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〈논 문〉

Vibration Analysis of Three Layer Sandwich Beam

3층 샌드위치보의 진동해석

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ABSTRACT

This paper proposes a new technique to formulate the finite element model of a sandwich beam by using GHM (Golla-Hughes-McTavish) internal auxiliary coordinates to account for frequency dependence. Through the use of auxiliary coordinates, the equation of motion of undamped mass and stiffness matrix form is extended to encompass viscoelastic damping matrix. However, this methods all suffer from an increase in order of the final finite element model which is undesirable in many applications. Here we propose to combine the GHM method with model reduction techniques to remove the objection of increased model order.

요 약

본 연구에서는 진동수에 종속된 GHM (Golla-Hughes-McTavish) 내부보조좌표를 사용하여, 3층 샌드위치보의 유한요소모델을 정식화하는 새로운 기법을 제안하였다. 내부보조좌표를 3층 샌드위치보에 사용하면, 비감쇠질량과 강성행렬의 운동방정식은 점성감쇠행렬이 포함되므로써 행렬의 요소들이 복잡하게 확장되어 진다. 따라서 이 방법은 실제의 많은 응용에 있어서 바람직하지 못한 유한요소모델의 행렬요소들의 증가에 따른 많은 단점을 갖게 된다. 따라서, 본 논문에서는 행렬요소들의 증가에 따른 여러 단점들을 제거하기 위하여, 행렬요소 감소방법을 GHM방정식과 합성된 운동방정식을 유도하는 새로운 방법을 제안한다.

Nomenclature

A_b, A_v, A_c	Cross sectional area of the beam, viscoelastic, and constraining layer
$[A], [B], [C]$	Original system, input and output matrices
$[\hat{A}][\hat{B}][\hat{C}]$	Internal balanced system, input and output matrices
$[\tilde{A}][\tilde{B}][\tilde{C}]$	Intermediate system, input and output matrices
b	Width of the sandwich beam
C	External damping coefficient
E_b, E_v, E_c	Young's modulus for the beam, viscoelastic, and constraining layer
E_0	Equilibrium value of the modulus

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E^{*n}	:	nth complex modulus in the Laplace domain
f	:	Externally applied forcing function
h	:	Thickness of the viscoelastic beam
$h(s)$	:	Material modulus function
I	:	Identity matrix
I_b, I_v, I_c	:	Area moment of inertia of the beam, viscoelastic, and constraining layer
$\bar{K}^n$	:	The contribution of nth stiffness modulus
K	:	Stiffness matrix of the element
l	:	Element length of plate
M	:	Mass matrix of the element
P	:	Linear transformation of the system
q	:	Elastic degree of freedom
w	:	Transverse deflection of viscoelastic beam
W_c	:	Controllability grammian
W_o	:	Observability grammian
z	:	Dissipation coordinates

Greek Symbols

$\hat{\alpha}$	:	Weighting constant on dissipation coordinate
$\hat{\zeta}$	:	Damping ratio of dissipation coordinate
$\hat{\omega}$	:	Natural frequency of dissipation coordinate
$[\sigma]$	:	Stress matrix
σ_i	:	Singular values of the grammians
$[\varepsilon]$	:	Strain matrix
ρ	:	Mass density of viscoelastic beam
Λ_c	:	Eigenvalues

1. Introduction

Energy dissipation of flexible structures plays a crucial role in the performance of a wide variety of engineering systems such as light space vehicles, automobiles, and communications satellite etc. Recently, many researchers show that passive surface treatments have been extensively utilized, as a simple and reliable means, for damping out the vibration of a wide variety of flexible structures. Such surface treatments rely in their operation on the use of viscoelastic damping layers which are bonded to the vibrating structures with a constrained configuration. However, accurate mathematical modeling of complex structures with viscoelastic materials is difficult because the measured dynamic properties of viscoelastic material are sensitive to frequency, temperature, type of deformation, and sometimes amplitude. Several researches have presented successful methods of modeling the effects of viscoelastic damping mechanisms which introduce hysterisis.

The Modal Strain Energy method (MSE) proposed by Rogers, et. al.[20] for estimating the damping of composite structures from the measured damping of constituent materials has been very popular. However, dynamic models based on the MSE method and "modal damping" are deficient in the some aspects which is especially important in structural control applications for high-fidelity dynamic models needed. Fractional derivative model developed by Bagley and Torvik[1] has been shown to closely fit experimental data over significant range of frequency. The strength of this method lies in its efficient way of modeling of viscoelastic material behavior. However, the use of numerical methods in performing the transforms to describe the frequency dependent mechanical properties of the viscoelastic materials is cumbersome. While dissatisfaction with available techniques has motivated several alternative lines of research, one pursued by Golla, Hughes, and McTavish[5,15] that has led to finite elements for modeling linear viscoelastic structures is most closely related to the subject Augmenting Thermodynamic Fields (ATF) method proposed by Lesieutre and his coworkers[12~14]. Like ATF, GHM employs additional coordinates to more acutely model damping. The GHM uses a second order physical coordinate system and the Leisutre approach uses a first order state space method. Both are superior to the Modal Strain Energy (MSE) as they capture the transient response characteristics of the material. These two complex approaches are able to account for damping effects over a range of frequencies, complex mode behavior, transient responses and both time and frequency domain modeling.

Inman[7] applied the GHM approach to simple beams and Banks and Inman[2] provide alternate time domain method for modeling hysterisis.

Here we examine the Golla-Hughes-McTavish (GHM) finite element modeling method[15] which represents a FEM model of viscoelastic structures by introducing a dummy variable and modeling the hysteresis with a transfer function. The main advantage in using the GHM method over the AFT method is that GHM keeps the system in second order, which makes it more readably adaptable to finite element analysis. However, if the damping is modeled using the GHM method, the structural dynamic parameters are at least doubled. This increases the calculation time in using finite element model. Thus we are motivated to consider the effects of model reduction techniques on the GHM model method to remove the objection of increased size. In particular we examine Guyan reduction and Internal Balancing reduction methods. Guyan reduction[6,9] removes some of the insignificant physical coordinates, thereby producing a model that has smaller mass and stiffness matrices but is not usually applied to systems with damping as it is based on static considerations. On the other hand the internal balancing method[17], it is possible to express the reduced model in terms of a subset of the original states with additional coordinate transformation. The model is then converted to state space form, and is reduced again by the internal balancing method. However, in the internal balanced coordinate system, the states of the reduced model have no apparent resemblance to those of the original model. Yae[22] produced a update version of the internal balancing method through another coordinate transformation derived from the states that are deleted during reduction. This reduced model is expressed by a subset of the original states.

This paper proposes a new technique to formulate the finite element model of a sandwich beam by using GHM (Golla-Hughes-McTavish) internal auxiliary coordinates to account for frequency dependence. Through the use of auxiliary coordinates, the equation of motion of undamped mass and stiffness matrix form is extended to encompass viscoelastic damping matrix. However, this methods all suffer from an increase in order of the final finite element model which is undesirable in many applications. Here we propose to combine the GHM method with model reduction techniques to remove the objection of increased model order.

2. Formulation of a Viscoelastic Beam

Following Inman[7,8] the equation of transverse vibration may be derived by a straight forward extension of the usual Euler-Bernoulli beam equation using the stress-strain relation given by

$$\sigma(x,t) = E\varepsilon(x,t) + \int_0^t g(t-s)ds \tag{1}$$

where $\sigma(x,t)$ is the stress, $x \in (o,l)$ is the distance along the beam, $t>0$ is the time, $\varepsilon(x,t)$ is the strain, E is the elastic modulus, and the kernel $g(t-s)$ describes the hysteresis as developed by Christensen[3], for example. With this modification the transverse vibration of a viscoelastic beam satisfies the following equation:

$$\rho A \partial w_{tt}(x,t) + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w(x,t)}{\partial x^2} + \int_0^t g(t-\tau) w_{xx}(x,\tau)d\tau \right] + c\frac{\partial w(x,t)}{\partial t} = f(x,t) \tag{2}$$

where $w(x,t)$ is the transverse displacement, ρ is the mass density, E is the elastic modulus, I is the moment of inertia, c is an external (air) damping coefficient, $f(x,t)$ is an externally applied load and the subscripts x and t denote partial differentiation. The initial conditions are assumed to all be zero to simplify the presentation. The external force is assumed to be an impulse.

The Golla-Hughes-McTavish method requires the representation of the material modulus function as a series of (damped) mini-oscillator terms or internal variables. GHM has been developed for direct incorporation into the finite element method. The material complex modulus can be written in the Laplace domain in the form

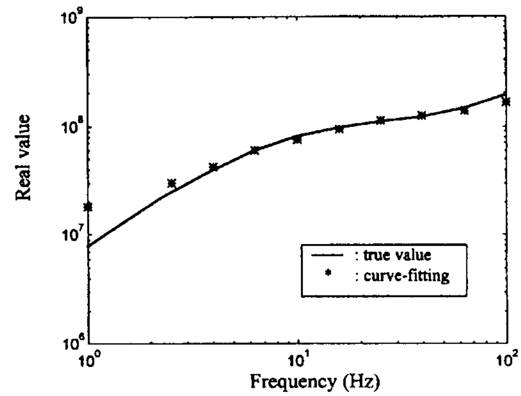
$$E^*(s) = E_0(1+h(s)) = E_0\left(1+\sum_{n=1}^{k}\hat{\alpha}_n \frac{s^2+2\hat{\zeta}_n\hat{\omega}_n s}{s^2+2\hat{\zeta}_n\hat{\omega}_n s+\hat{\omega}_n^2}\right) \quad (3)$$

where E_0 is the equilibrium value of the modulus, and s is the Laplace operator. The hatted terms are free variables for curve fitting to the complex data for a particular material at a given temperature. Also, the number of expansion terms, k, may be modified to represent the high or low frequency dependence of the complex terms. The expansion of $h(s)$ represents the material modulus as a series of the mini oscillator (second order equation) terms[15]. The real and imaginary parts of Young's modulus for DYAD-606 (SOUNDCOAT) at temperature 25°C are plotted in Fig. 1. These are compared to the corresponding curve fit values, indicated by the *, using two mini oscillators terms [$n=2$ in Eq. (3)].

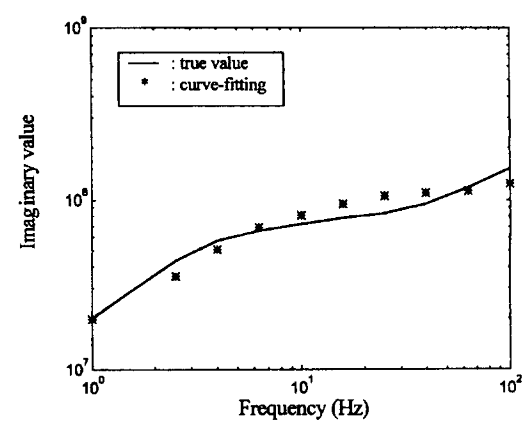
The constrained optimization algorithm is used MATLAB's constr command, a Sequential Quadratic Programming (SQP), to find the best choice of hatted mini-oscillator terms. It finds the constrained minimum of a objective function of the hatted terms α, ζ and ω, starting at initial estimates. That is, the material complex modulus in Eq. (3) is minimized with satisfying the constraints to find the optimum hatted terms of α, ζ and ω. This is mathematically stated as "minimize $E^*(s)$ subject to the constraints $g_i(s) \le 0$". The constraints is the hatted terms α_i in this case, this will give more accurate results since they are bounded by some limits. A method of solving optimization problems with inequality constraints is to use the Hessian of the Lagrangian function,

$$L(x,\lambda) = E^*(s) + \sum_{i=1}^{m}\lambda_i g_i(x)$$

This method defines a new vector λ_i, called the vector of Langrange multipliers, and the constraints are added directly to the objective function. The new cost function, $L(x,\lambda)$, is then minimized through iteration using a quasi-Newton updating method. This is then used to generate the Quadratic Programming (QP) sub-problem based on a quadratic approximation. The solution of the QP sub-problem is used to



a) Real value of DYAD-606



b) Imaginary value of DYAD-606

Fig. 1 Two-term GHM modulus function form a search direction for a line search procedure. The minimum along the line formed from this search direction is generally approximated using this search procedure or by a polynomial method involving interpolation or extrapolation. The problem is to find a new iterate x_{k+1} of the form, $x_{k+1} = x_k + \alpha^* d$ where x_k denotes the current iterate, d the search direction obtained by an appropriate method and α^* is a scalar step length parameter which is the distance to the minimum. When the objective function is minimized after some iterations, the approximate values of the hatted terms of α, ζ and ω are compared with the real values by giving the tolerance between them. When the tolerance is satisfied at a limit by which is given as a small value, the optimum hatted two mini-oscillator terms are obtained to be E_0=1.18E6, $\hat{\alpha}$ =[87.5 263.13], $\hat{\zeta}$ =[1344.6 129.6], $\hat{\omega}$ =[14945.5 39999.9]. The important effect of frequency is that the Young's modulus always increases with increasing frequency as shown Fig. 1.

Now the transverse vibration of a viscoelastic beam [Eq. (2)] can be rewritten as the following Laplace domain form:

$$\left\{\rho A s^2 + cs + E_0 I[1+h(s)]\frac{\partial^4}{\partial x^4}\right\} w(s) = f(s) \tag{4}$$

with
$$h(s) = \sum_{n=1}^{k} \hat{\alpha}_n \frac{s^2 + 2\hat{\zeta}_n \hat{\omega}_n s}{s^2 + 2\hat{\zeta}_n \hat{\omega}_n s + \hat{\omega}_n^2} \tag{5}$$

3. GHM Finite Element Model

The equation of motion for a finite element in the Laplace domain is

$$M\left(s^2 x(s) - s x_0 - s\dot{x}_0\right) + K(s)x(s) = f(s) \tag{6}$$

where M is the mass matrix, $x(s)$ is the displacement vector, x_0 and $\dot{x}_0$ are the initial displacement and initial velocity vectors respectively, $f(s)$ is the forcing function and

$$K(s) = \left(E^{*1}(s)\overline{K}^1 + E^{*2}(s)\overline{K}^2 + \ldots + E^{*n}(s)\overline{K}^n\right) \tag{7}$$

Here the variable $E^{*n}(s)$ represents the *nth* complex modulus in the Laplace domain and $\overline{K}^n$ is the contribution of the *nth* modulus to the stiffness matrix. Considering a single modulus model with multi expansion terms, and neglecting initial conditions,

$$Ms^2 x(s) + E_0\left(1 + \sum_{k=1}^{n} \hat{\alpha}_k \frac{s^2 + 2\hat{\zeta}_k \hat{\omega}_k s}{s^2 + 2\hat{\zeta}_k \hat{\omega}_k s + \hat{\omega}_k^2}\right)\overline{K} x(s) = f(s) \tag{8}$$

then the following Laplace domain element equation of motion is equivalent to Eq. (8) as follows:

$$M_v \begin{bmatrix} q \\ z_1 \\ \vdots \\ z_n \end{bmatrix} + D_v \begin{bmatrix} q \\ z_1 \\ \vdots \\ z_n \end{bmatrix} + K_v \begin{bmatrix} q \\ z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

where the GHM viscoelastic element matrices are

$$M_v = \begin{bmatrix} M & 0 & \cdots & 0 \\ 0 & \frac{\hat{\alpha}_1}{\hat{\omega}_1^2} E_0 \overline{K}^1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\hat{\alpha}_n}{\hat{\omega}_n^2} E_0 \overline{K}^n \end{bmatrix},$$

$$D_v = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{2\hat{\alpha}_1 \hat{\zeta}_1}{\hat{\omega}_1} E_0 \overline{K}^1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{2\hat{\alpha}_n \hat{\zeta}_n}{\hat{\omega}_n} E_0 \overline{K}^n \end{bmatrix},$$

$$K_v = \begin{bmatrix} \overline{K} E_0\left(1 + \sum_{k=1}^{n} \alpha_k\right) & -\hat{\alpha}_1 E_0 \overline{K}^1 & \cdots & -\hat{\alpha}_n E_0 \overline{K}^n \\ -\hat{\alpha}_1 E_0 \overline{K}^1 & -\hat{\alpha}_1 E_0 \overline{K}^1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\hat{\alpha}_n E_0 \overline{K}^n & 0 & 0 & -\hat{\alpha}_n E_0 \overline{K}^n \end{bmatrix} \tag{10}$$

There is no need to have additional dissipation coordinates for rigid body modes. It is also not favorable to add additional zero eigenvalues to the K matrix for each previously existing zero eigenvalues. This can be solved and a linear second order symmetric form can be obtained by applying singular value decomposition to the matrix $\overline{K}$ and separating the zero from non-zero eigenvalues. Using an elastic diagonal matrix of the nonzero eigenvalues Λ_e and corresponding eigenvectors R_e of the modulus-factored stiffness matrix and substituting

$$z(s) = R_e^T \hat{z}(s) \tag{11}$$

we arrive at the final form of the element viscoelastic matrices:

$$M_v = \begin{bmatrix} M & 0 & \cdots & 0 \\ 0 & \frac{\hat{\alpha}_1}{\hat{\omega}_1^2} E_0 \Lambda_{e1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\hat{\alpha}_n}{\hat{\omega}_n^2} E_0 \Lambda_{e_n} \end{bmatrix},$$

$$D_v = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{2\hat{\alpha}_1 \hat{\zeta}_1}{\hat{\omega}_1} E_0 \Lambda_{e1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{2\hat{\alpha}_n \hat{\zeta}_n}{\hat{\omega}_n} E_0 \Lambda_{en} \end{bmatrix},$$

$$K_v = \begin{bmatrix} \overline{K} E_0 \left(1 + \sum_{k=1}^{n} \alpha_k\right) & -\hat{\alpha}_1 E_0 R_{e1} \Lambda_{e1} & \cdots & -\hat{\alpha}_n E_0 R_{en} \Lambda_{en} \\ -\hat{\alpha}_1 E_0 \Lambda_{e1} R_{e\,1}^T & -\hat{\alpha}_1 E_0 \Lambda_{e1} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\hat{\alpha}_n E_0 \Lambda_{en} R_{e\,n}^T & 0 & 0 & -\hat{\alpha}_n E_0 \Lambda_{en} \end{bmatrix} \tag{12}$$

More complete details are given in the papers by Hughes et. al.[5,15]. The size of the additional coordinates z(s) depends on the nature of the material and on how many terms are needed to fit the particular material's loss modulus data.

4. Finite Element Model for Three Layer Sandwich Beam

In most existing finite element codes, the hysteretic damping is used to analyze a sandwich beam. It can be seen the peak response occurs at a frequency lower than the undamped natural frequency for the viscous case, but in the case of the hysteretic damping the resonant peak always occurs at the undamped natural frequency[18]. This means the hysteretic damping is independent of the value of loss factors. As a result, a finite element model is developed to describe the dynamics of three layer sandwich beam with viscous damping using the GHM method in this section.

Figure 2 indicates the theoretical parameters of a three layer sandwich beam in which the viscoelastic layer is sandwiched between two aluminum beam layers where t_b, t_v, and t_c, are the thickness of the base beam, the viscoelastic material, and constraining layer respectively. It is assumed that the shear strains in the constraining layer and in the base beam are negligible. The transverse displacement w of all points on any cross section of the sandwich beam are considered to be equal. Furthermore, the constraining layer and base beam are assumed to be elastic and dissipate no energy. In addition, each layer is considered to be perfectly bonded together and the thickness of bond is not considered in this model.

Figure 3 shows a finite element model of a sandwich beam. The element has two nodes with five degrees of freedom per node to describe the longitudinal displacements u_b, u_v, and u_c of the base beam, viscoelastic, and constraining layer respectively, the transverse deflection w, and the slopes $\partial w/\partial x$ of the deflection line. The important properties of one dimensional elements are axial and flexural deformations.

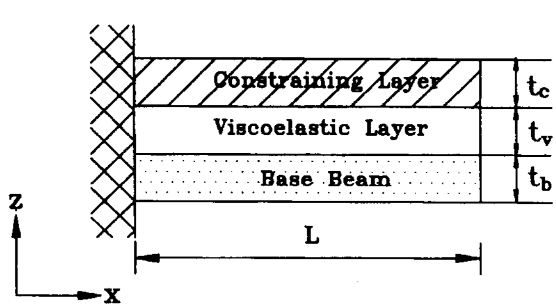


Fig. 2 A sandwich beam with geometric parameters

The axial displacements of a sandwich beam can be expressed as the following form of polynomials in the local variables x and y:

$$u_b = a_1 x + a_2 \, , \; u_v = a_3 x + a_4 \, , \; u_c = a_5 x + a_6 \tag{13}$$

Also, the transverse displacement w is expressed by the following polynomial:

$$w = a_7 x^3 + a_8 x^2 + a_9 x + a_{10} \tag{14}$$

The constants $\{a_1, a_2, \cdots, a_{10}\}$ are determined in terms of the ten components of the nodal deflection vector $\{q\}$ which is given by:

$$q^T = \left[u_{b1} \; u_{v1} \; u_{c1} \; w_1 \; \frac{dw_1}{dx} \; u_{b2} \; u_{v2} \; u_{c2} \; w_2 \; \frac{dw_2}{dx} \right] \tag{15}$$

where the subscript 1 refer to displacements on the left side of the element and the subscript 2 refers to displacements on the right side of the element. Therefore, the deflection $\{U\} = \{u_b(x), u_v(x), u_c(x), w(x)\}^T$ at any location (x, y) inside the i-th element can be determined from:

$$\{U\} = [\{N_1\}, \{N_2\}, \{N_3\}, \{N_4\}]^T \{q\} = [N]\{q\} \tag{16}$$

where q is given by Eq. (15) and $[N(x)]$ is a matrix of the spatial interpolation functions corresponding to $u_b(x)$, $u_v(x)$, $u_c(x)$, $w(x)$ as follows:

$$[N]^T = \begin{bmatrix} 1-\frac{x}{L} & 0 & 0 & 0 \\ 0 & 1-\frac{x}{L} & 0 & 0 \\ 0 & 0 & 1-\frac{x}{L} & 0 \\ 0 & 0 & 0 & 1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3} \\ 0 & 0 & 0 & x-\frac{2x^2}{L}+\frac{x^3}{L^2} \\ \frac{x}{L} & 0 & 0 & 0 \\ 0 & \frac{x}{L} & 0 & 0 \\ 0 & 0 & \frac{x}{L} & 0 \\ 0 & 0 & 0 & \frac{3x^2}{L^2}-\frac{2x^3}{L^3} \\ 0 & 0 & 0 & -\frac{x^2}{L}+\frac{x^3}{L^2} \end{bmatrix} \tag{17}$$

Using the strain-displacement relationships, the strain vector is obtained as follows:

$$\{\varepsilon\} = [d][N]\{q\} \tag{18}$$

where $[d]$ is the linear differential operator. $[L]$ and $[N]$ can be combined such that $[B] = [d][N]$. Therefore

$$\{\varepsilon\} = [B]\{q\} \tag{19}$$

Applying the principle of virtual work to a finite element of a sandwich beam system yields

$$\delta U_e = \delta W_e \tag{20}$$

where δU_e and δW_e are the virtual strain energy of internal stresses and virtual work of external actions.
or

$$\delta\{q\}^T \iiint_V [B]^T [D][B] dV \{q\} = \delta\{q\}^T \iiint_V [N]^T \{b(x,y,t)\} dV + \delta\{q\}^T \iiint_V [N]^T \{F\} dV - \delta\{q\}^T \iiint_V \rho [N]^T [N] dV \{\ddot{q}\} \tag{21}$$

where $\delta\{q\}$ is the virtual nodal deflection vector, $[D]$ is the rigidity operator matrix, $\{b(x,y,t)\}$ is the external body force vector and $\{F\}$ is the external force vector. Also, ρ and V denote the density and volume of element. Factoring out $\sigma\{q\}$, then the total stiffness matrix and the consistent mass matrix of a sandwich beam are defined as:

$$[K]_T = \iiint_V [B]^T [E][B] dV$$
$$[M]_T = \iiint_V \rho [N]^T [N] dV \tag{22}$$

Presenting the strain displacement matrix of each layer, the extension and bending stiffness matrices of each layer are obtained. The total stiffness matrix of the

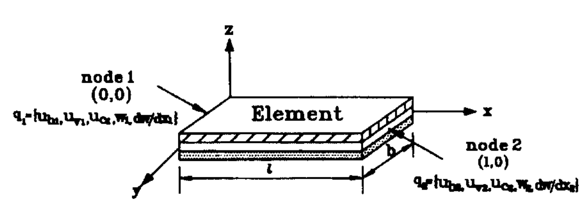


Fig. 3 Finite element model of a sandwich beam

sandwich beam is given by:

$$[K]_T = \sum_{i=b,c}([K_e]_i + [K_b]_i) + K_v \tag{23}$$

where $[K_e]_i$ denote the extension stiffness of i layer and $[K_b]_i$ denote the bending stiffness of i layer (subscript b refers to beam and c refers to constraining layer). These stiffness matrices can be written as follows:

$$[K_e]_i = E_i A_i \int_0^l [B_e]_i^T [B_e]_i dx\,,$$

$$[K_b]_i = E_i I_i \int_0^l [B_b]_i^T [B_b]_i dx\,. \tag{24}$$

$[K_v]$ denote the stiffness matrix of the viscoelastic layer which is modified using GHM method derived in the previous section

$$[K]_v = [K_e]_v + [K_b]_v \tag{25}$$

with

$$[K_e]_v = \left(E_v A_v \int_0^l [B_e]_v^T [B_e]_v dx\right)\left(1 + \sum_{k=1}^{n} \alpha_k\right),$$

$$[K_b]_v = \left(E_v I_v \int_0^l [B_b]_v^T [B_b]_v dx\right)\left(1 + \sum_{k=1}^{n} \alpha_k\right) \tag{26}$$

where A_b, A_v, and A_c is cross sectional area of the beam, viscoelastic layer, and constraining layer respectively and E_b, E_v, and E_c are Young's modulus for the beam, viscoelastic layer, and constraining layer respectively and I_b, I_v, and I_c are area moment of inertia of the beam, viscoelastic layer, and constraining layer respectively. Also, the strain displacement matrices $[B_e]_i$ and $[B_b]_i$ are given by:

$$[B_e] = [N_i]_{,x}\ [B_b] = -[N_4]_{,xx} \tag{27}$$

where the subscripts $,x$ and $,xx$ denote spatial first and second order differentiation with respect to x respectively.

The total mass matrix of the sandwich beam is given by:

$$[M]_T = \sum_{i=b,v,c}([M_e]_i + [M_b]_i) \tag{28}$$

where $[M_e]_i$ and $[M_b]_i$ denote the mass matrices due to extension and bending of the i layer. These mass matrices can be written by:

$$[M_e]_i = \rho_i A_i \int_0^l ([N_i]^T [N_i]) dx\,,$$

$$[M_b]_i = \rho_i A_i \int_0^l ([N_4]^T [N_4]) dx \tag{29}$$

where ρ_i represent the density of the i-th layer, respectively.

5. Brief Introduction of Reduction Methods

The model reduction methods are briefly introduced here as they have been developed in two different disciplines: finite element analysis and control theory. In the case of a condensation process or static reduction, such as Guyan reduction, some of the insignificant physical coordinates are removed such as rotational degrees of freedom at a node point[6]. On the other hand, in the internal balancing method of control theory, it is not directly possible to express the reduced model in terms of a subset of the original states. Hence an additional coordinate transformation is introduced and applied [21,22].

5.1 Guyan Reduction Method

The eigenvalue problem of mass-stiffness system can be partitioned as follows:

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \lambda \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right)\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

where the "2" degree of freedom $\{q_2\}$ are to be retained if it is critical (excited externally or controlled) for performance of the system and the "1" degree of freedom $\{q_1\}$ are to be removed by condensation. Thus, we temporarily ignore all mass but $[M_{22}]$, in order to obtain a relation between $\{q_1\}$ and $\{q_2\}$. From the upper partition of the Eq. (30),

$$\{q_1\} = -[K_{11}]^{-1}[K_{12}]\{q_2\} = [\bar{T}]\{q_2\} \tag{31}$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} \bar{T} \\ I \end{bmatrix}\{q_2\} = [T]\{q_2\} \tag{32}$$

where $[I]$ is an identity matrix. Substitution Eq. (32) into Eq. (30) and premultiplication by $[T]^T$ yields the condensed eigenvalue problem

$$\left([\bar{K}]-\lambda[\bar{M}]\right)\{q_2\}=\{0\} \tag{33}$$

where the reduced matrices are

$$[\bar{K}]=[T^T][K][T] \text{ and } [\bar{M}]=[T^T][M][T]. \tag{34}$$

For a damped system with a damping matrix $[D]$ and external loads $\{f\}$, a straight forward application of the above transformations yields that the condensed damping matrix and external loads are given by

$$[\bar{D}]=[T^T][D][T] \quad \text{and} \quad \{\bar{f}\}=[T^T]\{f\}. \tag{35}$$

5.2 Internal Balancing Method

Here the original equations of motion are taken to be

$$[M]\{\ddot{q}\}+[D]\{\dot{q}\}+[K]\{q\}=\{f\} \tag{36}$$

where M, D, and K are the $n \times n$ real, symmetric, positive definite matrices. The $n \times 1$ vector $\{q\}$ is the displacement vector. The overdots denote differentiation with respect to time. The $n \times 1$ vector $\{f\}$ represents the external forces applied to the structure. Eq. (36) is converted into the state space form such that

$$\dot{x}(t)=Ax(t)+Bu(t),$$
$$y(t)=Cx(t) \tag{37}$$

where $A=\begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix}$, $B=\begin{bmatrix} M^{-1}B_1 \\ 0 \end{bmatrix}$,

$$C=[C_1 \quad C_2] \tag{38}$$

and will be denoted by (A, B, C, x). It is assumed that the system (A, B, C, x) is controllable, observable and asymptotically stable. The idea used in this method is to reduce the order of a given model based on deleting those coordinates, or modes, that are the least controllable and observable. To implement this idea a measure of the degree of controllability and observability is needed. The useful measure is provided for asymptotically stable systems of the form given by Eq. (37) by defining the controllability and observability grammians, denoted by W_c and W_o, respectively and defined by

$$W_c=\int_0^\infty e^{At}BB^Te^{A^Tt}dt, \quad W_o=\int_0^\infty e^{A^Tt}C^TCe^{At}dt \tag{39}$$

where e^{At} is the state transition matrix of the open-loop system $\dot{x}(t)=Ax(t)$. W_c and W_o are the unique symmetric positive definite matrices which satisfy the Lyapunov matrix equations:

$$AW_c+W_cA^T=-BB^T, \; A^TW_o+W_oA=-C^TC \tag{40}$$

for asymptotically stable systems. Moore[17] has shown that there exists a coordinate system in which two grammians are equal and diagonal. Such a system is then called balanced. Let the matrix P denote a linear transformation of the system into the balanced coordinate system, which when applied to Eq. (37) yields the equivalent system

$$\dot{\hat{x}}(t)=\hat{A}\hat{x}(t)+\hat{B}u(t),$$
$$y(t)=\hat{C}\hat{x}(t). \tag{41}$$

These two balanced systems are related by

$$\hat{x}=P^{-1}x, \; \hat{A}=P^{-1}AP, \; \hat{B}=P^{-1}B, \; \hat{C}=CP \tag{42}$$

In addition, the two grammians are equal in this coordinate system:

$$\hat{W}_c=\hat{W}_o=diag[\sigma_1,\sigma_2,\cdots\sigma_{2n}] \tag{43}$$

where $\hat{W}_c=P^{-1}W_cP$, $\hat{W}_o=P^{-1}W_oP$ and σ_i's denote the singular values of the grammians. Applying the idea of singular values as a measure of rank deficiency to the controllability and observability grammians yields a systematic model reduction method. The matrix P that transforms the original system (A, B, C, x) into a balanced system $(\hat{A}, \hat{B}, \hat{C}, \hat{x})$ can be obtained using the following algorithm:

(a) The reduced order model can be

calculated by first calculating an intermediate transformation matrix P_1 based on the controll-ability grammians. Solving for W_c and find eigenvalues Λ_c and eigenvectors V_c such that $V_c^T W_c V_c = \Lambda_c$. Then define $P_1 = V_c \Lambda_c^{-1/2}$.

(b) The coordinate transformation $x = P_1 \tilde{x}$ yields an intermediate system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{x})$ calculated by $\tilde{A} = P_1^{-1} A P_1$, $\tilde{B} = P_1^{-1} B$, $\tilde{C} = C P_1$.

(c) To complete the balancing algorithm, these intermediate equations are balanced with respect to $\tilde{W}_o$. Solving for $\tilde{W}_o$ and find eigenvalues $\tilde{\Lambda}_o$ and eigenvectors $\tilde{V}_o$ such that $\tilde{V}_o^T \tilde{W}_o \tilde{V}_o = \tilde{\Lambda}_o$. Let $P_2 = \tilde{V}_o \Lambda_o^{-1/4}$.

(d) Another coordinate transformation $\tilde{x} = P_2 \hat{x}$ yields the desired balanced system $(\hat{A}, \hat{B}, \hat{C}, \hat{x})$:

$$\hat{A} = P_2^{-1} \tilde{A} P_2 = P_2^{-1}\left(P_1^{-1} A P_1\right) P_2, \quad \hat{B} = P_2^{-1} \tilde{B} = P_2^{-1} P_1^{-1} B,$$
$$\hat{C} = \tilde{C} P_2 = C P_1 P_2. \tag{44}$$

The transformation P is given by P_1 and P_2 as $P = P_1 P_2$. Using Equation (44), the balanced system $(\hat{A}, \hat{B}, \hat{C}, \hat{x})$ can be partitioned as

$$\begin{bmatrix} \dot{\hat{x}}_r \\ \dot{\hat{x}}_d \end{bmatrix} = \begin{bmatrix} \hat{A}_r & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_r \\ \hat{x}_d \end{bmatrix} + \begin{bmatrix} \hat{B}_r \\ \hat{B}_d \end{bmatrix} u, \quad y = \begin{bmatrix} \hat{C}_r & \hat{C}_d \end{bmatrix} \begin{bmatrix} \hat{x}_r \\ \hat{x}_d \end{bmatrix} \tag{45}$$

Deleting the k least controllable and observable states, i.e., $\hat{x}_d = 0$, yields

$$\dot{\hat{x}}_r(t) = \hat{A}_r \hat{x}_r(t) + \hat{B}_r u(t), \quad y_r(t) = \hat{C}_r \hat{x}_r(t) \tag{46}$$

a reduced model of order $(n-k)$. This produces the balanced system which can now be reduced by looking at the singular values of the balanced system and throwing away those coordinates which have relatively small singular values. This leaves a smaller order system with essentially the same dynamics as the full order system.

5.3 Modified Internal Balancing Method

Unfortunately the coordinates left after a balanced reduction are not a subset of the finite element nodal coordinates. Thus this is not simple to relate back to the original finite element model as is the case in Guyan reduction. This problem is solved by Yae[22] who introduced an additional coordinate transformation to produce a reduced order model in a coordinate system consisting of a subset of the original finite element coordinate system. For structural control and measurement applications, it is desirable to provide the designer with a clear, physical relationship between the original vector q in Eq. (36) and the reduced state vector $\hat{x}_r$. Such a relationship is found by using the fact that the balanced states are linear combinations of the original states. Symbolically this is written as:

$$\hat{x}_1 = \sum_{j=1}^{2n} c_{1j} x_j, \cdots, \hat{x}_{2n-k} = \sum_{j=1}^{2n} c_{(2n-k)j} x_j,$$
$$\hat{x}_{2n-(k-1)} = \sum_{j=1}^{2n} c_{(2n-k+1)j} x_j \to 0, \cdots, \hat{x}_{2n} = \sum_{j=1}^{2n} c_{2nj} x_j \to 0, \tag{47}$$

where c_{ij}'s are the coefficients in the linear combinations of $\{x_1, x_2, \cdots, x_{2n}\}$. Here the last k states are set to zero because they represent the least significant states in the balanced system[17]. Setting each of these summations equal to zero is equivalent to imposing k constraints on the original $2n$ states, which means that the modal reduction imposes dependencies on k number of the original states. In other words, one can construct a reduced order model by selecting $(2n-k)$ states out of the original $2n$ states. If the $(2n-k)$ selected states from the original system are denoted by $x_r = [x_{j1}\, x_{j2} \cdots x_{j2-k}]^T$ and the $(2n-k)$ states of the balanced system by $\hat{x}_r = [\hat{x}_1\ \hat{x}_2 \cdots \hat{x}_{2-k}]^T$, then the states in $\hat{x}_r$ are linear combinations of the states in x_r. Thus there exists a new transformation matrix P_r of order $(2n-k) \times (2n-k)$ such that $x_r = P_r \hat{x}_r$. The above constraints and the resulting transformation allow the designer to specify which nodes of the model to be retained in the model reduction. In the following it is shown that the matrix P_r consists of certain rows and columns of the original transformation matrix P, and that there is a systematic way of constructing P_r from P.

(a) Select the state variables to be retained from $\{x_1, x_2, \cdots, x_{2n-k}\}$. Let the indices of those selected be $\{j_1, \cdots, j_{2n-k}\}$ rows from P.

(b) The transformation matrix P_r can be obtained by selecting first $2n-k$ columns and $\{j_1, \cdots, j_{2n-k}\}$ rows from P.

(c) The reduced order system (A_r, B_r, C_r, x_r)

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t), \quad y_r(t) = C_r x_r(t) \tag{48}$$

is now expressed in terms of a subset x_r of the original state vector x, where

$$A_r = P_r \hat{A}_r P_r^{-1}, \; B_r = P_r \hat{B}_r, \; C_r = \hat{C}_r P_r^{-1}. \tag{49}$$

Thus we have provided a scheme that has the best feature of each reduction method: Here we are able to specify which coordinate to keep and provide a dynamically based reduction schemes. This will allow to remove the extended coordinate added to the system to build a damping matrix.

6. Numerical Examples

A numerical example is presented in order to demonstrate the use of GHM method in the finite element analysis of sandwich beam [Fig. 3] through the three reduction methods as described above. All the calculations are performed on IBM PC using MATLAB for windows by The Math Works, Inc. Table 1 shows the physical and geometrical parameters of the aluminum sheet as a base beam and a constraining layer and the DYAD-606 (SOUNDCOAT) as viscoelastic layer. The sandwich beam is equally divided into four elements so that it has four active node points. Each node point has three degrees of freedom for axial displacement, one degree of freedom for translational displacement, one degree of freedom for rotational displacement, and six additional viscoelastic auxiliary degrees of freedom. Hence, one element of the sandwich beam has sixteen degrees of freedom per node. The performance of viscoelastic material is affected by the temperature and frequency which, in turn, influence the shear modulus and the loss factor as shown in Fig. 1.

Table 1 Physical and geometrical properties of the sandwich beam

Layer	Thickness (m)	Length (m)	Young's modulus (Pa)	Width (m)	Density (kg/m³)	Posson's ratio
Aluminum	4.064E-4	0.125	7.1E10	0.01	2700	0.33
DYAD-606	5.08E-5	0.125	*	0.01	1105	0.49

* Depending on temperature and frequency

From the above section results, the equation of motion of the sandwich beam included viscoelastic layer may finally convert into a GHM finite element form with two mini-oscillator terms as follows:

$$M_e \ddot{q} + D_e \dot{q} + K_e q = f_e \tag{50}$$

where the finite element matrices are

$$M_v = \begin{bmatrix} M_T & 0 & 0 & 0 & 0 \\ 0 & \frac{\hat{\alpha}_1}{\hat{\omega}_1^2} E_0 \Lambda_{e1} & 0 & 0 & 0 \\ 0 & 0 & \frac{\hat{\alpha}_2}{\hat{\omega}_2^2} E_0 \Lambda_{e1} & 0 & 0 \\ 0 & 0 & 0 & \frac{\hat{\alpha}_1}{\hat{\omega}_1^2} E_0 \Lambda_{e2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\hat{\alpha}_2}{\hat{\omega}_2^2} E_0 \Lambda_{e2} \end{bmatrix},$$

$$D_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\hat{\alpha}_1\hat{\zeta}_1}{\hat{\omega}_1} E_0 \Lambda_{e1} & 0 & 0 & 0 \\ 0 & 0 & \frac{2\hat{\alpha}_2\hat{\zeta}_2}{\hat{\omega}_2} E_0 \Lambda_{e1} & 0 & 0 \\ 0 & 0 & 0 & \frac{2\hat{\alpha}_1\hat{\zeta}_1}{\hat{\omega}_1} E_0 \Lambda_{e2} & 0 \\ 0 & 0 & 0 & 0 & \frac{2\hat{\alpha}_2\hat{\zeta}_2}{\hat{\omega}_2} E_0 \Lambda_{e2} \end{bmatrix}$$

$$K_v = \begin{bmatrix} \overline{K}^T E_0\left(1+\sum_{k=1}^{n}\alpha_k\right) & -\hat{\alpha}_1 E_0 R_{e1}\Lambda_{e1} & -\hat{\alpha}_2 E_0 R_{e1}\Lambda_{e1} & -\hat{\alpha}_1 E_0 R_{e2}\Lambda_{e2} & -\hat{\alpha}_1 E_0 R_{e2}\Lambda_{e2} \\ -\hat{\alpha}_1 E_0 \Lambda_{e1} R_{e1}^T & -\hat{\alpha}_1 E_0 \Lambda_{e1} & 0 & 0 & 0 \\ -\hat{\alpha}_2 E_0 \Lambda_{e1} R_{e1}^T & 0 & -\hat{\alpha}_2 E_0 \Lambda_{e1} & 0 & 0 \\ -\hat{\alpha}_1 E_0 \Lambda_{e2} R_{e2}^T & 0 & 0 & -\hat{\alpha}_1 E_0 \Lambda_{e2} & 0 \\ -\hat{\alpha}_2 E_0 \Lambda_{e2} R_{e2}^T & 0 & 0 & 0 & -\hat{\alpha}_1 E_0 \Lambda_{e2} \end{bmatrix} \tag{51}$$

with the force vectors and coordinate

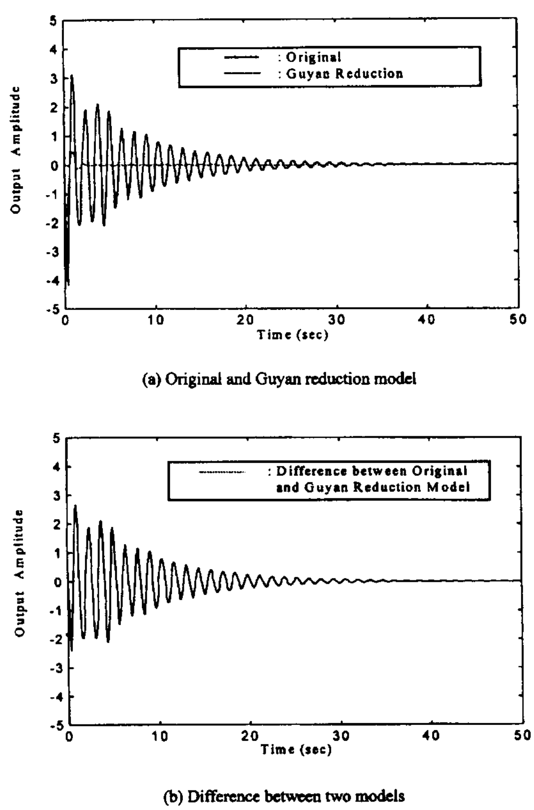
$$f_e = [f \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T,$$
$$q = [x \ \ z_1 \ \ z_2 \ \ z_3 \ \ z_4 \ \ z_5 \ \ z_6]^T. \tag{52}$$

Here the values of the hatted constants α, ζ and ω are obtained from curve fitting of the complex modulus data for the viscoelastic material provided by manufacturer as described before. And the submatrices Λ_{ei} and R_{ei} in GHM viscoelastic element mass, damping and stiffness matrices are found through spectral decomposition of the elastic component matrices [15]:

$$\Lambda_{e1} = \frac{2A}{l}, \quad R_{e1} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix},$$

$$\Lambda_{e2} = \begin{bmatrix} \frac{2I}{l} & 0 \\ 0 & \frac{6I(4+l^2)}{l^3} \end{bmatrix}, \quad R_{e2} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2+l^2/2}} \\ \frac{1}{\sqrt{2}} & \frac{l/2}{\sqrt{2+l^2/2}} \\ 0 & \frac{-1}{\sqrt{2+l^2/2}} \\ -\frac{1}{\sqrt{2}} & \frac{l/2}{\sqrt{2+l^2/2}} \end{bmatrix} \tag{53}$$

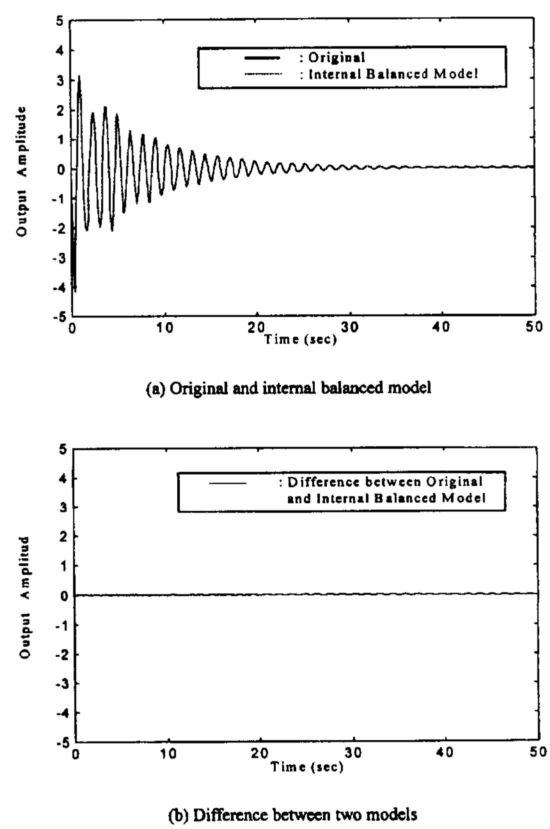
Now, all three procedures discussed in the section 5, Guyan, internally balanced and modified internal balanced reduction methods, are applied to this GHM finite element model of a cantilever sandwich beam. For the purpose of demonstration, the impulse input is placed on the node 2 and the displacement of the tip (node 4) is measured. In Fig. 4(a), the time response curves of the original model and reduced model by Guyan reduction method are plotted. Here in order to save the calculation time, the GHM degree of freedom vector is deleted. The difference is obtained by subtracting the output response of the system in the Guyan reduced model from that of the same output response in the original system as shown the dashed line in Fig. 4(b). A wide nonzero difference is detected in the transient region of the response from the output amplitude difference graph. The Guyan reduction method does not remove the GHM internal



(a) Original and Guyan reduction model

(b) Difference between two models

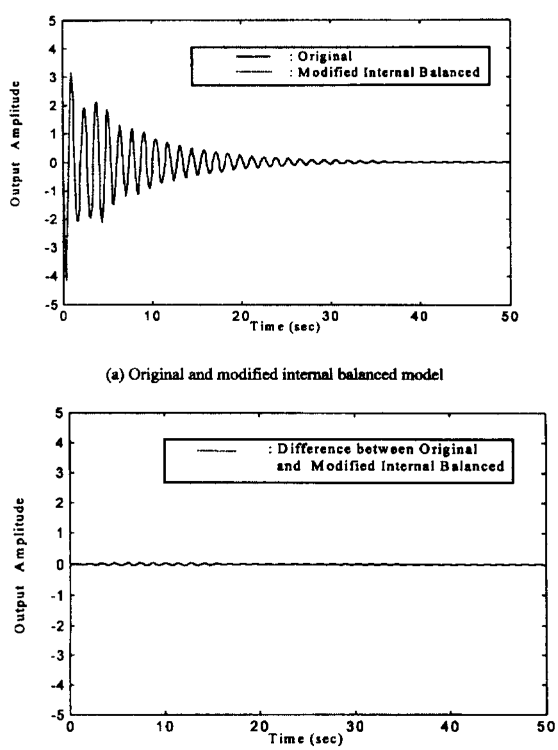
Fig. 4 Time response of output of a sandwich beam



(a) Original and internal balanced model

(b) Difference between two models

Fig. 5 Time response of output of a sandwich beam degrees of freedom properly through this technique as it does not take into account system responses. Next, the time response curves of the original model and the model reduced by internal balancing are plotted together in Fig. 5(a). Here we are able to delete the viscoelastic states, that is, GHM internal variable ones and extension elastic states at each node. The difference between the responses of the output amplitude in original and reduced model by internal balancing method shown by the dashed line of Fig. 5(b) is almost zero. In this case, the difference between the full and reduced system response lies between an upper limit of (0.1) and lower limit of (-0.1). In Fig. 6(a) the responses of the original states and those of the modified internal balanced states are plotted. Here we have also removed viscoelastic states and extension elastic states at each node. Again both responses and their difference are plotted in Fig. 6(b). In Figs. 5~6, it is shown that the differences are nearly zero in comparison to the response of the original state, indicating that the reduced models are indeed a respectable realization of the original system. Note also that Guyan method is not able to remove all of the internal variables as do the two balanced reduction methods. One more thing to be mentioned here, the system response without extension degrees of freedom of viscoelastic layer is almost same as that of with those degrees of freedom through the finite element method.



(a) Original and modified internal balanced model

(b) Difference between two models

Fig. 6 Time response of output of a sandwich beam

7. Conclusion

The technique proposed in this paper is a finite element model of the sandwich beam using GHM method that has been implemented to take viscous damping matrix. Unfortunately, this GHM method generates undesirable internal variables used to account for viscoelastic properties in finite element modeling. Therefore, three popular reduction methods has been introduced. The first method implemented is Guyan condensation method. However, the Guyan reduction method loses its fidelity as the FEM becomes more complex The other methods proposed here eliminate the need to increase the order over that of the original model. Internal variables are put into the model to add viscous damping, then taken back out to provide the original order and coordinates back again. The final method thereby provides a clear, physical relationship between the states in the reduced model and those in the original model. Thus the final reduced model is an excellent representation of the viscoelastic system and the reduced and full models yield similar time responses.

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