Comm. Korean Math. Soc. **13** (1998), No. 2, pp. 435–443

THE CATALAN'S CONSTANT AND SERIES INVOLVING THE ZETA FUNCTION

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ABSTRACT. Some mathematical constants have been used in evaluating series involving the Zeta function, the origin of which can be traced back to an over two centuries old theorem of Christian Goldbach. We show some of the series involving the Zeta function can be evaluated in terms of the Catalan's constant by using the theory of the double Gamma function.

1. Introduction and definitions

The Catalan's constant $\mathbf{G}$ is defined by

$$\mathbf{G} := \frac{1}{2}\int_0^1 \mathbf{K}(k)dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.915965\ldots, \tag{1.1}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind given by

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

The double Gamma function was defined and studied by Barnes [3-5] and others in about 1900, not appearing in the tables of the most well-known special functions, but cited in the exercise by Whittaker and Watson [23, p. 264]. Recently this function has been revived in the study of determinants of Laplacians [8, 15, 21, 22]. Shintani [17] also used this function to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou-Noguès [7] for the p-adic L-functions at the point 0.

Received July 3, 1997. Revised January 13, 1998.
1991 Mathematics Subject Classification: Primary 33B99, Secondary 11M99.
Key words and phrases: the Catalan's constant, series involving the Zeta function.

Before Barnes, these functions had been introduced under a different form by Alexeiewsky [2], Hölder [12] and Kinkelin [14].

Barnes [3] gave an explicit Weierstrass canonical product form of the double Gamma function $\Gamma_2 = 1/G$:

$$(1.2)\qquad \begin{aligned}\{\Gamma_2(z+1)\}^{-1} &= G(z+1)\\ &= (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}[(1+\gamma)z^2+z]} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}},\end{aligned}$$

where γ is the Euler-Mascheroni constant defined by

$$(1.3)\qquad \gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577215664\ldots.$$

The double Gamma function and the Gamma function satisfy the following relations:

$$(1.4)\qquad \begin{aligned}&\Gamma(1) = 1 \qquad \text{and} \qquad G(1) = 1\\ &\Gamma(z+1) = z\Gamma(z) \qquad \text{and} \qquad G(z+1) = \Gamma(z)G(z) \quad \text{for} \quad z \in \mathbf{C},\end{aligned}$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is

$$(1.5)\qquad \{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}.$$

The Stirling's formula for the G-function: For sufficiently large real x and $a \in \mathbf{C}$,

$$(1.6)\qquad \begin{aligned}\log G(x+a+1) &= \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax\\ &+ \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(1/x),\end{aligned}$$

where A is Glaisher's (or Kinkelin's) constant defined by

$$(1.7)\qquad \log A = \lim_{n\to\infty} \log(1^1 2^2 \cdots n^n) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4},$$

the numerical value of A being $1.282427130\cdots$. It is also known that [3]

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}} \text{ and } G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}}\pi^{-\frac{1}{4}}e^{\frac{1}{8}}A^{-\frac{3}{2}}. \tag{1.8}$$

Note [1, p. 189, Eq. (11)] that the partial fraction expansion for $\pi z \cot \pi z$ is given by

$$\pi z \cot \pi z = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2}; \tag{1.9}$$

and we find [1, p. 199, Eq. (30)] that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.10}$$

The Riemann Zeta function $\zeta(s)$ is defined, when $\mathrm{Re}(s) > 1$, by (see Titchmarsh [20] and Ivić [13])

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}. \tag{1.11}$$

Indeed it is meromorphic everywhere in the complex s-plane with a simple pole at $s = 1$ (with residue 1).

More recently Choi *et al.* ([10], [11]) showed that the theory of the double Gamma function turned out to be useful in evaluating some series involving the Zeta function, the origin of which can be traced back to an over two centuries old theorem of Christian Goldbach (1690-1764) as noted in Srivastava [19]. The Kinkelin's constant A was used in evaluating some series involving the Zeta function: For example,

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k+2} = -\frac{1}{2}\log\left(\frac{\pi}{2}\right) + \frac{\gamma}{3} - \frac{1}{3} + 2\log A, \tag{1.12}$$

which was shown by Choi *et al.* [10, p. 391, Eq. (2.31)];

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)} 2^{-k} = -\frac{\gamma}{4} + \frac{7}{12}\log 2 + \frac{1}{2}\log \pi - 3\log A, \tag{1.13}$$

$$\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{(k+1)(k+2)} = \frac{1}{2} + \frac{\gamma}{6} - 2\log A, \tag{1.14}$$

which were shown by Choi *et al.* ([11, p. 109, Eq. (2.23); p. 116, Eq. (2.63)]).

In this paper we evaluate some series involving the Zeta function in terms of the Catalan's constant, the results of which are presumably new since the author can not find any possible literature using the Catalan's constant on this subject.

2. Some special values of the G-function

We first give the following integral formula:

$$\int_0^z \pi t \cot \pi t dt = z \log(2\pi) + \log \frac{G(1-z)}{G(1+z)}, \tag{2.1}$$

which is originally due to Kinkelin. Indeed, it follows from (1.2) that

$$\Phi(z) := \frac{G(1+z)}{G(1-z)} = (2\pi)^z e^{-z} \prod_{k=1}^{\infty} \left[\left(\frac{1+\frac{z}{k}}{1-\frac{z}{k}} \right)^k e^{-2z} \right].$$

Taking logarithmic derivative of the resulting equation with respect to z, we get

$$\frac{d}{dz} \log \Phi(z) = \log(2\pi) - 1 - 2 \sum_{k=1}^{\infty} \frac{z^2}{z^2 - k^2}.$$

Combining Eq. (1.9) and the resulting equation, we obtain

$$\frac{d}{dz} \log \Phi(z) = \log(2\pi) - \pi z \cot \pi z.$$

Integrating and determining the constant of integration by the fact that $\Phi(0) = 1$, we get the desired equation (2.1).

Considering the relation

$$x \frac{d}{dx} \log \frac{\sin x}{x} = x \cot x - 1$$

and using Eq. (2.1), we obtain

$$\int_0^z \log \sin \pi t \, dt = z \log \frac{\sin \pi z}{2\pi} + \log \frac{G(1+z)}{G(1-z)}, \tag{2.2}$$

which was (with correction) recorded by Barnes [3, p. 279].

We now give two special values of the G-function. The following special value of the Γ-function is known (see [18, p. 1]):

$$\Gamma(1/4) \cong 3.62560\,99082\,21908 \ldots . \tag{2.3}$$

Setting $z = 1/4$ in Eq. (2.2) and considering (1.4) and the following known integral [6, Table 285, Eq. 1]:

$$\int_0^{\frac{\pi}{4}} \log \sin x \, dx = -\frac{\pi}{4} \log 2 - \frac{1}{2}\mathbf{G}, \tag{2.4}$$

we obtain

$$G(3/4) = 2^{-\frac{1}{8}} \pi^{-\frac{1}{4}} e^{\frac{\mathbf{G}}{2\pi}} \Gamma(1/4) G(1/4). \tag{2.5}$$

Recall a duplication formula for the G-function [9, p. 290]:

$$G(a)G(a + 1/2)^2 G(a+1) = e^{\frac{1}{4}} A^{-3} 2^{-2a^2+3a-11/12} \pi^{a-1/2} G(2a). \tag{2.6}$$

Setting $a = 1/4$ in Eq. (2.6) and using Eqs. (1.4) and (1.8), we obtain

$$\Gamma(1/4) G(1/4)^2 G(3/4)^2 = 2^{-\frac{1}{4}} \pi^{-\frac{1}{2}} e^{\frac{3}{8}} A^{-\frac{9}{2}}. \tag{2.7}$$

Combining Eq. (2.5) and Eq. (2.7), we obtain

$$G(1/4) = e^{\frac{3}{32} - \frac{\mathbf{G}}{4\pi}} A^{-\frac{9}{8}} \Gamma(1/4)^{-\frac{3}{4}} \cong 0.293756 \ldots ; \tag{2.8}$$

or, equivalently,

$$G(3/4) = 2^{-\frac{1}{8}} \pi^{-\frac{1}{4}} e^{\frac{3}{32} + \frac{\mathbf{G}}{4\pi}} A^{-\frac{9}{8}} \Gamma(1/4)^{\frac{1}{4}} \cong 0.848718 \ldots . \tag{2.9}$$

It follows from Eqs. (2.8), (2.9) and (1.4) that

$$\frac{G(3/4)}{G(5/4)} = 2^{-\frac{1}{8}} \pi^{-\frac{1}{4}} e^{\frac{\mathbf{G}}{2\pi}}. \tag{2.10}$$

Setting $z = 1/4$ in Eq. (1.10), we obtain

$$\Gamma(3/4) = \sqrt{2} \cdot \pi \cdot \Gamma(1/4)^{-1}. \tag{2.11}$$

3. Some series involving the Zeta function

As noted in Srivastava [19], this subject can be traced back to an over two centuries old theorem of Christian Goldbach (1690 – 1764) which has recently been posed as the following problem by Shallit and Zikan [16]: Show that

$$\sum_{\omega \in S} (\omega - 1)^{-1} = 1$$

the sum being extended over all members ω of S:

$$S = \{n^k \mid n \geq 2, \ \ k \geq 2\} = \{4, 8, 9, 16, 25, 27, 32, 36, \ldots\}.$$

Indeed, in terms of the Riemann ζ-function, this problem becomes

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1,$$

which can be found in various literatures (cf. Srivastava [19, p. 2]). From that time on, many mathematicians have developed lots of series involving the Zeta function. Srivastava [19] and others are concerned with this subject rather extensively. Zhang [24] evaluated some infinite series involving the Riemann Zeta function by using Maclaurin summation formula.

For further references on this subject, see Srivastava [19] and Choi *et al.* ([10], [11]). In this section, using only our results already developed, we evaluate some more series involving the Zeta function in terms of the Catalan's constant. For this purpose we summarize known results which were shown by Choi *et al.* [11]: For $|z| < 1$,

$$\text{(3.1)} \quad \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} z^{k+1} = [1 - \log(2\pi)] \frac{z}{2} + (1+\gamma) \frac{z^2}{2} + \log G(z+1);$$

(3.2)

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k(k+1)} z^{k+1} = \{\log(2\pi) - 1\} \frac{z}{2} + (\gamma - 1) \frac{z^2}{2} + z \log \Gamma(1+z) - \log G(1+z), \qquad |z| < 1;$$

(3.3)
$$\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)-1}{k(k+1)}z^{k+1} = \{\log(2\pi)-3\}\frac{z}{2}+(\gamma-2)\frac{z^2}{2} + (z+1)\log\Gamma(z+2)-\log G(z+2), \quad |z|<2;$$

$$(3.4)\quad \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+1)}z^{2k+1} = \{\log(2\pi)-1\}z+z\log\Gamma(1+z)\Gamma(1-z) + \log\frac{G(1-z)}{G(1+z)}, \qquad |z|<1;$$

(3.5)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k(2k+1)}z^{2k+1} = \{\log(2\pi)-3\}z+(z+1)\log\Gamma(2+z) + (z-1)\log\Gamma(2-z)+\log\frac{G(2-z)}{G(2+z)}, \qquad |z|<2.$$

Setting $z=1/4$ and $z=3/4$ in Eqs. (3.1)-(3.5) and using Eqs. (1.4) and (2.8)-(2.11), we obtain

$$(3.6)\quad \sum_{k=2}^{\infty}(-1)^k2^{-2k}\frac{\zeta(k)}{k+1} = 1+\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}-\frac{1}{2}\log\left(2\pi\cdot A^9\cdot\Gamma(1/4)^{-2}\right);$$

(3.7)
$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)}{k+1} = \frac{3}{4}+\frac{9}{32}\gamma+\frac{\mathbf{G}}{4\pi}+\frac{1}{8}\log\left(\pi^3\cdot A^{-9}\cdot\Gamma(1/4)^{-6}\right);$$

$$(3.8)\quad \sum_{k=2}^{\infty}(-1)^k2^{-2k}\frac{\zeta(k)}{k(k+1)} = -1+\frac{\gamma}{8}+\frac{\mathbf{G}}{\pi}+\frac{1}{2}\log\left(2^{-3}\cdot\pi\cdot A^9\right);$$

(3.9)
$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)}{k(k+1)} = -\frac{3}{4}+\frac{9}{32}\gamma-\frac{\mathbf{G}}{4\pi}+\frac{1}{8}\log\left(2^{-9}\cdot3^6\cdot\pi^3\cdot A^9\right);$$

(3.10)
$$\sum_{k=2}^{\infty}(-1)^k 2^{-2k}\frac{\zeta(k)-1}{k(k+1)} = -\frac{17}{8}+\frac{\gamma}{8}+\frac{\mathbf{G}}{\pi}+\frac{1}{2}\log\left(2^{-23}\cdot 5^{10}\cdot\pi\cdot A^9\right);$$

(3.11)
$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)-1}{k(k+1)} = -\frac{57}{32}+\frac{9}{32}\gamma-\frac{\mathbf{G}}{4\pi} + \frac{1}{8}\log\left(2^{-37}\cdot 3^6\cdot 7^{14}\cdot\pi^3\cdot A^9\right);$$

(3.12)
$$\sum_{k=1}^{\infty}2^{-4k}\frac{\zeta(2k)}{k(2k+1)} = -1+\frac{2\mathbf{G}}{\pi}+\log\left(\frac{\pi}{2}\right);$$

(3.13)
$$\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{2k}\frac{\zeta(2k)}{k(2k+1)} = -1-\frac{2\mathbf{G}}{3\pi}+\log\left(\frac{3\pi}{2}\right);$$

(3.14)
$$\sum_{k=1}^{\infty}2^{-4k}\frac{\zeta(2k)-1}{k(2k+1)} = -3+\frac{2\mathbf{G}}{\pi}+\log\left(2^{-5}\cdot 3^{-3}\cdot 5^5\cdot\pi\right);$$

(3.15)
$$\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{2k+1}\frac{\zeta(2k)-1}{k(2k+1)} = -\frac{9}{4}-\frac{\mathbf{G}}{2\pi}+\frac{1}{4}\log\left(2^{-15}\cdot 3^3\cdot 7^7\cdot\pi^3\right).$$

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