State Space Representation of the General Wiener-Hopf Controller

일반 위너-호프 제어기의 상태 공간 표현에 관한 연구

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요 약 : 이 논문에서는 다항식 서로소 인수로 표시되는 위너-호프 제어기의 계산상의 어려움을 극복하기 위하여 상태공간 변수 공식이 개발되었다. 위너-호프 인수 행렬을 이용하여 주어진 다항식 서로소 인수로부터 안정 유리행렬의 서로소 인수를 구하였으며 이 결과를 이용하여 위너-호프 제어기의 공식을 유리행렬의 서로소 인수로 표현한 후 이를 이용하여 상태공간 계수를 구하였다.

Keywords: Wiener-Hopf controller, H_2 design, state-space representation, polynomial coprime fraction, stable rational coprime fraction

I. Introduction

In the past decade, the H_2 design for the generalized plant model has been the interest of many researchers[1][2][3]. An important step in deriving the H_2 optimal controller formula is to describe the plant transfer matrix as a coprime fraction(CF). Two kinds of CFs, polynomial CF and stable rational CF, are widely used. The latter one is convenient to develop state-space representation of the controller formula and hence is popular nowadays. In this case, however, the plant transfer matrix and the controller are usually confined to be proper ones, which excludes many practical industrial applications such as PID controllers. In the contrast, the polynomial CF approach can treat the improper plant and controller cases. Furthermore, the polynomial CF approach is conceptually simple and its algebraic properties have long been studied[4], and hence found its applications in many areas[5][6][7]. The only disadvantage of the polynomial CF approach method seems to lie in its computational difficulty. The aim of this paper is to show that this difficulty of the polynomial CF method in the H_2 design problem can be circumvented when the plant has state-space representation.

Doyle et al[1] presented the H_2 controller formula in the state-space form and hence the plant is limited to be proper. Hunt et al[2] used polynomial methods to derive the optimal H_2 formula. Park and Bongiorno[3] derived the frequency domain Wiener-Hopf solutions to the generalized plant model by using polynomial CF method. It is often not easy to compute the optimal controller transfer matrix in [2][3] because efficient algorithms to calculate spectral factorization and pole-zero cancellation are lacking. However, the problem setting in [3] is far more general than that of [1] in that the improper plants and controllers are not excluded and the exogenous input to the generalized plant can include not only the white noises but also the shape-deterministic functions such as step or ramp reference inputs, which is not the case in [1]. In this paper, state-space representation of the frequency domain solutions in [3] is sought. A similar work was done in [8] for the three-degree-of-freedom configuration model[9]. The work in this paper is distinguished from the earlier one in that state-space parameters of the controllers are systematically obtained by using the technique of rationalizing polynomial CF descriptions.

Throughout the paper, only real rational matrices are considered and the notations G^{-1}, G^T and $Tr\,G$ are used for the inverse, transpose and trace of the matrix G, respectively. The matrix $G_*(s)$ stands for $G^T(-s)$. In the partial fraction expression of $G(s)$, the contributions made by all its finite poles in $Re\ s\leq 0$, $Re\ s>0$ and at $s=\infty$ are denoted by $\{G\}_+$, $\{G\}_-$ and $\{G\}_\infty$, respectively. The notation $\{G\}_s$ implies the strictly proper part of $G(s)$. It is obvious that $\{G\}_s=\{G\}_+ + \{G\}_-$. The order relationship $G(s)\leq O(s^k)$ means that no entry in $G(s)$ grows faster than s^k as $s\to\infty$. The conventional notation $\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right] = C(sI-A)^{-1}B+D$ is used.

II. Preliminaries

In this chapter, the main results of [3] concerning the optimal H_2 controller are summarized. Consider the standard feedback control system in Fig 1, where

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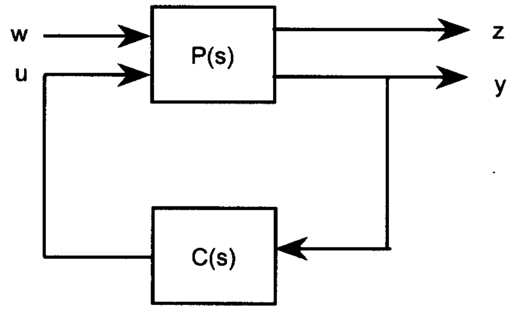

Fig. 1. The standard feedback system model.

$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}, \quad u(s)=C(s)\,y(s)$$

We assume that the exogenous input $w(s)$ is white noise and its power spectral density is the identity matrix. The H_2 optimization problem is to find the controller that minimizes the cost function

$$E=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} T_r[T_{zw*}(s)T_{zw}(s)]ds \qquad (1)$$

under the internal stability constraint where T_{zw} is the transfer matrix from w to z. The problem setting in [3] is far more general than the one above. Here, we sacrifice the generality to develop state-space representation.

Assumption 1 : The plant $P(s)$ is free of hidden poles in $Re\ s\geq 0$ and

$$\Psi_p^+ = \Psi_{p_{22}}^{\ +} \qquad (2)$$

where Ψ_p and $\Psi_{p_{22}}$ denote the characteristic denominators of $P(s)$ and $P_{22}(s)$, respectively and the monic polynomials $\Psi_p^{\ +}$ and $\Psi_{p_{22}}^{\ +}$ absorb all zeros of Ψ_p and $\Psi_{p_{22}}$ in $Re\ s\geq 0$, respectively. Let

$$P_{22}=A^{-1}(s)B(s)=B_1(s)A_1^{-1}(s) \qquad (3)$$

where (A,B) is a left and (B_1,A_1) is a right coprime pair of polynomial matrices. There always exist polynomial matrices X(s), Y(s), $X_1(s)$ and $Y_1(s)$ such that

$$\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}=I \qquad (4)$$

$$A_1Y_1=YA \qquad (5)$$

with $detX\cdot detX_1\neq 0$. Under the assumption 1, the set of all stabilizing controllers is characterized by

$$C(s)=-(Y(s)+A_1(s)K(s))(X(s)-B_1(s)K(s))^{-1} \qquad (6)$$

where $K(s)$ is any real rational matrix analytic in $Re\ s\geq 0$.Consider Wiener-Hopf spectral factorizations of

$$A_{1*}P_{12*}P_{12}A_1=\Lambda_*\Lambda \qquad (7)$$

and

$$AP_{21}P_{21*}A_*=\Omega\Omega_* \qquad (8)$$

Assumption 2 : The matrices $P_{12*}P_{12}$ and $P_{21}P_{21*}$ are full rank para-Hermitian and nonnegative definite on the finite $s=jw$ axis. The inverses Λ^{-1} and Ω^{-1} are analytic on the finite $s=jw$ axis.

Assumption 3 : The rational matrix $P_{11}(s)$ is strictly proper. That is,

$$P_{11}(s)\leq O(s^{-1}) \qquad (9)$$

Assumption 4 : The difference $T_r(P_{11}P_{11*})-T_r(\Gamma\Gamma_*)$ is analytic on the finite jw-axis where

$$\Gamma(s)=\Lambda_*^{-1}A_{1*}P_{12*}P_{11}P_{21*}A_*\Omega_*^{-1} \qquad (10)$$

Assumption 5 : $\Lambda^{-1}\Gamma\Omega^{-1}-A_1^{-1}Y$ is analytic on the finite jw-axis.

Assumption 6 : The following order conditions are satisfied;

$$(P_{21}P_{21*})^{-1}\leq O(s^0),\ (P_{12*}P_{12})^{-1}\leq O(s^0) \qquad (11)$$

$$P_{22}(s)\leq O(s^0) \qquad (12)$$

Theorem 1 : 1) The set of all stabilizing controllers that yield finite cost E is generated by the formula in (6) with

$$K(s)=-\Lambda^{-1}(\{\Lambda A_1^{-1}Y\Omega\}_\infty - \{\Gamma-\Lambda A^{-1}Y\Omega\}_+ + Z)\Omega^{-1} \qquad (13)$$

where $Z(s)$ is an arbitrary real rational matrix $\leq O(s^{-1})$ which is analytic in $Re\ s\geq 0$.

2) The stabilizing $C(s)$ that minimizes E is given by (6) with

$$K(s)=-\Lambda^{-1}(\{\Lambda A_1^{-1}Y\Omega\}_\infty - \{\Gamma-\Lambda A^{-1}Y\Omega\}_+)\Omega^{-1} \qquad (14)$$

3) Let the minimum cost E be denoted by $\hat{E}$. Then for any allowed $Z(s)$

$$E=\hat{E}+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} T_r(ZZ_*)ds \geq \hat{E} \qquad (15)$$

4) The following order relations hold for $C(s)$ generated by $K(s)$ in (13);

$$C(s)\leq O(s^{-1}) \qquad (16)$$

$$R(s)=C(I-P_{22}C)^{-1}\leq O(s^{-1}) \qquad (17)$$

The optimal formula in (13) requires the numerical calculations such as spectral factorization, partial fraction and pole-zero cancellation, which often make it difficult to compute the optimal controller transfer matrix directly in the frequency domain. In the next chapter, state-space representation of the controller transfer matrix in (13) is sought to overcome the computational difficulties.

III. Main results

The Wiener-Hopf controllers in (6) and (13) are described in terms of polynomial doubly coprime fraction (DCF) in (3) and (4). In the section 1 of this chapter, stable rational DCF representation of the Wiener-Hopf controllers will be sought as a intermediary to develop state-space representation. In section 2, state-space representation of the controllers in (6) and (13) are obtained.

1. Wiener-Hopf controllers in terms of stable rational doubly coprime fractions

We first construct a stable rational DCF for $P_{22}(s)$ from the polynomial DCF in (3) and (4).

Multiplying $\begin{bmatrix} \Lambda & -\{\Lambda A_1^{-1}Y\Omega\}_{\infty}\Omega^{-1} \\ 0 & \Omega^{-1} \end{bmatrix}$ on the left and $\begin{bmatrix} \Lambda^{-1} & \Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \\ 0 & \Omega \end{bmatrix}$ on the right to the equation (4) yields

$$\begin{bmatrix} \tilde{X}_1 & \tilde{Y}_1 \\ -\tilde{B} & \tilde{A} \end{bmatrix}\begin{bmatrix} \tilde{A}_1 & -\tilde{Y} \\ \tilde{B}_1 & \tilde{X} \end{bmatrix} = I \qquad (18)$$

where

$$\tilde{X}_1 = \Lambda X_1 + \{\Lambda A_1^{-1}Y\Omega\}_{\infty}\Omega^{-1}B$$

$$\tilde{Y}_1 = \Lambda Y_1 - \{\Lambda A_1^{-1}Y\Omega\}_{\infty}\Omega^{-1}A \qquad (19.1)$$

$$\tilde{X} = B_1\Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_{\infty} + X\Omega,$$

$$\tilde{Y} = Y\Omega - A_1\Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \qquad (19.2)$$

$$\tilde{B} = \Omega^{-1}B, \quad \tilde{A} = \Omega^{-1}A,$$

$$\tilde{A}_1 = A_1\Lambda^{-1}, \quad \tilde{B}_1 = B_1\Lambda^{-1} \qquad (19.3)$$

It should be noticed that

$$\tilde{A}^{-1}\tilde{B} = \tilde{B}_1\,\tilde{A}_1^{-1} = P_{22}(s) \qquad (20)$$

Lemma 1 : Suppose that Assumptions 2 and 6 are satisfied. Then, the expression in the equation (18) is a stable rational DCF for $P_{22}(s)$ and the eight matrices in (19) are irrelevant of a particular choice of polynomial DCF in (3) and (4).

Proof : Irrelevant property of the eight matrices will be shown first. Consider a polynomial DCF

$$\begin{bmatrix} \overline{X_1} & \overline{Y_1} \\ -\overline{B} & \overline{A} \end{bmatrix}\begin{bmatrix} \overline{A_1} & -\overline{Y} \\ \overline{B_1} & \overline{X} \end{bmatrix} = I \qquad (21)$$

which is different from the one in (4). The relation between two DCFs can be characterized by

$$\overline{X_1} = V^{-1}X_1, \quad \overline{Y_1} = V^{-1}Y_1,$$

$$\overline{A_1} = A_1V, \quad \overline{B_1} = B_1V \qquad (22.1)$$

$$\overline{X} = XU^{-1}, \quad \overline{Y} = YU^{-1},$$

$$\overline{A} = UA, \quad \overline{B} = UB \qquad (22.2)$$

where U and V are unimodular polynomial matrices. The Wiener-Hopf factors in (7) and (8) for $\overline{A_1}$, $\overline{B_1}$, $\overline{A}$ and $\overline{B}$ become ΛV and $U\Omega$, respectively, and it is a trivial practice to show that the eight matrices in (19) made by the DCF in (21) are not changed. Next, it will be shown that the eight matrices are all proper and analytic in $Re\ s \geq 0$. Obviously, the eight matrices are all analytic in $Re\ s \geq 0$ because Λ^{-1} and Ω^{-1} are analytic in $Re\ s \geq 0$ by assumption. The matrices $\tilde{A}_1$ and $\tilde{A}$ are proper. In fact, it follows from (7) and (8) that

$$(P_{12*}P_{12})^{-1} = A_1\Lambda^{-1}\Lambda_*^{-1}A_{1*} \qquad (23)$$

and

$$(P_{21}P_{21*})^{-1} = A_*\Omega_*^{-1}\Omega^{-1}A \qquad (24)$$

By the order conditions in (11), we can conclude that

$$A_1\Lambda^{-1} \leq O(s^0), \quad \Omega^{-1}A \leq O(s^0) \qquad (25)$$

The properness of $\tilde{B}_1$ and $\tilde{B}$ is obvious because $\tilde{B}_1 = P_{22}\tilde{A}_1$ and $\tilde{B} = \tilde{A}P_{22}$ where P_{22} is proper by assumption. As for $\tilde{X}_1$ in (19.1), applying the identity

$$\{\Lambda A_1^{-1}Y\Omega\}_{\infty} = \Lambda A_1^{-1}Y\Omega - \{\Lambda A_1^{-1}Y\Omega\}_s \qquad (26)$$

to the equation in (19.1) yields

$$\begin{aligned}\tilde{X}_1 &= \Lambda X_1 + \Lambda A_1^{-1}YB - \{\Lambda A_1^{-1}Y\Omega\}_s\Omega^{-1}B \\ &= \Lambda X_1 + \Lambda Y_1A^{-1}B - \{\Lambda A_1^{-1}Y\Omega\}_s\Omega^{-1}B \\ &= \Lambda(X_1 + Y_1B_1A_1^{-1}) - \{\Lambda A_1^{-1}Y\Omega\}_s\Omega^{-1}B \\ &= \Lambda A_1^{-1} - \{\Lambda A_1^{-1}Y\Omega\}_s\Omega^{-1}B \qquad (27)\end{aligned}$$

which is proper. In a similar way, we can show that $\tilde{Y}_1, \tilde{X}$ and $\tilde{Y}$ are proper and this completes the proof. Now, it is possible to describe the controller in (6) and (13) in terms of the stable rational DCF. Invoking the stable rational MFD, it follows from (6) and (13) that

$$\begin{aligned}C(s) &= -[\,Y - A_1\Lambda^{-1}(\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \\ &\qquad -\{\Gamma - \Lambda A_1^{-1}Y\Omega\}_{+} + Z)\Omega^{-1}] \\ &\quad\cdot [\,X + B_1\Lambda^{-1}(\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \\ &\qquad -\{\Gamma - \Lambda A_1^{-1}Y\Omega\}_{+} + Z)\Omega^{-1}]^{-1} \\ &= -[\,Y\Omega - A_1\Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \\ &\qquad + A_1\Lambda^{-1}(-Z + \{\Gamma - \Lambda A_1^{-1}Y\Omega\}_{+})] \\ &\quad\cdot [\,X\Omega + B_1\Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_{\infty} \\ &\qquad - B_1\Lambda^{-1}(-Z + \{\Lambda A_1^{-1}Y\Omega + \Gamma\}_{+})]^{-1} \\ &= -[\,\tilde{Y} + \tilde{A}_1(-Z + \{\Gamma - \Lambda A_1^{-1}Y\Omega\}_{+})] \\ &\quad\cdot [\,\tilde{X} - \tilde{B}_1(-Z + \{\Gamma - \Lambda A_1^{-1}Y\Omega\}_{+})]^{-1} \qquad (28) \blacksquare\end{aligned}$$

2. State-space representation of the Wiener-Hopf controller

Suppose that the plant $P(s)$ in Fig. 1 is described by the internal description

$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}=\left[\begin{array}{c|cc} F & G_1 & G_2 \\ \hline H_1 & 0 & J_{12} \\ H_2 & J_{21} & J_{22} \end{array}\right] \quad (29)$$

and the state-space parameters satisfy that

i) (F, G_2, H_2) is controllable and observable,

ii) J_{12} and J_{21} have column and row rank, respectively and

iii) $\begin{bmatrix} F-j\omega I & G_2 \\ H_1 & J_{12} \end{bmatrix}$ has full column rank for all w and $\begin{bmatrix} F-j\omega I & G_1 \\ H_2 & J_{21} \end{bmatrix}$ has full row rank for all w.

The three assumptions above are standard ones in H_2 or H_∞ problems [1]. In the expression in (29), it should be assured that the matrix F includes all modes of the all subsystems that consist of the generalized plant $P(s)$. Assumption i) is sufficient to the internal stabilizability condition in (2). Assumption ii) is always satisfied in correctly posed optimization problems. Assumption iii) is closely related to the assumption that Λ^{-1} and Ω^{-1} are jw-analytic in Assumption 2. The aim of this section is to show that the Wiener-Hopf controller formulas in (28) can be represented by state-space parameters under the standard conditions in i), ii), iii) and (29). We first find state-space representation of the stable rational DCF developed in section 3.1.

Lemma 2 : Let

$$(sI-F)^{-1}G_2=B_2(s)A_1^{-1}(s) \quad (30)$$

be a right coprime fractional description. Then $\Lambda(s)$ in (7) and $\bar{A}_1(s)=A_1(s)\Lambda^{-1}(s)$ are obtained by the formulas

$$\Lambda(s)=R_1^{\frac{1}{2}}(A_1(s)+K_1B_2(s)) \quad (31)$$

and

$$\begin{aligned}\bar{A}_1(s) &= A_1(s)\Lambda^{-1}(s) \\ &= [I+K_1(sI-F)^{-1}G_2]^{-1}R_1^{-\frac{1}{2}} \\ &= [I-K_1(sI-F+G_2K_1)^{-1}G_2]R_1^{-\frac{1}{2}} \quad (32)\end{aligned}$$

where

$$R_1=J_{12}^TJ_{12} \quad (33)$$

$$K_1=R_1^{-1}(J_{12}{}^TH_1+G_2{}^TM_1) \quad (34)$$

and M_1 is the stabilizing solution of the ARE

$$\begin{aligned}&(F-G_2R_1^{-1}J_{12}{}^TH_1)^TM_1 \\ &+M_1(F-G_2R_1^{-1}J_{12}{}^TH_1)-M_1G_2R_1^{-1}G_2^TM_1 \\ &+H_1^T(I-J_{12}R_1^{-1}J_{12}{}^T)H_1=0 \quad (35)\end{aligned}$$

The inverse $\Lambda^{-1}(s)$ is analytic on the finite $jw-$axis.

Proof : It should be noted that we can take the denominator matrices of right coprime fractions of $P_{22}(s)$ and $(sI-F)^{-1}G_2$ to be the same because (F, G_2, H_2) is minimal. Now a little algebra from (35) yields

$$\begin{aligned}P_{12*}(s)P_{12}(s)&=[K_1(sI-F)^{-1}G_2+I]_*J_{12}^TJ_{12} \\ &\cdot[K_1(sI-F)^{-1}G_2+I] \quad (36)\end{aligned}$$

and multiplying A_{1*} on the left and A_1 on the right yields

$$\begin{aligned}&A_{1*}(s)P_{12*}(s)P_{12}(s)A_1(s)= \\ &[A_1(s)+K_1B_2(s)]_*J_{12}^TJ_{12}[A_1(s)+K_1B_2(s)] \quad (37)\end{aligned}$$

It can be readily shown that $(A_1(s)+K_1B_2(s))^{-1}$ is analytic in $Re\ s\geq 0$. In fact, $det(K_1(sI-F)^{-1}G_2+I)=det(sI-F+G_2K_1)\,/\,det(sI-F)$ and at the same time $det(K_1(sI-F)^{-1}G_2+I)=det(A_1(s)+K_1B_2(s))\,/\,detA_1(s)$. Since $\eta det(sI-F)=det\ A_1(s)$, η a constant, it follows that $det(A_1(s)+K_1B_2(s))=\eta\, det(sI-F+G_2K_1)$ which is strict Hurwitz. Hence, it is clear that $(A_1(s)+K_1B_2(s))^{-1}$ is analytic in $Re\ s\geq 0$ and $(J_{12}^TJ_{12})^{\frac{1}{2}}(A_1(s)+K_1B_2(s))$ is a Wiener-Hopf factor $\Lambda(s)$ and it is obvious that $\Lambda^{-1}(s)$ is analytic on the finite $s=jw$ axis. The formula (32) is from (30) and (31). ■

In a similar way, we can obtain the following lemma and its proof is omitted.

Lemma 3 : Let

$$H_2(sI-F)^{-1}=A^{-1}(s)B_3(s) \quad (38)$$

be a left coprime fractional description. Then $\Omega(s)$ in (8) and $\bar{A}(s)=\Omega(s)^{-1}A(s)$ are obtained by the formulas

$$\Omega(s)=(A(s)+B_3(s)K_2)R_2^{\frac{1}{2}} \quad (39)$$

and

$$\begin{aligned}\bar{A}(s)&=\Omega^{-1}(s)A(s)=R_2^{-\frac{1}{2}}[I+H_2(sI-F)^{-1}K_2]^{-1} \\ &=R_2^{-\frac{1}{2}}[I-H_2(sI-F+K_2H_2)^{-1}K_2] \quad (40)\end{aligned}$$

where

$$R_2=J_{21}J_{21}^T \quad (41)$$

$$K_2=(M_2H_2^T+G_1J_{21}^T)R_2^{-1} \quad (42)$$

and M_2 is the stabilizing solution of the ARE

$$\begin{aligned}&(F-G_1J_{21}^TR_2^{-1}H_2)M_2 \\ &+M_2(F-G_1J_{21}^TR_2^{-1}H_2)^T-M_2H_2^TR_2^{-1}H_2M_2 \\ &+G_1(I-J_{21}^TR_2^{-1}J_{21})G_1^T=0 \quad (43)\end{aligned}$$

The matrix $\Omega^{-1}(s)$ is analytic on the finite $s=jw$ axis.

It should be remarked that the assumption iii) is required to guarantee the existence of the stabilizing solutions of the AREs in (36) and (43). The following lemma is needed to find state-space representation of the stable rational DCF in (19).

Lemma 4 :

$$\{\Lambda A_1^{-1} Y \Omega\}_s = R_1^{\frac{1}{2}} K_1 (sI-F)^{-1} K_2 R_2^{\frac{1}{2}} \quad (44)$$

Proof : For the proof, see Appendix.

Once state-space parameters of $\Omega^{-1}A$, $A_1\Lambda^{-1}$, and $\{\Lambda A_1^{-1}Y\Omega\}_s$ are obtained, finding state-space parameters of the eight matrices in (19) is trivial. State-space parameters of $\tilde{B}=\Omega^{-1}B$ and $\tilde{B}_1 = B_1\Lambda^{-1}$ are easily obtained by using the identities $\Omega^{-1}B = \Omega^{-1}AP_{22}$ and $B_1\Lambda^{-1} = P_{22}A_1\Lambda^{-1}$. As for $\tilde{X}_1$, $\tilde{Y}_1$, $\tilde{X}$ and $\tilde{Y}$, the identity in (27) is used. After straight algebra, we obtain the following formulas.

Lemma 5 : State-space parameters of the stable rational DCF in (18) and (19) are given by

$$\begin{bmatrix} \tilde{X}_1(s) & \tilde{Y}_1(s) \\ -\tilde{B}(s) & \tilde{A}(s) \end{bmatrix} = \left[\begin{array}{c|cc} F-K_2H_2 & G_2-K_2J_{22} & K_2 \\ \hline R_1^{\frac{1}{2}}K_1 & R_1^{\frac{1}{2}} & 0 \\ -R_2^{-\frac{1}{2}}H_2 & -R_2^{-\frac{1}{2}}J_{22} & R_2^{-\frac{1}{2}} \end{array}\right] \quad (45)$$

and

$$\begin{bmatrix} \tilde{A}_1(s) & -\tilde{Y}(s) \\ \tilde{B}_1(s) & \tilde{X}(s) \end{bmatrix} = \left[\begin{array}{c|cc} F-G_2K_1 & G_2R_1^{-\frac{1}{2}} & K_2R_2^{\frac{1}{2}} \\ \hline -K_1 & R_1^{-\frac{1}{2}} & 0 \\ H_2-J_{22}K_1 & J_{22}R_1^{-\frac{1}{2}} & R_2^{\frac{1}{2}} \end{array}\right] \quad (46)$$

It should be remarked that the above state-space parameters have been derived via the Wiener-Hopf factors in (7) and (8) and hence the above parameters yield the identities

$$\tilde{A}_{1*}P_{12*}P_{12}\tilde{A}_1 = I \quad (47)$$

and

$$\tilde{A}P_{21}P_{21*}\tilde{A}_* = I \quad (48)$$

That is, the matrices $P_{12}\tilde{A}_1$ and $\tilde{A}P_{21}$ are inner and coinner, respectively, and these properties play an important role in developing H_2 and H_∞ theories. Now we are ready to present the main theorems.

Theorem 2 : Suppose that the plant $P(s)$ in Fig. 1 has the internal description in (29) with the assumptions i), ii) and iii). Then

1) The assumptions 1 through 6 in chapter II are all satisfied.

2) $\{\Gamma - \Lambda A_1^{-1} Y \Omega\}_+ = 0 \quad (49)$

3) The set of all stabilizing controllers that yield finite cost E in (1) is generated by the formula

$$C(s) = (-\tilde{Y}(s) + \tilde{A}_1(s)Z(s)) \cdot (X(s) + \tilde{B}_1(s)Z(s))^{-1} \quad (50)$$

where $Z(s)$ is any arbitrary real rational matrix $\le O(s^{-1})$ which is analytic in $Re\ s \ge 0$.

4) The optimal controller $\hat{C}(s)$ that minimizes E is given by

$$\hat{C}(s) = -\tilde{Y}(s)\ \tilde{X}^{-1}(s) \quad (51)$$

which corresponds to the choice $Z(s) = 0$.

5) Let the minimum cost E be denoted by $\hat{E}$. Then for any allowed $Z(s)$,

$$E = \hat{E} + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} T_r(ZZ_*)ds \ge \hat{E} \quad (52)$$

Proof : For the proof, see Appendix.

To describe state-space parameters of the controller in (50), we define some notations. For given matrices $V(s)=\begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s)\end{bmatrix}$ and $M(s)=\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s)\end{bmatrix}$, let us denote the homographic transformation of $\phi(s)$ with respect to $V(s)$ as

$$V(s) \square \phi(s) = (V_{11}\phi + V_{12})(V_{21}\phi + V_{22})^{-1} \quad (53)$$

and denote the linear fractional transformation (LFT) of $\phi(s)$ with respect to $M(s)$ as

$$M(s) \circ \phi(s) = M_{11} + M_{12}\phi(I - M_{22}\phi)^{-1}M_{21} \quad (54)$$

A homographic transformation of $\phi(s)$ can be converted to a LFT of $\phi(s)$ by the relationship $V \square \phi = M \circ \phi$

$$\text{where} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix} = \begin{bmatrix} V_{12}V_{22}^{-1} & V_{11} - V_{12}V_{22}^{-1}V_{21} \\ V_{22}^{-1} & -V_{22}^{-1}V_{21} \end{bmatrix} \quad (55)$$

When state-space parameters of $V(s)$ are given by

$$V(s) = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22}\end{bmatrix} = \left[\begin{array}{c|cc} F & G_1 & G_2 \\ \hline H_1 & J_{11} & J_{12} \\ H_2 & J_{21} & J_{22} \end{array}\right], \quad (56)$$

state-space parameters of $M(s)$ in (55) are given by

$$M(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix} = \left[\begin{array}{c|cc} F-G_2J_{22}^{-1}H_2 & G_2J_{22}^{-1} & G_1-G_2J_{22}^{-1}J_{21} \\ \hline H_1-J_{12}J_{22}^{-1}H_2 & J_{12}J_{22}^{-1} & J_{11}-J_{12}J_{22}^{-1}J_{21} \\ -J_{22}^{-1}H_2 & J_{22}^{-1} & -J_{22}^{-1}J_{21} \end{array}\right] \quad (57)$$

Now, using the notation in (53), we can write from

(50) that

$$C(s) = \begin{bmatrix} \widehat{A}_1 & -\widehat{Y} \\ \widehat{B}_1 & \widehat{X} \end{bmatrix} \circ Z \tag{58}$$

which is a homographic transformation form. From an implementation point of view, LFT form is often more convenient. Applying the formula in (55) to (58) yields the LFT form for $C(s)$ where state-space parameters are direct form (46) and (57).

Theorem 3 : The controllers in (50) can be re-written as

$$C(s) = L(s) \circ Z \tag{59}$$

where

$$L(s) = \begin{bmatrix} -\widehat{Y}\widehat{X}^{-1} & \widehat{A}_1 + \widehat{Y}\widehat{X}^{-1}\widehat{B}_1 \\ \widehat{X}^{-1} & -\widehat{X}^{-1}\widehat{B}_1 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} F - G_2K_1 - K_2H_2 + K_2J_{22}K_1 & K_2 & (G_2 - K_2J_{22})R_1^{-\frac{1}{2}} \\ \hline -K_1 & 0 & R_1^{-\frac{1}{2}} \\ -R_1^{-\frac{1}{2}}(H_2 - J_{22}K_1) & R_2^{-\frac{1}{2}} & -R_2^{-\frac{1}{2}}J_{22}R_1^{-\frac{1}{2}} \end{array}\right] \tag{60}$$

Finally, it should be remarked that the controllers in (59) can be implemented as in Fig. 2 and it is free of unstable hidden modes.

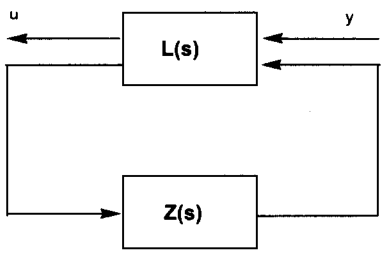

Fig. 2. Implementation of the controller.

IV. Conclusion

Computational difficulty of the Wiener-Hopf controller comes from the fact that it is described in terms of a polynomial doubly coprime fraction (DCF). In this paper, this computational difficulty is circumvented by developing state-space representation of the Wiener-Hopf controller. State-space parameters of the controller are obtained in two steps. Firstly, a stable rational DCF is constructed from the polynomial DCF of the given plant transfer matrix and state-space parameters of the state rational DCF are obtained. Secondly, the Wiener-Hopf controller formulas in terms of the polynomial DCF are transformed into the formulas in terms of the stable rational DCF constructed in the previous step and then the state-space parameters of the stable rational DCF are used to obtain state-space representation of the controller formulas. It is shown that the resulting state-space solutions are equivalent to the ones in [1].

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Appendix

Proof of Lemma 4 : From (32), (39), (30), and (5),

$$\begin{aligned} &\Lambda A_1^{-1} Y \Omega \\ &= R^{\frac{1}{2}}(I + K_1(sI-F)^{-1}G_2)Y(A + B_3K_2)R_2^{\frac{1}{2}} \\ &= R_1^{\frac{1}{2}} Y(A + B_3K_2)R_2^{\frac{1}{2}} + R_1^{\frac{1}{2}}K_1B_2Y_1R_2^{\frac{1}{2}} \\ &\quad + R_1^{\frac{1}{2}}K_1(sI-F)^{-1}G_2YB_3K_2R_2^{\frac{1}{2}} \end{aligned} \tag{a1}$$

Hence,

$$\begin{aligned} &\{\Lambda A_1^{-1} Y \Omega\}_s \\ &= R_1^{\frac{1}{2}} K_1\{(sI-F)^{-1}G_2YB_3\}_s K_2 R_2^{\frac{1}{2}} \end{aligned} \tag{a2}$$

It follows from (38) and (29) that $B = B_3G_2 + AJ_{22}$ and therefore $AX + BY = AX + B_3G_2Y + AJ_{22}Y = I$. Now, $XB_3 + A^{-1}B_3G_2YB_3 + J_{22}YB_3 = A^{-1}B_3$ so that

$$(X+J_{22}Y)B_3+H_2\,(sI-F)^{-1}G_2YB_3 = H_2\,(sI-F)^{-1} \quad \text{(a3)}$$

Multiplying s^k, k = 0, 1, 2, ⋯, on both sides an using the identity

$$s^k\,(sI-F)^{-1}=s^{k-1}I+s^{k-2}F + \cdots + F^{k-1}+F^k\,(sI-F)^{-1} \quad \text{(a4)}$$

yields the results

$$H_2F^k\{(sI-F)^{-1}G_2YB_3\}_s=H_2F^k\{(sI-F)^{-1}\}_s \quad \text{(a5)}$$

Since the observability matrix for (F, H_2) has column rank, we can conclude that

$$\{(sI-F)^{-1}G_2YB_3\}_s=\{(sI-F)^{-1}\}_s = (sI-F)^{-1} \quad \text{(a6)}$$

and this completes the proof. ■

Proof of Theorem 2 : 2) from (47), (48) and (10),

$$\Gamma-\Lambda\,A_1^{-1}Y\Omega = \Lambda_*^{-1}A_{1*}P_{12*}(P_{11}-P_{12}YAP_{21})P_{21*}A_*\Omega_*^{-1} \quad \text{(a7)}$$

It will be shown that $P_{11}-P_{12}YAP_{21}$ is a polynomial. Since

$$P_{12}=H_1(sI-F)^{-1}G_2+J_{12}=H_1B_2A_1^{-1}+J_{12} \quad \text{(a8)}$$

and

$$P_{21}=H_2(sI-F)^{-1}G_1+J_{21}=A^{-1}B_3G_1+J_{21}, \quad \text{(a9)}$$

$$\begin{aligned} P_{11}-P_{12}YAP_{21} &= H_1(sI-F)^{-1}G_1-H_1B_2A_1^{-1}Y(B_3G_1+AJ_{21}) \\ &\quad -J_{12}Y(B_3G_1+AJ_{21}) \\ &= H_1[(sI-F)^{-1}-B_2A_1^{-1}YB_3]G_1 \\ &\quad -H_1B_2A_1^{-1}YAJ_{21}-J_{12}Y(B_3G_1+AJ_{21}) \\ &= H_1[(sI-F)^{-1}-(sI-F)^{-1}G_2YB_3]G_1 \\ &\quad -H_1B_2Y_1J_{21}-J_{12}Y(B_3G_1+AJ_{21}) \end{aligned} \quad \text{(a10)}$$

By the identity in (a6), the first term of (a10) is a polynomial and hence $P_{11}-P_{12}YAP_{21}$ is a polynomial. From (a8) and (a9), we see that AP_{21} and $P_{12}A_1$ are polynomials so that $\{\Gamma-\Lambda\,A_1^{-1}Y\Omega\}_+=0$.

1) Assumptions 3 and 6 are obviously satisfied. Since the plant $P(s)$ in (29) is free of unstable hidden modes, it follows that $det(sI-F)=\Psi_p(s)\cdot h(s)$ where $h(s)$ is a strict Hurwitz polynomial. Hence the condition in (2) is equivalent to the one that $det(sI-F)/detA_1(s)$ is a strict Hurwitz polynomial, and this is true if and only if (F, G_2, H_2) is stabilizable and detectable. As for Assumption 2, the full rank properties of J_{21} and J_{12} assure that $P_{12*}P_{12}$ and $P_{21}P_{21*}$ are of full rank and para-Hermitian and ≥ 0 on the $s=jw$ axis. From now on, let us say that a rational matrix is 'good' if it is analytic on the finite part of the $s=jw$ axis. The inverses Λ^{-1} and Ω^{-1} are good by Lemmas 2 and 3. The matrix $\Lambda^{-1}\Gamma\Omega^{-1}-A_1^{-1}Y$ in assumption 5 is good since $\Lambda^{-1}(\Gamma-\Lambda A_1^{-1}Y\Omega)\Omega^{-1}$ is good by (49). It only remains to show that assumption 4 is satisfied. It is well known that when $P_{12}\tilde{A}_1$ is inner there always exists a stable $U_a(s)$ such that $[P_{12}\tilde{A}_1 \mid U_a]$ is square inner and hence $P_{12}\tilde{A}_1\tilde{A}_{1*}P_{12*}=I-U_aU_{a*}$. Similarly, we can find a stable $U_b(s)$ such that $\begin{bmatrix}\tilde{A}P_{21}\\ U_b\end{bmatrix}$ is square coinner and therefore $P_{21*}\tilde{A}_*\tilde{A}P_{21}=I-U_{b*}U_b$. Using these equations, we obtain after a little algebra that $T_r(P_{11}P_{11*})-T_r(\Gamma\Gamma_*) = T_r(P_{11}P_{11*}U_aU_{a*} + P_{11}U_{b*}U_bP_{11*}-P_{11}U_{b*}U_bP_{11*}U_aU_{a*})$. Next, we will show that $U_{a*}P_{11}$ and $P_{11}U_{b*}$ are all good. As a particular choice for $U_a(s)$, we can take $U_a(s)=(H_1-J_{12}K_1)(sI-F+G_2K_1)^{-1}G_a+J_a$ where $G_a^TM_1+J_a^TH_1=0$, $J_a^TJ_a=I$, $J_{12}^TJ_a=0$. We can obtain after a short algebra that $U_{a*}P_{11}=\{G_1^TM_1^T(sI-F+G_2K_1)^{-1}G_a\}_*$ which is good. In a similar way, we can show that $P_{11}U_{b*}$ is good and this completes the proof. The proofs of 3), 4) and 5) are direct from Theorem 1. ■

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