increased, and the algorithm can even become unstable. For these reasons, it is necessary to use the so-called Filtered-x LMS algorithm [4] for which the transfer characteristics between the output and the error signal of the adaptive canceler must be estimated and the result be used in the adaptive algorithm.

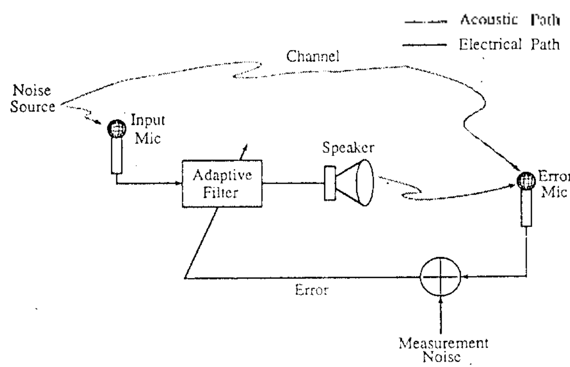

Figure 1. Basic adaptive active noise canceler configuration.

In many practical applications, the acoustic noise to be cancelled is generated by rotation machines and thus can be modeled as the sum of a fundamental sinusoid and its harmonics [5]. In this paper, we derive an adaptive canceller structure and analyze its convertgence behavior when the acoustic noise can be modeled as the sum of a fundamental sinusoid and its harmonics [6][7]. The convergence analysis is focused on the effects of parameter estimation inaccuracy on the performance.

Following the introduction, we give a brief description of the underlying system model in Section Ⅱ. The results of the convergence analysis and the simulation are presented in Sections Ⅲ and Ⅳ, respectively. Finally we make a conclusion in Section Ⅴ.

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II. System Model

Since the loud speaker-air-microphone path of Fig. 1 is linear, one can easily get the equivalent system as shown in Fig. 2. When the noise consists of the multiple sinusoids only, the acoustic and loudspeaker-acoustic-microphone paths can be described by the multiple in-phase (I) and quadrature (Q) weights as shown in the upper branch of Fig. 3. For the m-th sinusoidal noise the adaptive canceler structure also becomes to have two weights $\omega_{I,m}(n)$ and $\omega_{Q,m}(n)$, with I and Q inputs, $x_{I,m}(n)$ and $x_{Q,m}(n)$, respectively. Thus the output of the m-th canceler, $y_m(n)$, is expressed as

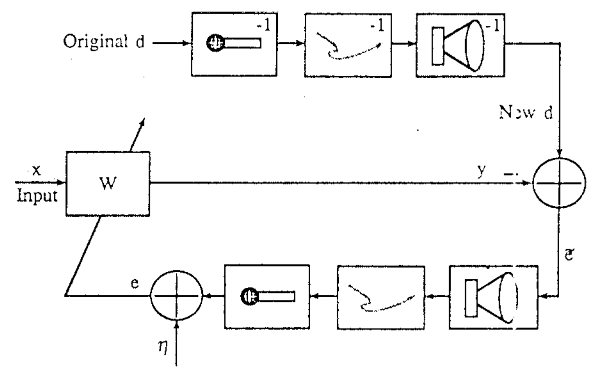


Figure 2 Rearranged form of canceler for linear system.

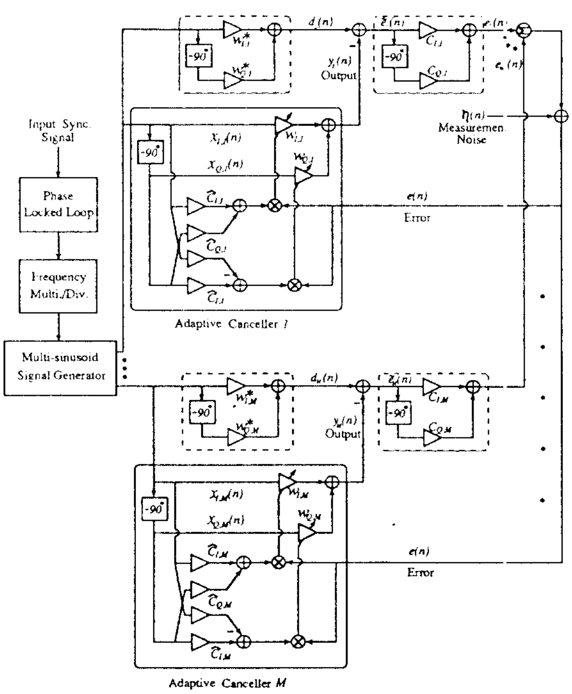


Figure 3 The diagram for the adaptive active noise cancellation system.

$$y_m(n) = \omega_{I,m}(n)\, x_{I,m}(n) + w_{Q,m}(n)\, x_{Q,m}(n) \tag{1}$$

where

$$x_{I,m}(n) = A_m \cos(\omega_m n + \varphi_m) \triangleq A_m \cos \Psi_m(n),$$
$$x_{Q,m}(n) = A_m \sin(\omega_m n + \varphi_m) \triangleq A_m \sin \Psi_m(n),$$

m : branch index = 1, 2, 3, ..., M,
n : discrete time index
A : amplitude,
ω_m : normalized frequency,
Ψ_m : random phase.

Also, referring to the notation in Fig. 2, the error signal $e(n)$ is represented by

$$\begin{aligned} e(n) = & \sum_{m=1}^{M} [c_{I,m} \tilde{e}_{I,m}(n) + c_{Q,m} \tilde{e}_{Q,m}(n)] + \eta(n) \\ = & -\sum_{m=1}^{M} [A_m \{c_{I,m} \cos \Psi_m(n) + c_{Q,m} \sin \Psi_m(n)\} \\ & \{\omega_{I,m}(n) - \omega^{*}_{I,m}\}] \\ & -\sum_{m=1}^{M} [A_m \{c_{I,m} \sin \Psi_m(n) + c_{Q,m} \cos \Psi_m(n)\} \\ & \{\omega_{Q,m}(n) - \omega^{*}_{Q,m}\}] + \eta(n) \end{aligned} \tag{2}$$

where

$\tilde{e}_I(n) \triangleq \tilde{e}(n) = \sum_{m=1}^{M} \{d_m(n) - y_m(n)\}$,

$\tilde{e}_Q$: 90° phase-shifted version of $\tilde{e}_I$

$\eta(n)$: zero-mean measurement noise.

Assuming that $\omega_{I,m}(n)$ and $\omega_{Q,m}(n)$ are slowly time-varying as compared to $x_{I,m}(n)$ and $x_{Q,m}(n)$, the phase-shifted output is given from (1) by

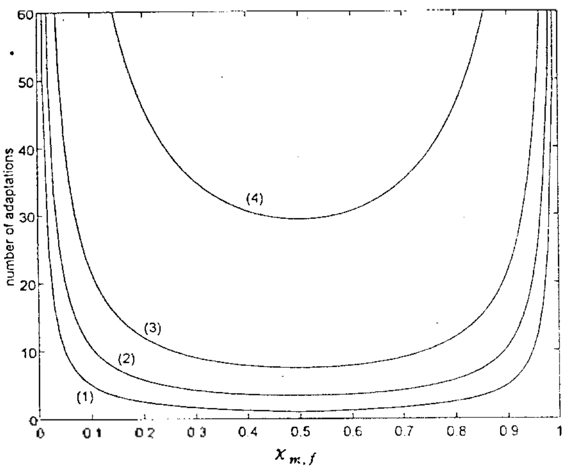
$$\begin{aligned} y_{Q,m}(n) = & \sum_{m=1}^{M} \{\omega_{I,m}(n)\, x_{Q,m}(n)\} \\ = & \sum_{m=1}^{M} A_m \{\omega_{I,m}(n) \sin \Psi_m(n) - \omega_{Q,m}(n) \cos \Psi_m(n)\}. \end{aligned} \tag{3}$$

From (1), (2), and (3), one can obtain as LMS weight update equation by minimizing $e^2(n)$ and using a gradient-descent method as

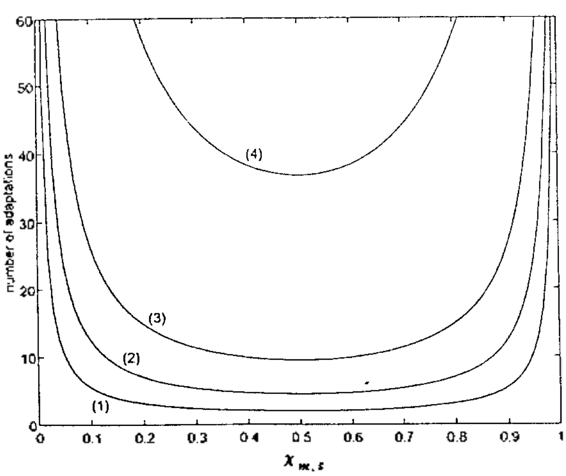
$$\begin{aligned} \omega_{I,m}(n+1) &= \omega_{I,m}(n) + \mu e(n) \{c_{I,m} x_{I,m}(n) + c_{Q,m} x_{Q,m}(n)\}, \\ \omega_{Q,m}(n+1) &= \omega_{Q,m}(n) + \mu e(n) \{c_{I,m} x_{Q,m}(n) + c_{Q,m} x_{I,m}(n)\}, \end{aligned} \tag{4}$$

where m=1, 2, ..., M and μ is a convergence constant.

It is noted that to implement the filtered-x LMS algorithm of (4), the values of c_I and c_Q must be estimated [8]. In the following, we analyze the effects of replacing $c_{I,m}$ and $c_{Q,m}$ in (4) with $\hat{c}_{I,m}$ and $\hat{c}_{Q,m}$ on the convergence behavior of the canceler.



(a)



(b)

Figure 4. Time constant;
(a) mean of the weight error magnitude,
(b) summed variance of the weight errors.

III. Convergence Analysis

A. The mean of weight error (Magnitude)

To see how the adaptive algorithm derived in (4) converges for inaccurate $\hat{c}_{I,m}$ and $\hat{c}_{Q,m}$, we first investigate the convergence of the expected values of the adaptive weights. From the underlying signal model (Fig. 2), $E[\omega_{I,m}(n)]$ and $E[\omega_{Q,m}(n)]$ are expected in the steady state to have $\omega_{I,m}$ and $\omega_{Q,m}$, respectively. To simplify the convergence equation, we may introduce two weight errors as

$$v_{I,m}(n) \triangleq \omega_{I,m}(n) - \omega_{I,m}$$
$$\text{and } v_{Q,m}(n) \triangleq \omega_{Q,m}(n) - \omega_{Q,m}. \tag{5}$$

Then, from (2), (5) and Fig. 2, we get

$$\begin{aligned} \hat{e}_{I,m}(n) &= -v_{I,m}(n)x_{I,m}(n) - v_{Q,m}(n)x_{Q,m}(n), \\ \hat{e}_{Q,m}(n) &= -v_{I,m}(n)x_{Q,m}(n) + v_{Q,m}(n)x_{I,m}(n). \end{aligned} \tag{6}$$

Inserting (5) into (4), we have

$$\begin{aligned} v_{I,m}(n+1) &= v_{I,m}(n) + \mu_m e(n)\{\hat{c}_{I,m}x_{Q,m}(n) + \hat{c}_{Q,m}x_{Q,m}(n)\}, \\ v_{Q,m}(n+1) &= v_{Q,m}(n) + \mu_m e(n)\{\hat{c}_{I,m}x_{Q,m}(n) + \hat{c}_{Q,m}x_{I,m}(n)\}. \end{aligned} \tag{7}$$

Rearranging (7) with (2) and (6), taking expectation both sides of the resultant two weight error equations, we can get the following convergence equation based on the independent assumption on the underlying signals; $x_m(n)$, $\eta(n)$, $v_{I,m}$ and $v_{Q,m}$. That is,

$$\begin{pmatrix} E[v_{I,m}(n+1)] \\ E[v_{Q,m}(n+1)] \end{pmatrix} = \begin{pmatrix} \alpha_m & \beta_m \\ -\beta_m & \alpha_m \end{pmatrix} \begin{pmatrix} E[v_{I,m}(n+1)] \\ E[v_{Q,m}(n+1)] \end{pmatrix} \tag{8}$$

where $\alpha_m \triangleq 1 - \frac{1}{2}\mu_m A_m^2 (c_{I,m}\hat{c}_{I,m} + c_{Q,m}\hat{c}_{Q,m})$,

$\beta_m \triangleq \frac{1}{2}\mu_m A_m^2 (\hat{c}_{I,m} c_{Q,m} - c_{I,m}\hat{c}_{Q,m})$.

Here, defining gain and phase response parameters as

$$g_m \triangleq \sqrt{c_{I,m}^2 + c_{Q,m}^2},$$
$$\hat{g}_m \triangleq \sqrt{\hat{c}_{I,m}^2 + \hat{c}_{Q,m}^2},$$
$$\theta_{c,m} \triangleq \tan^{-1}\left(\frac{c_{Q,m}}{c_{I,m}}\right),$$
$$\text{and } \hat{\theta}_{c,m} \triangleq \tan^{-1}\left(\frac{\hat{c}_{Q,m}}{\hat{c}_{I,m}}\right).$$

α_m and β_m in (8) can alternatively be expressed as

$$\begin{aligned} \alpha_m &\triangleq 1 - \frac{1}{2}\mu_m A_m^2 g_m \hat{g}_m \cos\Delta\theta_{c,m}, \\ \beta_m &\triangleq \frac{1}{2}\mu_m A_m^2 g_m \hat{g}_m \cos\Delta\theta_{c,m} \end{aligned} \tag{9}$$

where $\Delta\theta_{c,m} \triangleq \theta_{c,m} - \hat{\theta}_{c,m}$,

Also, using similarity transformation we can convert

(8) into the transformed domain as

$$\begin{pmatrix} E[\bar{v}_{I,m}(n+1)] \\ E[\bar{v}_{Q,m}(n+1)] \end{pmatrix} = \begin{pmatrix} 1-\lambda_{I,m} & 0 \\ 0 & 1-\lambda_{Q,m} \end{pmatrix} \begin{pmatrix} E[\bar{v}_{I,m}(n+1)] \\ E[\bar{v}_{Q,m}(n+1)] \end{pmatrix} \quad (10)$$

where

$$\lambda_{i,m} = \frac{1}{2}\mu_m A_m^2 g_m \hat{g}_m(\cos\triangle\theta_{c,m} \pm j\sin\triangle\theta_{c,m}) \quad i = I, Q.$$

It should be noted from (10) that since $\lambda_{i,m}$'s are complex values, so are the transformed weight errors. Therefore, we consider the convergence of the magnitude of the transformed error as

$$\rho_{i,m}(n+1) = |1-\lambda_{i,m}|\,\rho_{i,m}(n), \quad i = I, Q \quad (11)$$

where $\rho_{i,m}(n) \triangleq |E[\bar{v}_{I,m}(n)]|$.

We can see from (11) that the magnitude converges exponentially to zero ($i.e.$, $E[\omega_{I,m}(n)]$ to $\omega^*_{i,m}$) under the following condition

$$|1-\lambda_{i,m}| < 1 \qquad \forall, \qquad i = I, Q \quad (12)$$

Squaring both sides of (12) yields

$$1-\mu_m A_m^2 g_m \hat{g}_m \cos\triangle\theta_{c,m} + \frac{1}{4}\mu_m^2 A_m^2 g_m^2 \hat{g}_m < 1$$

$$0 < \mu_m < \frac{4\cos\triangle\theta_{c,m}}{A_m^2 g_m \hat{g}_m} \quad \text{or} \quad 0 < x_{f,m} < 1 \quad (13)$$

where

$$x_{f,m} \triangleq \frac{\mu_m A_m^2 g_m \hat{g}_m}{4\cos\triangle\theta_{c,m}}.$$

The time constant of the exponential convergence is derived from the following:

$$e^{-1/\tau_{i,m}} \cong 1-\frac{1}{\tau_{i,m}}$$
$$= |1-\lambda_{i,m}| \quad \text{for larg } \tau_{i,m}, \; i = I, Q \quad (14)$$

From (11) and (14) we get

$$\tau_{i,m} = \frac{1}{1-\sqrt{1-\mu_m A_m^2 g_m \hat{g}_m \cos\triangle\theta_{c,m} + \frac{1}{4}\mu_m^2 A_m^2 g_m^2 \hat{g}_m^2}}$$
$$= \frac{1}{1-\sqrt{1-4x_{f,m}(1-x_{f,m})\cos^2\triangle\theta_{c,m}}} \quad (15)$$

where $i = I$ and Q.

B. The sum of the squared weight errors

Next we investigate the convergence of the mean-square-error (MSE), $E[e^2(n)]$. Using (2), (6) and (8) we can express the MSE as

$$E[e^2(n)] = \sum_{m=1}^{M} e_m^2(n) + \sigma_\eta^2$$
$$= \frac{1}{2}\sum_{m=1}^{M} A_m^2 g_m^2 e_m(n) + \sigma_\eta^2 \quad (16)$$

where

$$\sigma_\eta^2 \triangleq E[\eta^2(n)],$$
$$\xi_m(n) \triangleq E[v_{I,m}^2(n)] + E[v_{Q,m}^2(n)].$$

It is noted from (16) that the convergence study for the MSE is equivalent to that for the sum of the mean-squared weight errors. Inserting (5), (2) and (6) into (4), squaring and taking expectation of both sides of the result yields

$$\xi_m(n+1) = \gamma_m \xi_m(n) + \delta_m \quad (17)$$

where

$$\gamma_m \triangleq 1-\mu_m A_m^2 g_m \hat{g}_m \cos\triangle\theta_{c,m} + \frac{1}{16}\mu_m^2 A_m^4 g_m^2 \hat{g}_m^2 [9-\cos(2\triangle\theta_{c,m})],$$
$$\delta_m \triangleq \mu_m^2 A_m^2 g_m^2 \hat{g}_m^2 \sigma_\eta^2.$$

Thus, when $|\gamma| < 1$, (17) has the solution as

$$\xi_m(n) = \gamma^n{}_m \xi_m(0) + \frac{1-\gamma_m^n}{1-\gamma_m}\delta_m. \quad (18)$$

Consequently, the convergence condition of the sum of the squared weight errors can be obtained from(18).

$$|\gamma_m| < 1. \quad (19)$$

Solving (19) yields

$$0 < \mu_m < \frac{16\cos\triangle\theta_{c,m}}{A_m^2 g_m \hat{g}_m(9-\cos 2\triangle\theta_{c,m})} \quad \text{or} \quad 0 < x_{m,s} < 1 \quad (20)$$

where

$$x_{m,s} \triangleq \mu_m A_m^2 \frac{g_m \hat{g}_m(9-\cos 2\triangle\theta_{c,m})}{16\cos\triangle\theta_c}.$$

The time constant of the exponential convergence is derived from (18) and (14).

$$\tau_{m,s} = \frac{1}{\mu_m A_m^2 g_m \hat{g}_m (\cos \Delta\theta_{c,m} - \varepsilon)} = \frac{9 - \cos 2\Delta\theta_{c,m}}{16 x_{m,s}(1 - x_{m,s}) \cos^2 \Delta\theta_{c,m}} \tag{21}$$

where

$$\varepsilon \triangleq \frac{1}{16} \mu_m A_m^2 g_m \hat{g}_m (9 - \cos 2\Delta\theta_{c,m}).$$

We can obtain the steady-state value as

$$\xi_m(\infty) = \delta_m / (1 - \gamma_m) = \frac{\mu_m \hat{g}_m \sigma_\eta^2}{g_m(\cos \Delta\theta_{c,m} - \varepsilon)} = \frac{16 x_{m,s} \sigma_\eta^2}{A_m^2 g_m^2 (1 - x_{m,s})(9 - \cos 2\Delta\theta_{c,m})} \tag{22}$$

The results of the convergence analysis are summarized in Table I.

IV. Simulation Results

Results of computer simulation are presented in this section along with those of the theoretical analysis of the Filtered-x LMS algorithm in section III. For convenience, we consider a single sinusoid case. The input signal $x(n)$ and desired signal d(n) are given as

$$x(n) = \sqrt{2}\left\{\cos\left(\frac{240\pi n}{2000} + \varphi_1\right) + \cos\left(\frac{480\pi n}{2000} + \varphi_2\right)\right\},$$
$$d(n) = \sum_{m=1}^{2} \{\omega_{I,m}^{*} x_{I,m}(n) + \omega_{Q,m}^{*} x_{Q,m}(n)\}$$
$$= 0.6 x_{I,1}(n) - 0.1 x_{Q,1}(n) + 0.3 x_{I,1}(n) - 0.3 x_{Q,1}(n). \tag{23}$$

where the sinusoidal frequencies of the sinusoid and sampling were 120 Hz, 240Hz and 2KHz, respectively. The variance of the zero-mean measurement noise was 0.001. The initial weight values were all zero. The simulation results were obtained by ensemble averaging 1000 independent runs. The convergence constant μ was 0.002.

Fig. 5 show the learning curves obtained from the analysis and simulation of the summed variance of weight errors when the phase errors $|\Delta\theta_c|$ are (1) 0°, (2) 45°, (3) 60°, and (4)75°.

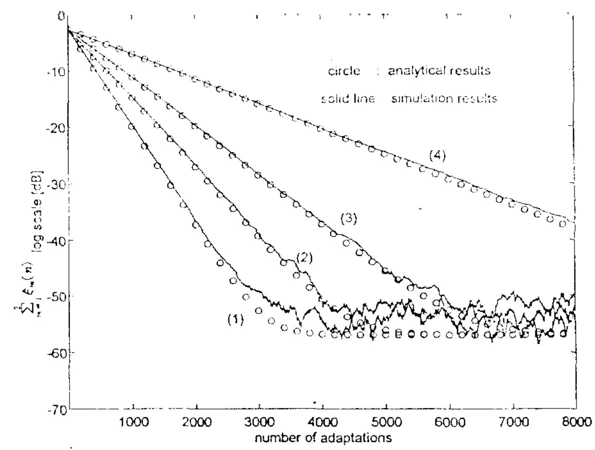


Figure 5. Learning curves of the summed variance of the weight errors.
(1) $|\Delta\theta_c| = 0°$, (2) $|\Delta\theta_c| = 45°$, (3) $|\Delta\theta_c| = 60°$, (4) $|\Delta\theta_c| = 75°$.

It can be seen from the figure that the theoretical results for the summed variance behavior agree well with the simulation result. We can also see that, the convertgence speed is the fastest for $|\Delta\theta_c| = 0°$.

V. Conclusion

We can easily see from Table I that the effects of parameter estimation inaccuracy on the convergence be-

Table 1. The results of the convergence analysis of the Filtered-x LMS algorithm.

	mean of weight error (Magnitude) $\chi_{m,f} \doteq \frac{\mu_m A_m^2 g_m \hat{g}_m}{4\cos\Delta\theta_{c,m}}$	Summed variance of weight errors $\chi_{m,s} \doteq \frac{\mu_m A_m^2 g_m \hat{g}_m [9 - \cos(2\Delta\theta_c)]}{16\cos\Delta\theta_{c,m}}$
Stability condition	$0 < \mu_m < \frac{4\cos\Delta\theta_{c,m}}{A_m^2 g_m \hat{g}_m}$ or $0 < \chi_{m,f} < 1$	$0 < \mu_m < \frac{16\cos\Delta\theta_{c,m}}{A_m^2 g_m \hat{g}_m [9 - \cos(2\Delta\theta_{c,m})]}$ or $0 < \chi_{m,s} < 1$
Time constant	$\frac{1}{1 - \sqrt{1 - 4\chi_{m,f}(1 - \chi_{m,f})\cos^2\Delta\theta_{c,m}}}$	$\frac{9 - \cos(2\Delta\theta_{c,m})}{16\chi_{m,s}(1 - \chi_{m,s})\cos^2\Delta\theta_{c,m}}$
Steady-state value	0	$\frac{16\chi_{m,s}\sigma_\eta^2}{A_m^2 g_m^2 (1 - \chi_{m,s})[9 - \cos(2\Delta\theta_{c,m})]}$

havior of the filtered-x LMS algorithm are characterized by two distinct components:Phase estimation error $\Delta\theta_\epsilon$ and estimated magnitude $\hat{g}$. In particular, $|\Delta\theta_\epsilon|$ should be less than 90° for convergence. It is, however, noted that once x_f or x_s is selected, the convergence turns out to be determined only by $\Delta\theta_\epsilon$. The convergence speed is the fastest for $x=1/2$ regardless of $\Delta\theta_\epsilon$. When $\Delta\theta_\epsilon=0$ and $\hat{g} = g$ the convergence result becomes the same as the LMS case. In conclusion, the convergence of the Filtered-x LMS algorithm is shown to be strongly affected by the accuracy of the phase response estimate.

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