병렬 상호 연결망을 위한 초집중기의 구성

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요 약

병렬 컴퓨터 구조의 통신 시스템에 있어서 수많은 반도체 소자의 연결을 가능하게 하는 선형 사이즈의 팽창기가 병렬 상호 연결망과 관련된 여러 분야에서 활발히 연구 되어왔다. 그러나 이러한 병렬 컴퓨터 구성의 주요한 단점은 프로세서와 메모리간의 병렬 상호 연결망 구성에 있어서 요구되는 비용이 크다는 것이다. 선형 사이즈의 팽창기를 이용한 집중기는 기존의 병렬 상호 연결망 보다 이론적으로 최적의 병렬 상호 연결망 구조로 구성 될 수 있다. 현존하는 구조는 커다란 팽창 상수를 갖는 팽창기에 근거한다. 이는 현실적으로 반도체 기술에 부합하는 네트워크의 구성에 비현실성을 내포한다. 팽창 상수를 줄임으로서 현실성 있는 팽창기에 근거하여 집중기를 구성하는 것이 바람직하다. 본 논문은 식, $|\Gamma_x| \geq [1 + d(1 - |X|/n)]\ |X|$을 만족하는 향상된 팽창 상수를 찾기 위한 증명 과정에서 퍼뮤테이션 함수의 일치점을 세분화하여 이용하였고, 그 팽창 상수를 집중기 구성에 적용하여 회귀적 네트워크의 구조를 갖는 보다 현실성있는 초집중기의 구성을 제안한다. 결과적으로, $(n, 5, 1-\sqrt{3}/2)$로 구성된 팽창기를 이용하여, Gabber와 Gali의 구조에 적용 함으로서 $209n$의 복잡도를 갖는 초집중기를 구성한다.

An Explicit Superconcentrator Construction for Parallel Interconnection Network

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ABSTRACT

Linear size expanders have been studied in many fields for the practical use, which make it possible to connect large numbers of device chips in both parallel communication systems and parallel computers. One major limitation on the efficiency of parallel computer designs has been the highly cost of parallel communication between processors and memories. Linear order concentrators can be used to construct theoretically optimal interconnection network schemes. Existing explicitly defined constructions are based on expanders, which have large constant factors, thereby rendering them impractical for reasonable sized networks. For these objectives, we use the more detailed matching points in permutation functions, to find out the bigger expansion constant from an equation, $|\Gamma_x| \geq [1 + d(1 - |X|/n)]\ |X|$. This paper presents an improvement of expansion constant on constructing concentrators using expanders, which realizes the reduction of the size in a superconcentrator by a constant factor. As a result, this paper shows an explicit construction of $(n, 5, 1-\sqrt{3}/2)$ expander. Thus, superconcentrators with $209n$ edges can be obtained by applying to the expanders of Gabber and Galil's construction.

1. Introduction

One of the most important issues in parallel high computing system is the communication between processors in a message-passing architecture, and between

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processors and memory in a shared global memory machine. In order to achieve high-bandwidth parallel communication, it is necessary to be able to move information in parallel along separate, or disjoint, communication pathways. So as to do in a point-to-point fashion, multistage rearrangeable networks such as the Benes network[4], which has hardware complexity $O(N \log N)$ and is extracted from the three-stage Clos network[5] have been proposed, as well as single-stage recirculating network such as the shuffle-exchange network[15]. Such networks have limitations in either the complexity of the routing computation[11], or in the time required to move the information[3][12]. Because of these problems, many researches have been focused on finding some new interconnection structures, which realize optimal hardware complexity and which might be easier to route than the existing configurations.

In order to route n streams of information efficiently in a parallel computer, it is necessary to construct a network with n disjoint paths from source to destination. One way to build such an interconnection is to use a superconcentrator to divide the input stream into two output parts, then to recursively divide each part of the output with two additional superconcentrators until each stream has been connected to its specific destination[9].

To build optimal such an interconnection scheme, it is necessary to use linear superconcentrator with hardware complexity $O(n)$. Pinsker[13] and Pippenger [14], showed how to build such a superconcentrator, by using a more primitive two-stage structure called a concentrator.

Pinsker[13] proved that there exists an (n, m) concentrator with $29n$ edges. Pippenger[14] gave a new version of the nonconstructive existence theorem and a simple recursive construction with $39n$ edges with the good and the bad one argument. However, the explicit construction is still needed for many applications. Chung[6] improved it to $261.5n$ edges and Alon and Milman[1] to $175n$ edges again.

So as to develop the construction of explicit linear sized concentrators and superconcentrators, Margulis [10] was the first to describe a family of linear size expanders in bipartite graph $\hat{G}_n$ and proved $|\Gamma_x| \geq [1 + d(1 - |X|/n)]\ |X|$ for any subset X of input vertices with $|X| \leq n/2$. It is an (n, k, d) expander having n input vertices, n output vertices and at most kn edges. Furthermore, an explicit construction is built by Gabber and Galil[7] with $404n$ edges in a family of explicit graph G_n. By using it, this paper shows that an improvement of expansion constant d can reduce the size of the resulting concentrator built from any given expander. Thus, it is possible to have superconcentrators with $209n$ edges by applying these expanders to Gabber and Galil's construction.

The basic expander is described in section 2. For an $(n, 5, 1-\sqrt{3}/2)$ expander, we will provide some proofs in the section 3 and section 4 to find d by using Gabber and Galil's method, which is the principal contribution of this paper. The section 5 shows that the result can be improved by applying it to them.

2. Preliminary

The basic expander defined by Gabber and Galil[7] consists of a set of n inputs, where $n = m^2$, m is any integer, and an equal number of outputs. The inputs are connected to the outputs by a set of five permutations, which shift each row right or left several columns, with wrap-around. Each successive row is shifted one more places until the middle row is shifted back onto itself, and so on. Identity and column increment permutations are included, and a separate set of similar permutations shifts the columns by rows. These permutations are described by the following functions, where $+$ is mod m:

$$\sigma_0(x, y) = (x, y),\quad \sigma_1(x, y) = (x, x+y),\ \sigma_2(x, y) = (x, x+y+1),$$
$$\sigma_3(x, y) = (x+y, y),\ \ \sigma_4(x, y) = (x+y+1, y)$$

First of all, a concentrator is defined to show how to build a superconcentrator through it. An (n, θ, k)

concentrator is a directed acyclic graph with n input vertices, θn output vertices($\theta<1$), and at most kn connections from the inputs to the outputs, and it has property that, for every subset of inputs X such that $|X| \leq n/2$, there exist at least X flows (vertex-disjoint directed paths) connected from input to output vertices. In order to build such concentrators explicitly from expanders, an (n, k, d) expander, as used through this paper, is a two-stage network with n inputs and n outputs, with each inputs connected by links to kn outputs. The links are chosen in such a way that, for every set of inputs X, where $|X| \leq n/2$, the set of outputs Γ_x which are connected by links to X, observe the rule that:

$$|\Gamma_x| \geq [1+d(1-|X|/n)]\,|X| \qquad (1)$$

One of the major difficulties in this theory is to find out d value which satisfies (1) and to construct linear families of superconectrators according to $n \rightarrow \infty$ linearly. Therefore, the solution of estimating the expansion constant obtained by the five permutation functions, beginning at Theorem 2.1 is considered step by step with a relatively straight forward analysis.

Theorem 2.1: For $\theta_1=(1-\kappa)/(1+\kappa)$, $\theta_2=\kappa/(1-\kappa^2)$, and $0<\kappa<1$, we have

$$|z_0|^2 \leq \theta_1 \sum_{-\infty<n<\infty} |z_n|^2 + \theta_2 \sum_{-\infty<n<\infty} |z_{n+1}-z_n|^2, \quad (2)$$

where $\{z_n\}_{-\infty<n<\infty}$ be a series of complex numbers[7].

Definition 2.2: For any $m, n \in \mathbf{Z}$, a function $\varphi_i: \mathbf{Z}^2 \rightarrow \mathbf{Z}^2$, $\{\varphi_i : i \in 1, 2\}$ defines $\varphi_1(m, n)=(m, n+\lambda m)$, and $\varphi_2(m, n)=(m+\lambda n, m)$, where $\mathbf{Z}$ denotes the set of integers.

Proposition 2.3: Let $\Lambda_1=\{(m, n): |m|<|n|\} \cup \{(m, n): m=n\}$, and $\Lambda_2=\{(m, n): |m|>|n|\} \cup \{(m, n): m=-n\}$ but $m \neq 0$ and $n \neq 0$. Then, the condition that $\varphi_i^l(\Lambda_{3-i})$ have always disjointness for each case i and $l \in \mathbf{Z}$ is at least $\lambda \geq 2$.

Proof: For simplicity, it will be only argued that the result for $i=1$. The proof for $i=2$ is identical. This will be proved by induction on $l: \varphi_1^l(m, n)=(m, 2lm+n)$. So, for $l=0$, the result follows the fact that all elements in a set Λ_2 are not changed. Now assume that the result is true for $l=k-1$. Then, points at $(m, 2(k-1)m+n)$ are all disjoint. If all points move along n-axis in equal, it is obvious all elements are disjoint. Therefore, $(m, 2(k-1)m-2m+n)=\varphi_1^k(m, n)$ gives the required result for $l=k$. Eventually, that λ is at least 2 should be required also for disjointness. □

3. Expansion Constant in Complex System

Let $a_{m,n}$ denote a system of complex numbers in the range $-\infty<m, n<\infty$. Through conversions of $a_{m,n}$ by virtue of functions $\{\varphi_i : i \in 1, 2\}$, we are capable of deciding the expansion constant.

Lemma 3.1: Assume that $a_{0,0}=0$, and $\sum_{m,n} |a_{m,n}|^2 < \infty$. Then,

$$\sum_{m,n} |a_{m,n+\lambda m}-a_{m,n}|^2 + \sum_{m,n} |a_{m+\lambda n,n}-a_{m,n}|^2 \geq 2\lambda d \sum_{m,n} |a_{m,n}|^2 \quad (3)$$

Proof: In order to use (2), assume that we have $z_j = a_{\varphi_i^j(m,n)}$ for $j \in \mathbf{Z}$. Then, (2) can be written into following equation:

$$|a_{m,n}|^2 \leq \theta_1 \sum_j |a_{\varphi_i^j(m,n)}|^2 + \theta_2 \sum_j |a_{\varphi_i^{j+1}(m,n)} - a_{\varphi_i^j(m,n)}|^2, \quad (4)$$

where $(m, n) \in \{\Lambda_{3-i} : i \in 1, 2\}$,

If right-side on (4) is considered with $(m, n) \in \Lambda_{3-i}$ according to i, these points will be distributed with regular distance. Disjointness can be guaranteed by Proposition 2.3. Therefore, (4) is equivalent to:

$$|a_{m,n}|^2 \leq \theta_1 \cdot |a_{m,n}|^2 + \theta_2 \cdot |a_{\varphi_i(m,n)} - a_{m,n}|^2$$

For any $(m, n) \in \Lambda_{3-i}$, we have:

$$\sum_{m,n} |a_{m,n}|^2 \leq 2\theta_1 \sum_{m,n} |a_{m,n}|^2 + \theta_2 \Big(\sum_{m,n} |a_{m,n+\lambda m} - a_{m,n}|^2 + \sum_{m,n} |a_{m+\lambda n,n} - a_{m,n}|^2 \Big)$$

Next, if the argument, $2\theta_1 \sum_{m,n} |a_{m,n}|^2$, from right side moves to left side, similarly we obtain:

$$(1-2\theta_1)(\sum_{m,n} |a_{m,n}|^2)/\theta_2 \le \sum_{m,n} |a_{m,n+\lambda m} - a_{m,n}|^2 + \sum_{m,n} |a_{m+\lambda n,n} - a_{m,n}|^2 \quad (5)$$

When the condition of (3) is satisfied with (5), we can get:

$$2\lambda d = (1-2\theta_1)/\theta_2 \quad (6)$$

Thus, as long as we choose a value of $2\lambda d$ from (6), the inquality (3) is satisfied and the theorem holds. This completes the proof □

The maximum expansion constant d from Theorem 2.1, can be taken from $\lambda=2$ and $\kappa=1/\sqrt{3}$ such that: $d=1-\sqrt{3}/2$.

Theorem 3.2: Let $A_{m\times m}$ be the integer sets in [0, m) ×[0, m)-plane. For $m=1, 2, \cdots$, the bipartite graph that is capable of being obtained from the given permutations $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ on $A_{m\times m}$ is an $(n, 5, d)$ expander that satisfies equation, $|\Gamma_x| \ge [1 + d(1-|X|/n)]\ |X|$, where $n=m^2$, $d=1-\sqrt{3}/2$, and Γ_x denotes the nodes in output that are adjacent to nodes in X.

4. Expansion Constant

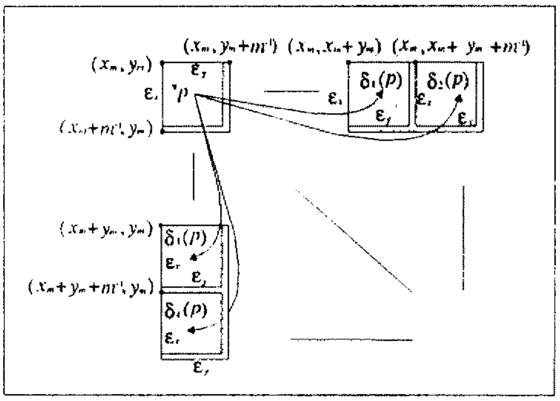
First of all, we need a definition on permutations for the proofs of the next lemmas. (Fig. 1) shows positions to which these permutations are connected. These are for matching discrete functions into continuous.

Definition 4.1: $\delta_i: A \to A$, $0 \le \varepsilon_x < m^{-1}$, $x/m = x_m$, $0 \le \varepsilon_y < m^{-1}$, $y/m = y_m$,

$$\delta_1(x_m+\varepsilon_x,\ y_m+\varepsilon_y) = (x_m+\varepsilon_x,\ x_m+y_m+\varepsilon_y)$$
$$\delta_2(x_m+\varepsilon_x,\ y_m+\varepsilon_y) = (x_m+\varepsilon_x,\ x_m+y_m+m^{-1}+\varepsilon_y)$$
$$\delta_3(x_m+\varepsilon_x,\ y_m+\varepsilon_y) = (x_m+y_m+\varepsilon_x,\ y_m+\varepsilon_y)$$
$$\delta_4(x_m+\varepsilon_x,\ y_m+\varepsilon_y) = (x_m+y_m+m^{-1}+\varepsilon_x,\ y_m+\varepsilon_y)$$

where A is defined as the [0, 1)×[0, 1) *torus*, and the + is modulo 1.

The concept of the measure $\mu(A)$ of a set A is a natural generalization for the length, the area, the volume and so on. The common value $\mu(A)$ of the outer and inner measure for a countable set A is called its *Lebesgue measure*.



(Fig. 1) Matching points with δ_i

Showing Theorem 3.2, the next Lemma that required the assumption of *Lebesgue Measure* should be satisfied.

Lemma 4.2: If $\int_A f d\mu = 0$ and $\int_A |f|^2 d\mu < \infty$ are satisfied with measurable function f on A according to the definitions of *Lebesgue measure*, we obtain:

$$\sum_{i=0}^{4} \int |\delta_i^{-\lambda}(f) - f|^2 d\mu \ge 4\lambda d \int |f|^2 d\mu \quad (7)$$

Proof: In order to make sure the assumption, we have:

$$\int_A f(X)\,d\mu(X) = \int_{A_x} \mu(X^c)\,d\mu(X) - \int_{A_{x^c}} \mu(X)\,d\mu(X^c)$$

Since $\mu(X^c)$ and $\mu(X)$ are measurable, it follows:

$$\int_A f(X)\,d\mu(X) = \mu(X^c)\mu(X) - \mu(X)\mu(X^c) = 0.$$

Similarly, we have: $\int_A |f(X)|^2 d\mu(X) = \mu(X^c)\mu(X) < \infty$

By assumptions and $f \in L^2(A)$, the condition for existence Fourier coefficient is satisfied. So, the Fourier coefficient of function f exists on A. Then, $a_{m,n}(f) = \int_A f(p)\,\xi_{m,n}(p)\,d\mu(p)$, where $\xi_{m,n}(p) = e^{-2\pi i(mx+ny)}$ and $p=(x, y)$. Fourier coefficient of $\delta_1^{-\lambda}(f)$ and $\delta_2^{-\lambda}(f)$ are:

$$a_{m,n}(\delta_1^{-\lambda}(f)) = \int_A f(\delta_1^{\lambda}(p))\, \xi_{m,n}(p)\, d\mu(p) = a_{m+\lambda n,\, n}(f)$$
$$a_{m,n}(\delta_2^{-\lambda}(f)) = \int_A f(\delta_2^{\lambda}(p))\, \xi_{m,n}(p)\, d\mu(p) = a_{m+\lambda n,\, n}(f) \cdot e^{-2\pi i \lambda n}$$

These denote that the coefficient is not changed because it is magnitude. Therefore, the results can be simplified to: $a_{m,n}(\delta_1^{-\lambda}(f)) = a_{m,n}(\delta_2^{-\lambda}(f)) = a_{m+\lambda n,\, n}(f)$ Similarly, we have: $a_{m,n}(\delta_3^{-1}(f)) = a_{m,n}(\delta_4^{-1}(f)) = a_{m,\, n+\lambda m}(f)$

By using the *Parsaval's equality*, Lemma 3.1 drives such that: $\sum_{m,n} |a_{m,n}|^2 = \int_A |f|^2 d\mu < \infty$

$$2\int |f - \delta_3^{-\lambda}(f)|^2 d\mu + 2\int |f - \delta_1^{-\lambda}(f)|^2 d\mu \geq 4\lambda d \int |f|^2 d\mu \quad (8)$$

From (8), we obtain:

$$\sum_{i=1}^{4} \int |\delta_i^{-\lambda}(f) - f|^2 d\mu \geq 4\lambda d \int |f|^2 d\mu \quad \square$$

Definition 4.3: If $X \subseteq A$ and $x \in X$, then the *Characteristic Function* χ_A defined by $\chi_A(x) = 1$ on $x \in A$ or $\chi_A(x) = 0$ on $x \notin A$ is measurable. And for every $Y \subseteq A$, the permutation δ has the property that $\delta^{-1}(\chi_Y) = \chi_{\delta^{-1}(Y)} = \chi_{\delta(Y)}$.

Lemma 4.4: If subset $X \subseteq A$ is measurable, we have

$$\sum_{i=1}^{4} \mu(X - \delta_i^{-\lambda}(X)) \geq 2\lambda d \mu(X) \mu(X^c). \quad (9)$$

Proof: This will be proved by induction on λ. First of all, in order to use characteristic function from Definition 4.3 assume $f = \chi_X - \mu(X)$ for a measurable subset $X \subseteq A$. Then, $\chi_X = 1$ on X and $\chi_X = 0$ on X^c.

For $\lambda = 1$, the argument, $\delta_i^{-1}(f) - f$, in (7) can be rewritten as follows:

$$\delta_i^{-1}(f) - f = \chi_{\delta_i(X)} - \chi_X \quad (10)$$

From (10), we have:

$$|\delta_i^{-1}(f) - f|^2 = \begin{cases} 1 & on \quad [\delta_i(X) - X] \cup [X - \delta_i(X)] \\ 0 & otherwise \end{cases}$$

Since δ_i is preserving measure, we get:

$\int_A |\delta_i^{-1}(f) - f|^2 d\mu = \mu(\delta_i(X) - X) + \mu(X - \delta_i(X))$ or
$\int_A |\delta_i^{-1}(f) - f|^2 d\mu = 2\mu(X - \delta_i^{-1}(X))$

If this equation is substituted to (7), we have:

$$\sum_{i=1}^{4} \int |\delta_i^{-1}(f) - f|^2 d\mu = \sum_{i=1}^{4} 2\mu(X - \delta_i^{-1}(X)) \geq 4d \int |f|^2 d\mu$$

Also, Lemma 4.2 yields: $\sum_{i=1}^{4} \mu(X - \delta_i^{-1}(X)) \geq 2d\mu(X)\mu(X^c)$

As a result, this shows that it is true for $\lambda = 1$. Now, in order that the result is true for $\lambda = k$ on (7), therefore we have:

$$\delta_i^{-k}(f) - f = \chi_{\delta_i^k(X)} - \chi_X \quad (11)$$

The absolute value is:

$$|\delta_i^{-k}(f) - f|^2 = \begin{cases} 1 & on \quad [\delta_i^k(X) - X] \cup [X - \delta_i^k(X)] \\ 0 & otherwise \end{cases}$$

also, $\int_A |\delta_i^{-k}(f) - f|^2 = 2\mu(X - \delta_i^{-k}(X))$

Assume that $\sum_{i=1}^{4} \mu(X - \delta_i^{-k}(X)) \geq 2kd\mu(X)\mu(X^c)$ it is true for $\lambda = k$. Finally we have only to show that the result is true for $\lambda = k+1$. Similarly, from (10) and (11), it drives:

$$\delta_i^{-(k+1)}(f) - f = \delta_i^{-1}(\delta_i^{-k}(f) - f) + \delta_i^{-1}(f) - f = \chi_{\delta_i^{k+1}(X)} - \chi_X$$

From Definition 4.3, we have:

$$|\delta_i^{-(k+1)}(f) - f|^2 = \begin{cases} 1 & on \quad [\delta_i^{k+1}(X) - X] \cup [X - \delta_i^{k+1}(X)] \\ 0 & otherwise \end{cases}$$

also,

$$\int_A |\delta_i^{-(k+1)}(f) - f|^2 = 2\mu(X - \delta_i^{-(k+1)}(X)) \quad (12)$$

From Lemma 4.2 and (12), we obtain:

$$\sum_{i=1}^{4} \mu(X - \delta_i^{-(k+1)}(X)) \geq 2(k+1) d\mu(X)\mu(X^c)$$

Therefore, this proof is verified by the inductive hypothesis. □

Lemma 4.5: Assume $|X| \le n/2$ and every subset $X \subseteq A_{m\times m}$. Then,

$$\sum_{i=1}^{2} [|\sigma_{2i-1}(X)-X| + |\sigma_{2i}(X)-X| - |Y_i|] \ge 2d|X||X^c|/n,$$

where $Y_i = [\sigma_{2i-1}(X)\backslash X] \cap [\sigma_{2i}(X)\backslash X]$

Proof: From Definition 4.1, we have $(x_m, y_m) \in A$. Let

$\tilde{A}_{(x_m, y_m)} = \{x_m + \varepsilon_x, y_m + \varepsilon_y | 0 \le \varepsilon_x, \varepsilon_y < m^{-1}\}$ and $\tilde{X} = \bigcup_{\alpha \in X} \tilde{A}_\alpha$ for a subset $X \subseteq A_{m\times m}$. Let $\alpha \in X$ and $\beta \in \tilde{X}$. Then, we get:

$$A_\beta = \{x_m + \varepsilon_x(\beta), y_m + \varepsilon_y(\beta) : 0 \le \varepsilon_x(\beta) < m^{-1}, 0 \le \varepsilon_y(\beta) < m^{-1}\}$$

The permutations $(i=1, 2)$ on $\beta \in \tilde{X}$ are:

$$\delta_1(\beta) = \{x_m + \varepsilon_x(\beta), x_m + y_m + \varepsilon_y(\beta)\}$$
$$\delta_2(\beta) = \{x_m + \varepsilon_x(\beta), x_m + y_m + m^{-1} + \varepsilon_y(\beta)\}$$

If $\alpha \in X - \sigma_1^{-1}(X)$ and $\alpha \in X - \sigma_2^{-1}(X)$, these imply

$$\alpha \in X \;\; and \;\; \left\{ \begin{array}{l} \alpha \in \sigma_1^{-1}(X) \leftrightarrow \sigma_1(\alpha) \notin X \\ \alpha \notin \sigma_2^{-1}(X) \leftrightarrow \sigma_2(\alpha) \notin X \end{array} \right\}, \text{ and}$$

$$\beta \in \tilde{X} \;\; and \;\; \left\{ \begin{array}{l} \delta_1(\beta) \notin \tilde{X} \leftrightarrow \beta \notin \delta_1^{-1}(\tilde{X}) \\ \delta_2(\beta) \notin \tilde{X} \leftrightarrow \beta \notin \delta_2^{-1}(\tilde{X}) \end{array} \right\}$$

Thus, above relationship results in $\beta \in \tilde{X} - \delta_1^{-1}(\tilde{X})$ and $\beta \in \tilde{X} - \delta_2^{-1}(\tilde{X})$. However, the overlapped range can exist in $\delta_2(\beta_1)$ *and* $\delta_1(\beta_2)$, which should be removed such that:

$$\mu\lfloor(\tilde{X}\backslash\delta_1^{-1}(\tilde{X})) \cup (\tilde{X}\backslash\delta_2^{-1}(\tilde{X}))\rfloor = \mu\lfloor\tilde{X}\backslash\delta_1^{-1}(\tilde{X})\rfloor + \mu\lfloor\tilde{X}\backslash\delta_2^{-1}(\tilde{X})\rfloor - \mu\lfloor\tilde{X}\backslash\delta_1^{-1}(\tilde{X})\rfloor \cap \mu\lfloor\tilde{X}\backslash\delta_2^{-1}(\tilde{X})\rfloor \quad (13)$$

By Definition:

$$\mu(\tilde{X}) = |X|/n \text{ and } \mu(\tilde{X}^c) = |X^c|/n \quad (14)$$

Therefore, (13) can be written by (14):

$$\frac{1}{n}\,(|X-\sigma_1^{-1}(X)| + |X-\sigma_2^{-1}(X)| - |Y_i'|),$$

where $Y_i' = (X\backslash\sigma_{2i-1}^{-1}(X)) \cap (X\backslash\sigma_{2i}^{-1}(X))$.

If similar method can be applied to δ_3 and δ_4:

$$\bigcup_{i=1}^{4} \mu\{\tilde{X}\backslash\delta_i^{-1}(\tilde{X})\} = \frac{1}{n}\sum_{i=1}^{2} [|\sigma_{2i-1}(X)-X| + |\sigma_{2i}(X)-X| - |Y_i|]$$

If $\hat{X}$ is substituted for X on Lemma 4.4 with $\lambda = 1$, we obtain: $\sum_{i=1}^{4} \mu(\tilde{X} - \delta_i^{-1}(\tilde{X})) \ge 2d\mu(\tilde{X})\mu(\tilde{X}^c)$

Finally, it is simplified to:

$$\sum_{i=1}^{2} [|\sigma_{2i-1}(X)-X| + |\sigma_{2i}(X)-X| - |Y_i|] \ge 2d\mu|\tilde{X}||\tilde{X}^c|/n$$ □

Proof of Theorem 3.2: For $i = 1, 2$, the relationship between permutations $\sigma_{2i-1}(X)$, $\sigma_{2i}(X)$ (σ_{2i-1}, σ_{2i}: for simplicity) and Γ_x is $\Gamma_x \supseteq X \cup \sigma_{2i-1} \cup \sigma_{2i}$. If rewrite,

$$\Gamma_x\backslash X \supseteq (\sigma_{2i-1}\backslash X) \cup (\sigma_{2j}\backslash X)$$
$$\Gamma_x\backslash X \supseteq [(\sigma_{2i-1}\backslash X) \cup (\sigma_{2j}\backslash X)]\backslash X[(\sigma_{2i-1}\backslash X) \cap (\sigma_{2j}\backslash X)]$$

By using Lemma 4.5, we have for an i and every subset

$$X \subseteq A_{m\times m}: |\sigma_{2i-1} - X| + |\sigma_{2i} - X| - |Y_i| \ge d|X|(1-|X|/n)$$

From these inequalities, we get:

$$|\Gamma_x - X| \ge |\sigma_{2i-1} - X| + |\sigma_{2i} - X| - |Y_i| \ge d|X|(1-|X|/n)$$
$$|\Gamma_x| \ge [1 + d(1-|X|/n)]|X|$$ □

5. Bound on Superconcentrators

In order to construct a superconcentrator from concentrator, we build a network with n inputs and n outputs, with a direct connection from each input to a corresponding output. To superconcentrate a set of inputs A to a set of outputs B where $|A|=|B|$, connect any inputs in A to any output in B that happens to be linked by the direct connection. If $|A|>n/2$, then at most $n/2$ of these inputs will fail to link using the direct connection. These are then passed through an (n, θ, k) concentrator, while on the output side a mirror image structure feeds the outputs. Between these two structures, a recursion of the entire superconcentrator structure is implemented, but with θn inputs and θn outputs. This structure is illustrated in (Fig. 2). The total hardware cost $S(n)$ of this structure, in terms of the number of links, is given by:

$$S(n)=n+2kn+S(\theta n) \tag{15}$$

or, after solving the recursion (15) (ignoring the minor impact of restrictions on the number of inputs to the concentrators):

$$S(n)/n=(2k+1)/(1-\theta) \tag{16}$$

Theorem 5.1[7]: Assume an $(n, k, 2/(p-1))$ expander exists. Then, the linear order superconcentrators with $[(2k+3)p+1]n$ edges are able to be built for every n.

Gabber and Galil construct a obvious family of linear superconcentrators with (16) and Theorem 5.1 for an $(n, \theta_n, k, 1/2)$ bounded concentrator as shown in (Fig. 2). For example, applying this formula to Pippenger's $(n, 2/3, 6, 1/2)$ bounded concentrator, that proved its nonconstructive existence, yields $S(n)=39$ n. It is simple recursive construction that $39n$ edges drive a family of linear superconcentrators. Therefore, by applying an $(n, 5, 1-\sqrt{3}/2)$ expander, proved on Theorem 3.2, to Gabber and Galil's construction, we get better result of density in superconcentrator.

The expansion value, Gabber and Galil ($d=0.0670$), Jimbo and Maruoka ($d=0.1161$), and the improved value($d=1340$), directly impacts the density of the resulting superconcentrator according to the number of permutation. Using the formula developed by Gabber and Galil[7], they found that for an expander structure with $k=5$, and $d=(2-\sqrt{3})/4$, $p=31$ and the resulting superconcentrator has density:

$$S(n)/n=(2k+3)p+1=(2\cdot5+3)\cdot31+1=404$$

Using the improved value($d=1-\sqrt{3}/2$, $p=16$), it can be obtained that:

$$S(n)/n=(2k+3)p+1=(2\cdot5+3)\cdot16+1=209$$

Gabber and Galil also developed an expander with $k=7$ and $d=(2-\sqrt{3})/2$, for which they found $p=16$, and $S(n)/n=273$. Using the new value, we found:

$$2/(p-1)\le 2d \Rightarrow p\ge 8.46$$

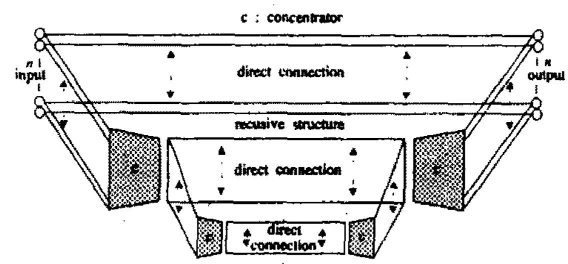
Therefore, if we take $p=9$ since p is the smallest natural number, the density is:

$$S(n)/n=(2k+3)p+1=(2\cdot7+3)\cdot9+1=154$$

Furthermore these comparisons can be applied to in a more recent result of Alon and Galil[2]. The expansion of $\widetilde{T}_n$ is approximately $0.411\cdots$. If our new concentration coefficient is applied for $\widetilde{T}_n$, we can get it as follows:

$$2/(p-1)\le 0.411 \Rightarrow p\ge 5.886. \tag{17}$$

By choosing p from eq.(17), the density is:



(Fig. 2) A recursive structure of superconcentrator

$$S(n)/n=(2k+3)p+1=(2\cdot 7+3)\cdot 6+1=103,$$

while they found the density is 122.74⋯.

6. Conclusions

This paper proposes a new formula to find expansion constant; which improves the density of the concentratorm and superconcentrator composed of expander. The better result is important only if it projected to general routing scheme and to the well-known concentrators such as Pippenger's network. Furthermore, we might have such a valuable result, only if our approach to an expansion constant through hypothesis which is represented by analyzing the distribution of output can be proved theoretically. It, however, remains to be proven or disproved whether the improved bound is the best possible construction of a concentrator using an expander or not. Supposed that an expander on the hardware complexity of the Pippenger structure can be explicitly constructed, this improvement will have useful importance in switching systems. For instance, the asynchronous transfer mode switching system promises to be the ultimate on-premise internetworking technology. Its high-bandwidth uniform switching can transfer graphics, audio, video, and text from application to application at much higher speeds than now available.

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