Journal of Control, Automation and Systems Engineering, Vol. 4, No. 1, February, 1998

Reconstruction of the State Variables from the Low Order Controller

저차원제어기로부터 상태변수를 재구성하는 방법

Kwang Young Jeong
(정 광 영)

요 약 : 발란싱 축소모델의 상태변수는 플랜트의 상태변수에 대한 정보를 주지 못한다. 발란싱 축소모델을 좌표변환하여 얻은 상태유지 축소모델은 그 축소오차가 발란싱 축소모델의 축소오차와 같음을 증명하였다. 상태축소오차를 정의하였고, 이 오차를 구하는 방법을 제시하였는데, 이 오차는 축소모델의 차수가 정해지면 불변임을 증명하였다. 상태유지 축소모델의 상태변수는 그 상태축소오차가 작은 경우 원 시스템의 상태변수를 근사하는 장점이 있다. 상태유지 축소모델을 저차원제어기 설계에 적용하여 저차원 상태관측기의 상태변수가 플랜트의 상태변수를 근사하는 예를 보여주었다.

Keywords : model reduction, balancing coordinates, state retaining reduced model, state reduction error, low order controller

I. Introduction

One of the most fundamental problems in dynamic system theory is to approximate a high order system with a low order model. The resulting reduced model can be used usually to approximate the high order plant or to design and implement low order controllers. Among many reduction methods, balancing reduction is the most popular one and was first introduced by Moore[1]. When the system is transformed to the balanced coordinates, each state is equally controllable and observable and the reduced model is obtained by truncating the least controllable and observable states. Although lots of modification have been made for better impulse responses or for improved low frequency behaviour[2]-[4], the resulting reduced model usually does not give the physical information of the states of the system.

Estimation of some desired states is required in many applications for monitoring and/or for decision making process. LQG controller can work as an estimator because it is divided into an estimator and a controller. The estimator estimates all states of the plant and the estimated states serve as input of the controller[5]. When balancing reduction is applied to reduce the order of the controller, the resulting low order controller does not have the information of the states of the plant because the balancing coordinates are different from the coordinates of the given system. There are attempts to represent the reduced model in the subset of the original coordinate system[6][7]. In their paper, however, it is not shown which state should be retained or how well the states of the reduced model approximate those of the given system.

In this paper, it is shown that the state retaining reduced model is obtained by transforming the balanced reduced model back to a subset of the original coordinates. Both reduced models are proved to have the same reduction error. The former has advantage that its states approximate the states of the given system. In order to see how well its states approximate those of the given system, the state reduction error is defined. The state reduction error of the state retaining reduced model is proved to be independent of the choice of the other retained states as long as the order of the reduced model is the same. The algorithms to get the state reduction errors and to apply this to model reduction are shown. Finally, the low order controller to use state retaining reduced model is proposed.

접수일자 : 1997. 7. 29., 수정완료 : 1998. 1. 20.
정광영 : 공주대학교 기계공학과
※ This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1996.

II. State retaining balancing reduction

Consider a linear time-invariant stable system (A, B, C) given by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where

$$A \in R^{n\times n},\ B \in R^{n\times p},\ C \in R^{m\times n}$$

It is well known that the controllability and the observability grammians of the system satisfy the following Lyapunov equations:

$$AW_c + W_cA^T + BB^T = 0 \tag{3}$$

$$A^TW_o + W_oA + C^TC = 0 \tag{4}$$

where W_c and W_o are the controllability and the observability grammians, respectively.

There always exists an equivalent system for which the grammians are equal and diagonal. Such a representation is called *balanced* over the interval $(0,\infty)$. Let P denote the transformation matrix from the original system to a balanced system.

$$x = P\bar{x} \tag{5}$$

The balanced system can be written

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{6}$$

$$y = \overline{C}\,\overline{x} \tag{7}$$

where

$$\overline{A} = P^{-1}AP,\ \overline{B} = P^{-1}B,\ \overline{C} = CP \tag{8}$$

$$\overline{W}_c = \overline{W}_o = diag\ \{\ \sigma_1, \sigma_2, \ldots, \sigma_n\ \} \tag{9}$$

The diagonal elements of the grammians of the balanced system are called Hankel singular values. The transformation matrix P can be obtained using Laub's algorithm[8]. The reduced model of order r is obtained by eliminating $(n-r)$ number of the least controllable and observable states corresponding to smaller Hankel singular values.

$$\dot{\overline{x}}_r = \overline{A}_r\,\overline{x}_r + \overline{B}_r u \tag{10}$$

$$y_r = \overline{C}_r \overline{x}_r \tag{11}$$

Note that the system given in (10) and (11) is asymptotically stable and balanced.

Since the coordinates of the reduced model are different from those of the original state space model, a coordinate transformation to the original coordinates is needed. Since the least controllable and observable states are eliminated,

$$\overline{x}_{r+1} \rightarrow 0,\ \overline{x}_{r+2} \rightarrow 0,\ \ldots,\ \overline{x}_n \rightarrow 0$$

Therefore, the original states $\{\ x_1, x_2, \ldots, x_n\ \}$ are approximated as linear combinations of the states $\{\ \overline{x}_1,\ \overline{x}_2, \ldots,\ \overline{x}_r\ \}$.

$$x = P_{nr}\,\overline{x}_r \tag{12}$$

where

$$x = \{\ x_1, x_2, \ldots, x_n\}^T$$

$$\overline{x}_r = \{\ \overline{x}_1,\ \overline{x}_2, \ldots,\ \overline{x}_r\ \}^T$$

P_{nr} is obtained by deleting the last $(n-r)$ columns of P. Suppose that we want to estimate the r states

$$x_{j1}, x_{j2}, \ldots, x_{jr}\ ,\ \ 1 \le j_k \le n\ ,\ \ k = 1, \ldots, r$$

By selecting $\{\ j_1, j_2, \ldots, j_r\}$ rows from P_{nr}, the transformation matrix P_r between $\overline{x}_r$ and x_r is obtained. In summary, P_r is obtained from P by choosing the first through r^{th} columns and $\{\ j_1, j_2, \ldots, j_r\}$ rows.

$$x_r = P_r\,\overline{x}_r \tag{13}$$

The reduced model in the orignal coordinates is

$$\dot{x}_r = A_r x_r + B_r u \tag{14}$$

$$y_r = C_r x_r \tag{15}$$

where

$$A_r = P_r\,\overline{A}_r P_r^{-1},\ B_r = P_r\,\overline{B}_r,\ C_r = \overline{C}_r P_r^{-1} \tag{16}$$

We call the reduced model $(A_r,\ B_r,\ C_r)$ the *state retaining reduced model* since the state variable x_r approximates the r states of x and the output y_r approximates y.

The reduction error of $(A_r,\ B_r,\ C_r)$ is defined as follows[9]:

$$J = \| g - g_r \|_2 = \{trace(\ \tilde{Q}\tilde{R})\}^{1/2} \tag{17}$$

where g and g_r are the impulse responses of the original system and the reduced model and where $\tilde{Q}$ and $\tilde{R}$ are obtained by considering the augmented system.

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{18}$$

$$\tilde{R} = \tilde{C}^T\tilde{C} \tag{19}$$

where

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A_r \end{pmatrix},\ \tilde{B} = \begin{pmatrix} B \\ B_r \end{pmatrix},\ \tilde{C} = (\ C\ \ -C_r) \tag{20}$$

The normalized error is defined by dividing the reduction error by L_2 norm of the system impulse response as follows:

$$J_n = \frac{\| g - g_r \|_2}{\| g \|_2} = \frac{trace(\ \tilde{Q}\tilde{R})^{1/2}}{trace(C^T C W_c)^{1/2}} \tag{21}$$

Proposition 1 : The reduction error of the state retaining reduced model $(A_r,\ B_r,\ C_r)$ is the same as that of the balanced reduced model $(\ \overline{A}_r,\ \overline{B}_r,\ \overline{C}_r)$.

Proof : The impulse response of the state retaining reduced model is equal to that of $(\ \overline{A}_r,\ \overline{B}_r,\ \overline{C}_r)$ because the impulse response is invariant under coordinate transformation. This proves the theorem. ■
Although the state retaining reduced model is not balanced, its error is the same as that of $(\ \overline{A}_r,\ \overline{B}_r,\ \overline{C}_r)$ Moreover, state retaining reduced model has better property that its states approximate the r states of the original system.

Definition 1 : The *state reduction error* is defined as the impulse response difference norm between a state of the reduced model and the corresponding state of the original system.

The above definition implies that the state in the reduced model which has a smaller state reduction error approximates the corresponding state in the given system better than the one with a larger state reduction error does. It is recommended, therefore, that the reduced model should consist of the states with smaller state reduction errors. In this paper, the state reduction error is obtained from (17) and (20) by substituting 1 and -1 into the corresponding states of $\tilde{C}$. If j^{th} state of the reduced model approximate i^{th} state of the original system, the state reduction error is calculated by considering $\tilde{C}$ whose i^{th} and $(n+j)^{th}$ elements are 1 and -1, respectively, and the other elements are 0's.

Proposition 2 : The state reduction error of the state retaining reduced model is independent of the selection of the retained states as long as the order of the reduced model is fixed.

Proof : See Appendix

With the proof of proposition 2, the algorithms to get each state reduction error and the state retaining reduced model are summarized as follows:

① Obtain the balancing transformation matrix and Hankel singular values.
② Decide the order of the reduced model and obtain P_{nr} by choosing the r columns of P.
③ Obtain the balanced reduced model $(\ \overline{A}_r,\ \overline{B}_r,\ \overline{C}_r)$.
④ Calculate $\overline{Q}_{12}$ and $\overline{Q}_{22}$ from (A.8) and (A.11).
⑤ Obtain each state reduction error from (A.15) by selecting a row vector from P_{nr}.
⑥ Choose r states to be retained.

⑦ Obtain P_r by choosing the r rows of P_{nr} and obtain (A_r, B_r, C_r).

Since the reduction error does not depend on the coordinate transformation, it is independent of the choice of the retained states of the state retaining reduced model as long as the order of the reduced model is fixed. For monitoring and decision making process, it is desirable for reduced model to retain the states that have less state reduction errors.

To obtain each state reduction error involves a solution of the Lyapunov equation given in (18). Therefore, to get all the state reduction errors of very large order system needs enormous calculations. In the algorithms, the state reduction errors are obtained in a closed form as shown in (A.15), thereby saving lots of computing time. That is to say, all state reduction errors are obtained by substituting a suitable row vector P_{nri} to (A.15) without deriving the state retaining reduced model once the order of the reduced model is decided.

Example : Consider the 6th order system[10] described by the following equations:

$$\dot{x} = \begin{pmatrix} -0.2105 & -0.1056 & -0.0007 & 0 & -0.0706 & 0 \\ 1 & -0.0354 & -0.0001 & 0 & -0.0004 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -605.1 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3906.3 & -12.5 \end{pmatrix} x + \begin{pmatrix} -7.211 \\ -0.0523 \\ 0 \\ 794.7 \\ 0 \\ -448.5 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0.0003 & 0 & -0.0077 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} x$$

By transforming these equations into balanced coordinates, Hankel singular values are obtained as follows:

$$\sigma = \{\ 63.9398\ 33.6618\ 0.0024\ 0.0020\ 0.0010\ 0.0009\ \}$$

By inspecting Hankel singular values, the second order reduced model is selected to describe the given system equation. The reduced model in the balanced coordinates is as follows:

$$\dot{\bar{x}}_r = \begin{pmatrix} -0.0822 & 0.3035 \\ -0.3281 & -0.1637 \end{pmatrix} \bar{x}_r + \begin{pmatrix} 3.2415 \\ 3.320 \end{pmatrix} u$$

$$y_r = \begin{pmatrix} -0.8505 & -1.3392 \\ -3.1279 & 3.0380 \end{pmatrix} \bar{x}_r$$

Before this second order reduced model is represented in the subset of the original coordinates, the normalized state reduction errors are calculated as follows:

$$J_{sn} = \{0.00016\ 0.00001\ 0.99601\ 1.0000\ 0.99843\ 1.0000\}$$

By looking at the errors, it is natural for reduced model to retain 1st and 2nd states of the given system. The reduced model, which retains states 1 and 2 of the given system, is obtained as follows:

$$\dot{x}_r = \begin{pmatrix} -0.2105 & -0.1056 \\ 1.0000 & -0.0354 \end{pmatrix} x_r + \begin{pmatrix} -7.2037 \\ -0.0525 \end{pmatrix} u$$

$$y_r = \begin{pmatrix} 0.9999 & 0.0000 \\ 0.0000 & 1.0000 \end{pmatrix} x_r$$

This reduced model has the same reduction error as the reduced model in the balanced coordinates. The merit of this reduced model is that its states approximate the states of the given system. When the impulse are given as an input, the state histories of the given system and reduced model are shown on Figs. 1 and 2. Since the state reduction errors of state 1 and 2 are very small, two curves seem to be identical on both figures.

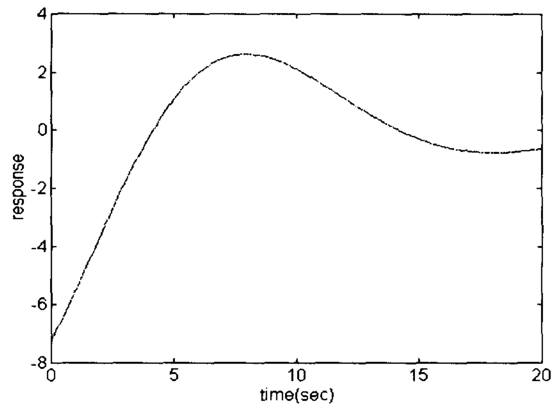

Fig. 1. Time history of state 1 (——, reduced model; ······, full order model).

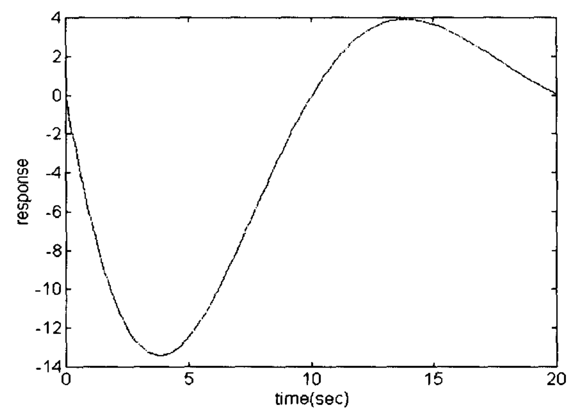

Fig. 2. Time history of state 2 (——, reduced model; ······, full order model).

III. Low order controller

In this section, the reduction scheme derived in the previous section is applied to get low order controller. Linear Quadratic Gaussian theory is briefly reviewed by considering a linear time-invariant stable system of state $\boldsymbol{x}$, output $\boldsymbol{y}$ and measurement $\boldsymbol{z}$,

$$\dot{x} = Ax + Bu + Dw \tag{22}$$

$$y = Cx \tag{23}$$

$$z = Mx + v \tag{24}$$

where $\boldsymbol{w}$ and $\boldsymbol{v}$ are white Gaussian noise with zero mean and covariance matrices W and V, respectively. The noise is assumed to be stationary having the following properties:

$$E[\,w(t)\,w^T(\tau)] = W\delta(t-\tau) \tag{25}$$

$$E[\,v(t)\,v^T(\tau)] = V\delta(t-\tau) \tag{26}$$

$$E[\,w(t)\,v^T(\tau)] = 0 \tag{27}$$

Define the quadratic performance index as

$$J = E\left[\int_0^{\infty}(y^TQy + u^TRu)dt\right] \quad (28)$$

where Q and R are positive definite weighting matrices.

The problem to find the particular admissible control to minimize the performance index is called Linear Quadratic Gaussian (LQG) problem. The minimization results in the optimal control vector $\boldsymbol{u}$ [11][12] given by

$$\boldsymbol{u} = G\hat{\boldsymbol{x}} \quad (29)$$

where the controller gain G is expressed as

$$G = -R^{-1}B^TK \quad (30)$$

with K satisfying the steady state algebraic Riccati equation

$$KA + A^TK - KBR^{-1}B^TK + C^TQC = 0 \quad (31)$$

The estimated states are obtained from

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + F(\boldsymbol{z} - M\hat{\boldsymbol{x}}) + B\boldsymbol{u} \quad (32)$$

where the estimator gain F is shown as

$$F = PM^TV^{-1} \quad (33)$$

with P satisfying

$$AP + PA^T + DWD^T - PM^TV^{-1}MP = 0 \quad (34)$$

By substituting (8) into (11),

$$\dot{\hat{\boldsymbol{x}}} = H\hat{\boldsymbol{x}} + F\boldsymbol{z} \quad (35)$$

where

$$H = A - FM + BG \quad (36)$$

LQG controller is divided into two parts, the estimator and the controller. (35) and (29) imply that the measurement acts as the input of estimator and that controller output acts as the input of the plant. All states are well estimated in the optimal sense and the controller output is obtained from the controller gain multiplied by estimated states that approximate the real state of the plant.

In order to get the low order controller, the order of LQG controller given by (35) and (29) is to be reduced. There are some methods to get low order controllers[13]. A common thing of these methods is that the states of the low order controllers do not approximate those of the plant. Among some controller reduction methods, BCRA (Balanced controller reduction algorithms, [5]) is chosen to apply the state retaining algorithms given in the previous section. The algorithms, however, can be applied to other controller reduction methods based on balancing reduction. BCRA transforms the system (H, F, G) into balanced coordinates and truncates the least controllable and observable states. By following the algorithms used in the previous section, the state reduction errors are obtained. The low order controller is derived by choosing the retaining states and transforming the balanced reduced model into the state retaining model as follows:

$$\dot{\hat{\boldsymbol{x}}}_r = H_r\hat{\boldsymbol{x}}_r + F_r\boldsymbol{z} \quad (37)$$

$$\boldsymbol{u}_r = G_r\hat{\boldsymbol{x}}_r \quad (38)$$

The combined equations are derived by substituting (24) into (37) and by substituting (38) into (22).

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\hat{\boldsymbol{x}}}_r \end{pmatrix} = \begin{pmatrix} A & BG_r \\ F_rM & H_r \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \hat{\boldsymbol{x}}_r \end{pmatrix} + \begin{pmatrix} D & 0 \\ 0 & F_r \end{pmatrix}\begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{pmatrix} \quad (39)$$

Example : The example considered is a 20-member truss structure. Fig. 3 shows the geometry and dimensions of the structure. The objective of this example is to show how the low order controller works to suppress the vibration of the structure. Three actuators are located on trusses 1-2 2-6 and 6-7. The planar truss actuators are used because they give global truss motions and do not require the added mass necessary for the operation of inertia-type actuators. Two displacement sensors are located on nodes 3 and 5 to measure x displacement.

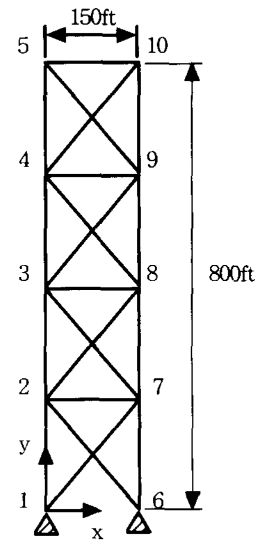

Fig. 3. Geometry and dimensions of a truss.

The discretized equations of motion for a flexible structure are usually written as

$$M\ddot{\boldsymbol{q}} + C\dot{\boldsymbol{q}} + K\boldsymbol{q} = L \quad (40)$$

where M, C and K are mass, damping, and stiffness matrices, respectively; L is the load vector; and $\boldsymbol{q}$ is the structural response vector. This equation can be reduced to a general first order form by defining the augmented vector $\boldsymbol{x}^T = (\dot{\boldsymbol{q}}^T \ \boldsymbol{q}^T)$ as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f} \quad (41)$$

with

$$A = \begin{pmatrix} -M^{-1}C & -M^{-1}L \\ I & 0 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} -M^{-1}L \\ 0 \end{pmatrix} \quad (42)$$

Considering the control problem, we limit ourselves to a linear control so that $\boldsymbol{f}$ is given as

$$\boldsymbol{f} = B\boldsymbol{u} + D\boldsymbol{w} \quad (43)$$

where $\boldsymbol{u}$ is a control vector; $\boldsymbol{w}$ is a vector of white noise in the control commands which is assumed to be a zero mean Gaussian stationary process; B and D are determined from the locations of the controllers and influence of the noise to the state variables. (41) and (43) yield (22) where output and measurements equations are given (23) and (24). with noise properties are given in (25), (26) and (27).

The mass and stiffness matrices of the structure are determined from structure analysis programs such as ANSYS. The damping matrix, which is assumed to be proportional, can be calculated from the mass and stiffness matrices when damping ratio is given[14]. In this simulation, a small value of the damping ratio (=0.005) is used. By changing the equations of motion to state space form, we have a first order linear system with 32 states, which are x and y displacements and velocities of nodes 2, 3, 4, 5, 7, 8, 9 and 10.

The controller is required to bring the system to the vicinity of equilibrium. For simulation purposes, the initial displacements from the static equivalent positions of nodes 2, 3, 4, and 5 are set as (2, 0), (6, 0), (11, 0), and (18, 0). Also, the initial conditions of nodes 7, 8, 9, and 10 are the same as those of nodes 2, 3, 4 and 5, respectively. Later, the process and measurement noise are incorporated as random numbers with specified statistical properties. Covariance matrices W and V in (25) and (26) are assumed to be diagonal whose diagonal elements are 10 and 0.05. Also, Q and R in (28) are set to be diagonal matrices whose diagonal elements are 12,000's and 1's, respectively. In this simulation, noise is generated by the random number generator using the built-in function of Matlab.

Out of 32 system states, 16 states are retained in the low order controller. The effect of this controller is shown on Fig. 4 where solid and dotted lines are x displacements of node 5 with and without controller, respectively. It is shown that the controller brings the truss to the equilibrium very quickly.

In order to know how well the state estimator of the lower order controller approximates the states of the plant, the states of the plant and those of the estimator should be plotted. x displacement of node 4 is shown on Fig. 5 where the solid and dotted curves are the states of the low order controller and plant, respectively. By comparing with a full order (LQG) controller, as shown on Fig. 6, the low order controller works as good as the full order controller. When the velocities of some nodes are included in the retained states of the low order controller, we can estimate not only the displacements but also the velocities of the nodes. On Fig. 7, x velocity of node 5 is shown, where solid and dashed lines are states of low order controller and LQG controller, respectively. As shown on the figure, the estimated x velocity of the low order controller well approximates that of LQG controller.

If BCRA[5] is applied to this example among some existing low order controller algorithms, the states of BCRA controller can not approximate any state of the plant. By transforming BCRA controller into the state retaining form, the state retaining reduced controller is obtained. This controller can estimate some states of the plant because its states approximate the corresponding states of LQG controller and because the states of LQG controller estimate plant states well in the optimal sense. Since the reduction error is invariant under coordinate transformation, the performances of BCRA controller and the state retaining reduced controller are the same except that the latter has a merit that each state of the controller estimates the corresponding state of the plant.

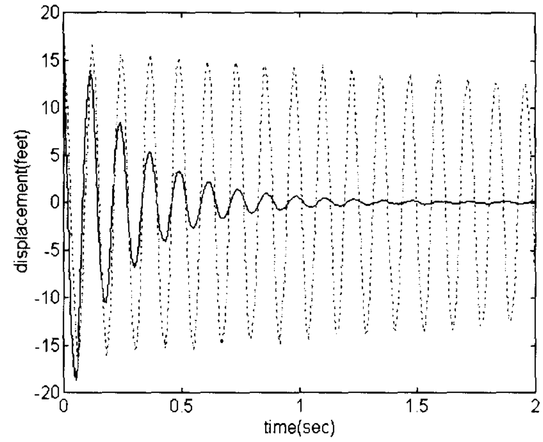

Fig. 4. X displacement of node 5 (——, with the low order controller; ⋯⋯, without control).

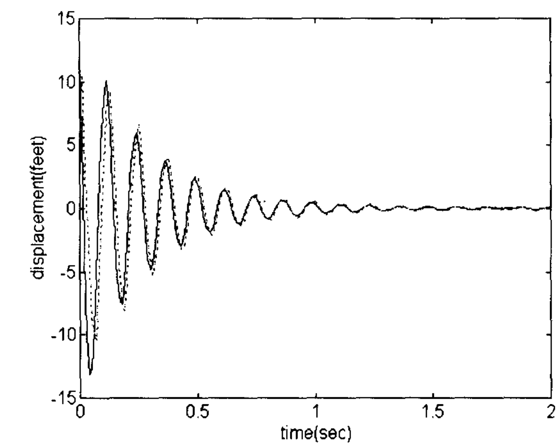

Fig. 5. X displacement of the low order controller and the plant for node 4 (——, low order controller; ⋯⋯, plant).

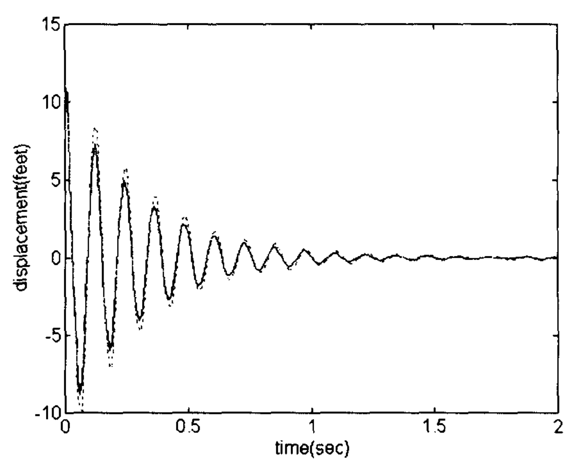

Fig. 6. X displacements of the low order controller and LQG controller for node 4 (——, low order controller; ⋯⋯, LQG controller).

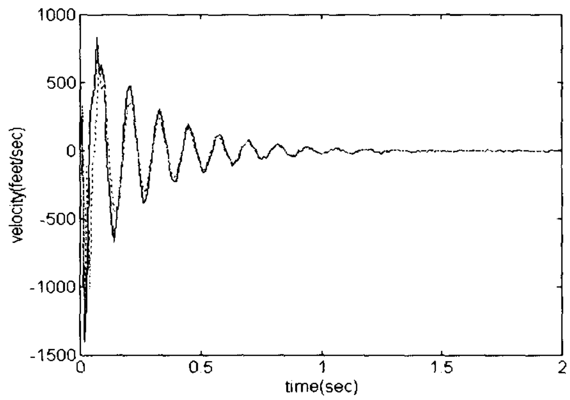

Fig. 7. X velocity of the low order controller and LQG controller for node 5 (——, low order controller; ······, LQG controller).

IV. Conclusions

The state retaining reduced model is analyzed. This model has the same reduction error as the balanced reduced model. Moreover, it has a merit that its states approximate the original states, which is required in many applications for monitoring, for tracking, or for decision making process. The reduction error of the reduced model is proved to be the same as that of the state retaining reduced model.

State reduction errors are introduced. Each state reduction error is proved to be invariant without regarding to the selection of the other retaining states when the order of the reduced model is fixed. The algorithms to get each state reduction error are presented. To obtain each state reduction error involves a solution of the Lyapunov equation. To get all the state reduction errors of very large order system, therefore, enormous calculations are needed. In this paper, the state reduction errors are obtained in a closed form, thereby saving lots of computing time.

The existing controller reduction algorithms such as BCRA do not estimate the states of the plant because the coordinates of the states of the controller obtained from balancing reduction are different from those of the plant. By applying state retaining reduction algorithms to the controller design, the states of the controller are reconstructed so that the states of the controller can approximate those of the plant.

Two examples are presented. One example shows that state reduction errors are obtained without deriving reduced model and that the states of the reduced model with small state reduction errors well approximate the states of the given system. As another example, vibration suppression of a 20-member truss is considered. A low order controller with 16 states is designed out of the plant with 32 states. The simulation shows that the low order controller works as good as the full order controller and that its states estimate the corresponding states of the plant.

References

[1] B. C. Moore, "Principal component analysis in linear systems : controllability, observability and model reduction," *IEEE Trans. Automat. Cont.*, vol. AC-26, no. 1, pp. 17-32, 1981.

[2] B. Clapperton, F. Creusca and M. Aldeen, "Bilinear transformation and generalised singular perturbation model reduction," *Proc. of the 33rd Conf. on Decision and Control*, Lake Buena Vista, Fl, pp. 3737-3742, Dec., 1994.

[3] G. Muscato and G. Nunnari, "On the σ-reciprocal system for model order reduction," *Proc. of the 33rd Conf. on Decision and Control*, Lake Buena Vista, Fl, pp. 3743-3745, Dec., 1994.

[4] V. Sreeram and P. Agathoklis, "Model reduction using balanced realizations with improved low frequency behaviour," *System and Control Letters*, vol. 12, pp. 33-38, 1989.

[5] A. Yousuff and R. E. Skelton, "A note on balanced controller reduction," *IEEE Trans. Automat. Cont.*, vol. AC-29, no. 3, pp. 254-257, 1984.

[6] A. Kumar, "An algorithm for model and controller reduction to retain desired states," *Proc. American Control Conf.*, Minneapolis, MN, pp. 1532-1533, June, 1987.

[7] K. H. Yae and K. Y. Jeong, "Model reduction in the physical coordinate system," *Proc. 3rd Annual Conf. on Aerospace Computational Control*, vol. 2, pp. 792-804, 1989.

[8] A. J. Laub, "Computation of balancing transformation," *Proc. JACC*, San Francisco, CA, session FA8-E, 1980.

[9] D. S. Bernstein and D. C. Hyland, "The optimal projection equations for reduced-order state estimation," *IEEE Trans. Automat. Cont.*, vol. AC-30, no. 6, pp. 583-585, 1985.

[10] D. K. Linder, J. Babendreier and A. M. A. Hamdan, "Measures of controllability and observability and residues," *IEEE Trans. Automat. Cont.*, vol. AC-34, no. 6, pp. 648-650, 1989.

[11] R. E. Skelton, *Dynamics Systems Control*, John Wiley & Sons, 1988.

[12] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, New Jersey, 1995.

[13] J. A. Davis and R. E. Skelton, "Another balanced controller reduction algorithm," *System and Control Letters*, vol. 4, pp. 79-83, 1984.

[14] R. W. Clough and J. Penzien, *Dynamics of Structure*, McGraw-Hill, New York, 1975.

Appendix

Proof of proposition 2 : Suppose r^{th} order reduced model (A_r, B_r, C_r) is obtained from n^{th} order system (A, B, C) by reducing in the balanced coordinates and transforming to the original system using P_r matrix. When i^{th} state of the original system is approximated by j^{th} state of the reduced model, the state reduction error for state i is defined as:

$$J_s = \{trace(\tilde{Q}\,\tilde{R}_s)\}^{1/2} \tag{A.1}$$

where non-zero elements of R_s are

$$\tilde{R}_s(i,\ i) = \tilde{R}_s(n+j,\ n+j) = 1 \tag{A.2}$$

$$\tilde{R}_s(n+i,\ j) = \tilde{R}_s\ (j,\ n+i) = -1 \tag{A.3}$$

The symmetric matrix $\tilde{Q}$ is obtained by partitioning (18) as follows:

$$AQ_{11} + Q_{11}A^T + BB^T = 0 \tag{A.4}$$

$$AQ_{12} + Q_{12}A_r^T + BB_r^T = 0 \tag{A.5}$$

$$A_rQ_{22} + Q_{22}A_r^T + B_rB_r^T = 0 \tag{A.6}$$

with

$$\tilde{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \tag{A.7}$$

Since Q_{11} is the controllability grammian of the original system, its elements are not affected by reduction.

Substituting (16) into (A.5) and postmultiplying P_r^{-T}, we get

$$A\,\overline{Q}_{12} + \overline{Q}_{12}\,\overline{A}_r^{\,T} + B\,\overline{B}_r = 0 \tag{A.8}$$

where

$$\overline{Q}_{12} = Q_{12}P_r^{-T} \tag{A.9}$$

Since $\overline{Q}_{12}$ is obtained from A, B, $\overline{A}_r$ and $\overline{B}_r$, it does not affected by the choice of retained states. Once $\overline{Q}_{12}$ is known, Q_{12} is obtained by postmultiplying transformation matrix P_r^T.

$$\begin{aligned} Q_{12} &= \overline{Q}_{12}P_r^T \\ &= \{\ \overline{Q}_{12}P_{r1}^T\ \ \overline{Q}_{12}P_{r2}^T\ \ldots\ \overline{Q}_{12}P_{rr}^T\ \} \end{aligned} \tag{A.10}$$

where P_{rj} is j^{th} row of P_r.

Premultiplying (A.6) by P_r^{-1}, postmultiplying by P_r^{-T}, and considering (16),

$$\overline{A}_r\,\overline{Q}_{22} + \overline{Q}_{22}\,\overline{A}_r^{\,T} + \overline{B}_r\,\overline{B}_r^{\,T} = 0 \tag{A.11}$$

where

$$\overline{Q}_{22} = P_r^{-1}Q_{22}P_r^{-T} \tag{A.12}$$

Since $\overline{Q}_{22}$ is calculated from balancing coordinates, it does not depend on choice of retained states as long as the order of the reduced model is fixed. Once $\overline{Q}_{22}$ is known, Q_{22} is obtained from the following equation.

$$Q_{22} = P_r\,\overline{Q}_{22}P_r^T \tag{A.13}$$

Considering the structure of matrix $\tilde{R}_s$, nonzero columns of $\tilde{Q}\tilde{R}_s$ are i^{th} and $(n+j)^{th}$ columns, which are equal except the sign is reversed. Since the trace of a matrix is the sum of diagonal elements, which are $(i,\ i)$ and $(n+j,\ n+j)$ elements for $\tilde{Q}\tilde{R}_s$. Therefore,

$$\begin{aligned} J_s &= \{\ trace(\ \tilde{Q}\tilde{R}_s)\ \}^{1/2} \\ &= \{\ \tilde{Q}(i,\ i) - \tilde{Q}(i,\ n+j) - \tilde{Q}(n+j,\ i) + \tilde{Q}(n+j,\ n+j)\}^{1/2} \\ &= \{\ Q_{11}(i,\ i) - 2Q_{12}(i,\ j) + Q_{22}(j,\ j)\ \}^{1/2} \\ &= \{\ Q_{11}(i,\ i) - 2\,\overline{Q}_{12i}P_{rj}^T + P_{rj}\,\overline{Q}_{22}\,P_{rj}^T\ \}^{1/2} \end{aligned} \tag{A.14}$$

where $\overline{Q}_{12i}$ is i^{th} row of $\overline{Q}_{12}$, $Q_{11}(i,\ i)$ is i^{th} diagonal element of the controllability grammian of the given system, and $\overline{Q}_{22}$ is a matrix which is obtained from (A.11). Since $\overline{Q}_{22}$ is the controllability grammians of the truncated balanced system, it is a diagonal matrix. $\overline{Q}_{12}$ is obtained from (A.8). Since P_{rj} is j^{th} row of P_r, which is a sub-matrix of P_{nr}, P_{rj} is i^{th} row of P_{nr}, that is to say,

$$J_s = \{\ Q_{11}(i,\ i) - 2\,\overline{Q}_{12i}P_{nri}^T + P_{nri}\,\overline{Q}_{22}\,P_{nn}^T\ \}^{1/2} \tag{A.15}$$

where P_{nri} is i^{th} row of P_{nr}.

(A.15) says that all terms of J_s do not depend on the selection of the state but on the order of the reduced model. This proves the theorem that the state reduction error of a reduced model depends on the order of the reduced model not on the choice of the other states retained in the reduced model. ■

정 광 영

1980년 서울대학교 기계공학과 학사. 1982년 한국과학기술원 기계공학과 석사. 1991년 Univ. of Iowa 기계공학과 박사. 1982년~1986년, 1991년~1993년 삼성중공업 근무. 1993년~현재 공주대학교 기계공학과 조교수. 관심분야는 모델축소화, 제어시스템 설계 및 응용.