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$M_1,M_2/M/1$ RETRIAL QUEUEING SYSTEMS WITH TWO CLASSES OF CUSTOMERS AND SMART MACHINE

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ABSTRACT. We consider $M_1,M_2/M/1$ retrial queues with two classes of customers in which the service rates depend on the total number of the customers served since the beginning of the current busy period. In the case that arriving customers are blocked due to the channel being busy, the class 1 customers are queued in the priority group and are served as soon as the channel is free, whereas the class 2 customers enter the retrial group in order to try service again after a random amount of time. For the first N ($N \geq 1$) exceptional services model which is a special case of our model, we derive the joint generating function of the numbers of customers in the two groups. When $N = 1$ i.e., the first exceptional service model, we obtain the joint generating function explicitly and if the arrival rate of class 2 customers is 0, we show that the results for our model coincide with known results for the M/M/1 queues with smart machine.

1. Introduction

Retrial queueing systems are characterized by the feature that arriving customers who find the server busy join the retrial group to try again for their requests in random order and at random intervals. Retrial queues have been widely used to model many problems in telephone switching systems, computer and communication systems. For comprehensive surveys of retrial queues with one type of customers, see Yang and Templeton[5] and Falin[3]. For survey of retrial queues with two classes of customers, see Choi and Chang[1].

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In this paper, we consider M_1,M_2/M/1 retrial queues with two classes of customers in which the service rates depend on the number of the customers served since the beginning of the current busy period. Such a system is called queues with a smart machine. In the case that arriving customers are blocked due to the channel being busy, the class 1 customers are queued in the priority group and are served as soon as the channel is free, whereas the class 2 customers enter the retrial group in order to try service again after a random amount of time. All customers in the retrial group attempt to get service to the server directly and return to the retrial group when the server is busy, so that they will be served only when there are no customers in the priority group. This rule implies that customers in the priority group have non-preemptive priority over those in the retrial group. The arrival processes are assumed to be independent of service times in the system. Li et al.[4] studied M/M/1 queues with smart machine.

This paper is organized as follows. In the section 2, we describe the mathematical queueing model in detail. For the first N ($N \geq 1$) exceptional services model which is a special case of our model, we derive the joint generating function of the numbers of customers in the two groups. In the section 3, we obtain the joint generating function explicitly when $N = 1$ i.e., the first exceptional service model. Finally, if the arrival rate of class 2 customers is 0, we show that the results for our model coincide with known results for the M/M/1 queues with smart machine[4].

2. Joint probabilities of queue lengths

We consider M_1,M_2/M/1 retrial queues with two classes of customers in which the service rates depend on the number of the customers served since the beginning of the current busy period. The busy period is defined the duration to begin with the arrival of a customer to an idle server and to end the server becomes idle. Let μ_k be the service rate when k customers have been served since the beginning of the current busy period.

Class 2 customers arrive at the system according to a Poisson process with rate λ_2. If a class 2 customer upon arrival finds the server free, he immediately occupies the server and leaves the system after service. If he finds the server busy on his arrival, he enters the retrial group in order to seek service again after a random amount of time. He persists this way until he succeeds the connection. The retrial time (the time interval between two successive attempts made by a cus tomer in the retrial group) is exponentially distributed with mean $1/\nu$ and is independent of all previous retrial times and all other stochastic processes in the system.

Class 1 customers arrive at the system according to a Poisson process with rate λ_1. They are queued in the priority group after blocking and then are served in accordance with some discipline such as FIFO or random order. As soon as the server is free, a class 1 customer occupies the server immediately, so class 2 customers in the retrial group will be served only when there are no class 1 customers in the priority group. According to the above rule class 1 customers in t he priority group have non-preemptive priority over class 2 customers.

At an arbitrary time, the steady state of the system can be characterized by the random variables;

N_1 = the number of class 1 customers in priority group (including the customer in service),

N_2 = the number of class 2 customers in retrial group,

S = the number of customers served since the beginning of the current busy period.

Define the probabilities;

$q_j = P\{N_1 = 0, N_2 = j\}, j \geq 0,$

$p_{ijk} = P\{N_1 = i, N_2 = j, S = k\}, i \geq 1, j \geq 0, k \geq 0.$

Note that q_j is the steady state probability that there are 0 customer in the priority group, j customers in the retrial group and the server is idle, and p_{ijk} is the steady state probability that there are $i-1$ customers in the priority group, j customers in the retrial group, k customers have been served since the beginning of the current busy period and the server is busy. Then q_j and p_{ijk} satisfy the following balance equations, for $j \geq 0$, $k \geq 1$,

(2.1a) $$(\lambda + j\nu)q_j = \sum_{n=0}^{\infty} \mu_n p_{1jn}, \ j \geq 0,$$

(2.1b) $$(\lambda + \mu_0)p_{1j0} = \lambda q_j + (j+1)\nu q_{j+1} + \lambda_2 p_{1,j-1,0},$$

(2.1c) $(\lambda+\mu_0)p_{ij0} = \lambda_1 p_{i-1,j0} + \lambda_2 p_{i,j-1,0},\ i \geq 2,$

(2.1d) $(\lambda+\mu_k)p_{ijk} = \lambda_1 p_{i-1,jk} + \lambda_2 p_{i,j-1,k} + \lambda_2 p_{i,j-1,k} + \mu_{k-1} p_{i+1,j,k-1},\ i \geq 1,$

where $\lambda = \lambda_1 + \lambda_2$, $p_{0jk} = 0$, $p_{i,-1,k} = 0$ and $p_{ij,-1} = 0$. Multiplying equations (2.1) by z_2^j and summing over j, we have the following equations, for $k \geq 1$,

(2.2a) $\lambda Q(z_2) + \nu z_2 Q'(z_2) = \sum_{n=0}^{\infty} \mu_n P_{1n}(z_2),$

(2.2b) $(\lambda+\mu_0-\lambda_2 z_2)P_{10}(z_2) = \lambda Q(z_2) + \nu Q'(z_2),$

(2.2c) $(\lambda+\mu_0-\lambda_2 z_2)P_{i0}(z_2) = \lambda_1 P_{i-1,0}(z_2),\ i \geq 2,$

(2.2d) $(\lambda+\mu_k-\lambda_2 z_2)P_{ik}(z_2) = \lambda_1 P_{i-1,k}(z_2) + \mu_{k-1} P_{i+1,k-1}(z_2),\ i \geq 1,$

where for $|z_2| \leq 1$

$$Q(z_2) = \sum_{j=0}^{\infty} q_j z_2^j,$$

$$P_{ik}(z_2) = \sum_{j=0}^{\infty} p_{ijk} z_2^j,\ i \geq 1,\ k \geq 0.$$

Next we introduce the generating function of $P_{ik}(z_2)$

$$P_k(z_1, z_2) = \sum_{i=1}^{\infty} P_{ik}(z_2) z_1^i,\ k \geq 0.$$

Note that $P_k(z_1,z_2) = E[z_1^{N_1} z_2^{N_2}\ ;\ S = k]$ is the joint generating function of (N_1, N_2) when the number of customers served since the beginning of the current busy period is k. Multiplying equations (2.2b), (2.2c) and (2.2d) by z_1^i respectively and summing over i, we have

(2.3a) $(\lambda+\mu_0-\lambda_1 z_1-\lambda_2 z_2)P_0(z_1,z_2) = \lambda z_1 Q(z_2) + \nu z_1 Q'(z_2),$

(2.3b) $(\lambda+\mu_k-\lambda_1 z_1-\lambda_2 z_2)P_k(z_1,z_2)$

$$= \mu_{k-1}\left(\frac{P_{k-1}(z_1,z_2)}{z_1} - P_{1,k-1}(z_2)\right),\ k \ge 1.$$

In the remainder of this paper, we consider a special case in which the service rates are same after fixed number of customers have been served in the current busy period. That is, there is a positive integer $N \ge 1$ such that $\mu_k = \mu_N$ for $n \ge N$. Rewrite (2.3) as

$$(2.4a)\quad (\lambda + \mu_0 - \lambda_1 z_1 - \lambda_2 z_2)P_0(z_1,z_2) = \lambda z_1 Q(z_2) + \nu z_1 Q'(z_2),$$

$$(2.4b)\quad (\lambda + \mu_k - \lambda_1 z_1 - \lambda_2 z_2)P_k(z_1,z_2) = \mu_{k-1}\left(\frac{P_{k-1}(z_1,z_2)}{z_1} - P_{1,k-1}(z_2)\right),\ 1 \le k \le N,$$

$$(2.4c)\quad (\lambda + \mu_N - \lambda_1 z_1 - \lambda_2 z_2)P_k(z_1,z_2) = \mu_N\left(\frac{P_{k-1}(z_1,z_2)}{z_1} - P_{1,k-1}(z_2)\right),\ k > N.$$

Denote

$$\bar{P}_N(z_1,z_2) = \sum_{k=N}^{\infty} P_k(z_1,z_2).$$

Summing (2.4b) and (2.4c) over k and using (2.2a), we obtain

$$(2.5)\quad \begin{aligned}(\lambda + \mu_N - \lambda_1 z_1 - \lambda_2 z_2)\bar{P}_N(z_1,z_2) &= \frac{\mu_{N-1}}{z_1}P_{N-1}(z_1,z_2) \\ &+ \frac{\mu_N}{z_1}\bar{P}_N(z_1,z_2) + \sum_{k=0}^{N-2}\mu_k P_{1k}(z_2) - \lambda Q(z_2) - \nu z_2 Q'(z_2).\end{aligned}$$

From (2.2), we see that

$$(2.6)\quad P_{1k}(z_2) = f_{1k}(z_2)Q(z_2) + f_{2k}(z_2)Q'(z_2)$$

and $f_{ik}(z_2)$ $(i = 1,2,\ k \ge 0)$ can be determined recursively. By substituting (2.6) into (2.2a), we obtain differential equation for $Q(z_2)$

$$\left(\lambda - \sum_{k=0}^{\infty}\mu_k f_{1k}(z_2)\right)Q(z_2) + \left(\nu z_2 - \sum_{k=0}^{\infty}\mu_k f_{2k}(z_2)\right)Q'(z_2) = 0$$

whose solution is

$$Q(z_2) = c \cdot \exp\left(\int_{z_2}^{1} \frac{\lambda - \sum_{k=0}^{\infty} \mu_k f_{1k}(x)}{\nu x - \sum_{k=0}^{\infty} \mu_k f_{2k}(x)} \, dx \right), \tag{2.7}$$

where c is a constant.

The rest of this section is devoted to obtain the solution $P_k(z_1, z_2)$ from (2.4), (2.5) and (2.6) in closed form. Thus, $P_k(z_1, z_2)$ $(1 \le k \le N-1)$ and $\bar{P}_N(z_1, z_2)$ can be expressed as

$$P_k(z_1, z_2) = \bar{f}_{1k}(z_1, z_2)Q(z_2) + \bar{f}_{2k}(z_1, z_2)Q'(z_2), \tag{2.8a}$$

$$\bar{P}_N(z_1, z_2) = \bar{f}_{1N}(z_1, z_2)Q(z_2) + \bar{f}_{2N}(z_1, z_2)Q'(z_2). \tag{2.8b}$$

From (2.4), (2.5) and (2.6), $\bar{f}_{ik}(z_1, z_2)$ $(1 \le k \le N)$ satisfy the following system equation

$$A(z_1, z_2)X_i(z_1, z_2) = b_i(z_1, z_2), \ i = 1, 2,$$

where

$$A(z_1, z_2) = \begin{pmatrix} h_0(z_1, z_2) & 0 & \cdots & 0 & 0 \\ -\frac{\mu_0}{z_1} & h_1(z_1, z_2) & \cdots & 0 & 0 \\ 0 & -\frac{\mu_1}{z_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{N-1}(z_1, z_2) & 0 \\ 0 & 0 & \cdots & -\frac{\mu_{N-1}}{z_1} & h_N(z_1, z_2) - \frac{\mu_N}{z_1} \end{pmatrix},$$

$$X_i(z_1, z_2) = \left[\bar{f}_{i0}(z_1, z_2), \bar{f}_{i1}(z_1, z_2), \cdots \bar{f}_{iN}(z_1, z_2)\right]^T, \ i = 1, 2,$$

$$b_1(z_1, z_2) = \left[\lambda z_1, -\mu_0 f_{10}(z_2), \cdots, -\mu_{N-2} f_{1,N-2}(z_2), \sum_{k=0}^{N-2} \mu_k f_{1k}(z_2) - \lambda\right]^T,$$

$$b_2(z_1, z_2) = \left[\nu z_1, -\mu_0 f_{20}(z_2), \cdots, -\mu_{N-2} f_{2,N-2}(z_2), \sum_{k=0}^{N-2} \mu_k f_{2k}(z_2) - \nu z_2\right]^T,$$

$$h_k(z_1, z_2) = \lambda + \mu_k - \lambda_1 z_1 - \lambda_2 z_2, \ k = 0, 1, \cdots, N.$$

We can solve $\bar{f}_{ik}(z_1,z_2)$ using Cramer's rule. That is,

$$\bar{f}_{ik}(z_1,z_2)=\frac{\det A_{i,k+1}(z_1,z_2)}{\det A(z_1,z_2)},\ i=1,2,\ k=0,1,\cdots,N,$$

where $A_{i,k+1}(z_1,z_2)$ is matrix $A(z_1,z_2)$ with column $k+1$ replaced by the vector $b_i(z_1,z_2)$, $i=1,2$. We have, for $i=1,2,\ k=0,1,\cdots,N-1$,

$$\det A(z_1,z_2)=\left(h_N(z_1,z_2)-\frac{\mu_N}{z_1}\right)\prod_{n=0}^{N-1}h_n(z_1,z_2),$$

$$\det A_{i,k+1}(z_1,z_2)=\left(h_N(z_1,z_2)-\frac{\mu_N}{z_1}\right)\sum_{n=0}^{k}(-1)^{k+n}b_{in}(z_1,z_2)\prod_{i=0}^{n-1}h_i(z_1,z_2)\prod_{j=n}^{k-1}\left(-\frac{\mu_j}{z_1}\right)\prod_{l=k+1}^{N-1}h_l(z_1,z_2),$$

$$\det A_{i,N+1}(z_1,z_2)=\sum_{n=0}^{N}(-1)^{N+n}b_{in}(z_1,z_2)\prod_{i=0}^{n-1}h_i(z_1,z_2)\prod_{j=n}^{N-1}\left(-\frac{\mu_j}{z_1}\right),$$

where

$$b_i(z_1,z_2)\equiv[b_{i0}(z_1,z_2),b_{i2}(z_1,z_2),\cdots,b_{iN}(z_1,z_2)]^T$$

and with convention

$$\prod_{i=n}^{n-1}\left(-\frac{\mu_i}{z_1}\right)=1,\quad \prod_{i=N}^{N-1}h_i(z_1,z_2)=1.$$

Thus, for $i=1,2,\ k=0,1,\cdots,N-1$,

$$\text{(2.9a)}\quad \bar{f}_{ik}(z_1,z_2)=\sum_{n=0}^{k}(-1)^{k+n}\frac{b_{in}(z_1,z_2)}{h_k(z_1,z_2)}\prod_{j=n}^{k-1}\left(-\frac{\mu_j}{z_1h_j(z_1,z_2)}\right),$$

$$\text{(2.9b)}\quad \bar{f}_{iN}(z_1,z_2)=\sum_{n=0}^{N}(-1)^{N+n}\frac{b_{in}(z_1,z_2)z_1}{z_1h_N(z_1,z_2)-\mu_N}\cdot\prod_{j=n}^{N-1}\left(-\frac{\mu_j}{z_1h_j(z_1,z_2)}\right).$$

To find the probability c that the server is idle, we use identity,

$$Q(1) + \sum_{k=0}^{N-1} P_k(1,1) + \bar{P}_N(1,1) = 1. \tag{2.10}$$

By substituting (2.7) and (2.8) into (2.10), we have

$$\begin{aligned} \frac{1}{c} =& 1 + \sum_{k=0}^{N-1} \bar{f}_{1k}(1,1) + \bar{f}_{1N}(1,1) \\ &+ \frac{\sum\limits_{k=0}^{\infty} \mu_k f_{1k}'(1)}{\nu - \sum\limits_{k=0}^{\infty} \mu_k f_{2k}'(1)} \left[\sum_{k=0}^{N-1} \bar{f}_{2k}(1,1) + \bar{f}_{2N}(1,1) \right]. \end{aligned}$$

Note that $Q(1) = c$ is the probability that the server is idle and

$$\sum_{k=0}^{N-1} P_k(1,1) + \bar{P}_N(1,1) = 1 - c$$

is the probability that the server is busy.

Thus, we have obtained our main results.

THEOREM 2.1. *The stationary distribution of* (N_1, N_2, S) *is given by the following generating functions*

$$Q(z_2) = c \cdot \exp\left(\int_{z_2}^{1} \frac{\lambda - \sum\limits_{k=0}^{\infty} \mu_k f_{1k}(x)}{\nu x - \sum\limits_{k=0}^{\infty} \mu_k f_{2k}(x)} \, dx \right),$$

$$P_k(z_1, z_2) = \bar{f}_{1k}(z_1, z_2) Q(z_2) + \bar{f}_{2k}(z_1, z_2) Q'(z_2), \ 0 \le k \le N-1,$$

$$\bar{P}_N(z_1, z_2) = \bar{f}_{1N}(z_1, z_2) Q(z_2) + \bar{f}_{2N}(z_1, z_2) Q'(z_2),$$

where

$$\frac{1}{c} = 1 + \sum_{k=0}^{N-1} \bar{f}_{1k}(1,1) + \bar{f}_{1N}(1,1) + \frac{\sum_{k=0}^{\infty} \mu_k f'_{1k}(1)}{\nu - \sum_{k=0}^{\infty} \mu_k f'_{2k}(1)} \left[\sum_{k=0}^{N-1} \bar{f}_{2k}(1,1) + \bar{f}_{2N}(1,1) \right],$$

$$\bar{f}_{ik}(z_1,z_2) = \sum_{n=0}^{k} (-1)^{k+n} \frac{b_{in}(z_1,z_2)}{h_k(z_1,z_2)} \prod_{j=n}^{k-1} \left(-\frac{\mu_j}{z_1 h_j(z_1,z_2)} \right),$$

$$\bar{f}_{iN}(z_1,z_2) = \sum_{n=0}^{N} (-1)^{N+n} \frac{b_{in}(z_1,z_2) z_1}{z_1 h_N(z_1,z_2) - \mu_N} \cdot \prod_{j=n}^{N-1} \left(-\frac{\mu_j}{z_1 h_j(z_1,z_2)} \right).$$

3. Special cases

(a) Consider the case $N = 1$ which is well-known as the first exceptional service model. To determine the function $Q(z_2)$, we shall need the following lemma.

LEMMA 3.1. *For a given $|z_2| < 1$, the following equation in z_1*

$$\lambda + \mu_1 - \lambda_1 z_1 - \lambda_2 z_2 - \frac{\mu_1}{z_1} = 0$$

$$i.e.\quad z_1 - \frac{\mu_1}{\mu_1 + \lambda - \lambda_1 z_1 - \lambda_2 z_2} = 0$$

has exactly one solution $\phi(z_2)$ in the region $\{z \in \mathbb{C} \mid |z| < 1 \text{ or } z = 1\}$.

By substituting $z_1 = \phi(z_2)$ into (2.5), $\bar{P}_N(z_1,z_2)$ is eliminated and we have

$$\lambda Q(z_2) + \nu z_2 Q'(z_2) = \frac{\mu_0 P_0(\phi(z_2), z_2)}{\phi(z_2)}. \tag{3.1}$$

From (3.1) and (2.4a), we obtain

$$\begin{aligned} Q(z_2) &= c \exp\left(\frac{\lambda}{\nu} \int_{z_2}^{1} \frac{\lambda - \lambda_1 \phi(x) - \lambda_2 x}{x(\lambda + \mu_0 - \lambda_1 \phi(x) - \lambda_2 x) - \mu_0} \, dx \right) \\ &= c \exp\left(\frac{\lambda}{\nu} \int_{z_2}^{1} \frac{\mu_1 (1 - \phi(x))}{\mu_1 x (1 - \phi(x)) - \mu_0 \phi(x)(1 - x)} \, dx \right). \end{aligned} \tag{3.2}$$

Substituting (2.9) into (2.8) yields

$$P_0(z_1,z_2) = \frac{\lambda(z_1h_1(z_1,z_2)-\mu_1)Q(z_2)+\nu(z_1h_1(z_1,z_2)-\mu_1)Q'(z_2)}{h_0(z_1,z_2)\left(h_1(z_1,z_2)-\frac{\mu_1}{z_1}\right)},$$

$$\bar{P}_1(z_1,z_2) = \frac{\lambda(\mu_0-h_0(z_1,z_2))Q(z_2)+\nu(\mu_0-z_2h_0(z_1,z_2))Q'(z_2)}{h_0(z_1,z_2)\left(h_1(z_1,z_2)-\frac{\mu_1}{z_1}\right)}.$$

To find c which is the probability that the server is idle, we evaluate the above equations and (3.2) at $z_1=1$, $z_2=1$, then we obtain

$$c = \frac{\mu_0(\mu_1-\lambda_1)-\lambda_2\mu_1}{\mu_0(\mu_1-\lambda_1)+\lambda_1\mu_1}.$$

Since c is the probability that the server is idle, $c>0$ *i.e.* $\frac{\lambda_2\mu_1}{\mu_0(\mu_1-\lambda_1)}<1$ and $\frac{\lambda_1}{\mu_1}<1$.

REMARK. If $\mu_0=\mu_1=\mu$, the model becomes an ordinary M/M/1 retrial queue with two classes of customers and $c=1-\frac{\lambda}{\mu}$[2].

(b) When $\lambda_2=0$, our model becomes the ordinary M/M/1 queue with smart machine. In this case $N_2=0$. Equations (2.8) are reduced to

$$P_k(z_1,z_2) = \bar{f}_{1k}(z_1,0)q_0,$$
$$\bar{P}_N(z_1,z_2) = \bar{f}_{1N}(z_1,0)q_0.$$

From equations (2.9), we obtain

$$\bar{f}_{1k}(z_1,0) = \sum_{n=0}^{k}(-1)^{k+n}\frac{b_{1n}(z_1,0)}{\lambda_1+\mu_k-\lambda_1z_1}\prod_{j=n}^{k-1}\left(-\frac{\mu_j}{z_1(\lambda_1+\mu_j-\lambda_1z_1)}\right),$$

$$\bar{f}_{1N}(z_1,0) = \sum_{n=0}^{N}(-1)^{N+n}\frac{b_{1n}(z_1,0)z_1}{(z_1-1)(\mu_N-\lambda_1z_1)} \cdot\prod_{j=n}^{N-1}\left(-\frac{\mu_j}{z_1(\lambda_1+\mu_j-\lambda_1z_1)}\right).$$

This results agree in Li et al.[4].

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