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SLLN FOR WEIGHTED SUMS OF STOCHASTICALLY DOMINATED PAIRWISE INDEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $\{X_n, n \geq 1\}$ be a sequence of stochastically dominated pairwise independent random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n \neq 0$ and $0 < b_n \uparrow \infty$. A strong law of large numbers of the form $\sum_{j=1}^{n} a_j X_j / b_n \to 0$ almost surely is obtained.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and let$\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying $a_n \neq 0$ and $0 < b_n \uparrow \infty$. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general strong law of large numbers(SLLN) with norming constants $\{b_n, n \geq 1\}$ if the normed weighted sum $\sum_{j=1}^{n} a_j X_j / b_n$ converges to 0 almost surely. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a constant $0 < D < \infty$ such that

$$P(|X_n| > t) \leq DP(|DX| > t) \text{ for } t \geq 0 \text{ and } n \geq 1. \tag{1}$$

Feller[5] proved that

$$\frac{\sum_{j=1}^{n} X_j}{b_n} \to 0 \text{ almost surely}$$

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if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables such that $\sum_{n=1}^{\infty} P(|X_n| > b_n) < \infty$ and $\{b_n, n \geq 1\}$ is a sequence of positive constants with $b_n/n \uparrow \infty$. Rosalsky[6] showed Feller's SLLN under the weaker condition that $\{X_n, n \geq 1\}$ is a sequence of pairwise independent and identically distributed random variables. Adler and Rosalsky[2] generalized Feller's SLLN to the weighted sum of i.i.d. random variables. Furthermore Adler, Rosalsky, and Taylor[3] extended Adler and Rosalsky's theorem to the weighted sum of stochastically dominated independent random variables.

In this paper, we extend Adler, Rosalsky, and Taylor's result to pairwise independent random variables.

2. Main result

To prove the main result we will need the following two lemmas. Lemma 1 is due to Adler and Rosalsky[1].

LEMMA 1. (Adler and Rosalsky[1]). *Let X_0 and X be random variables such that X_0 is stochastically dominated by X in the sense that for some constant $0 < D < \infty$*

$$P(|X_0| > t) \leq DP(|DX| > t) \text{ for } t \geq 0.$$

Then for all $p > 0$

$$E|X_0|^p I(|X_0| \leq t) \leq Dt^p P(|DX| > t) + D^{p+1} E|X|^p I(|DX| \leq t) \text{ for } t \geq 0.$$

The following lemma plays an essential role in our main theorem.

LEMMA 2. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables which are stochastically dominated by a random variable X in the sense that (1) holds for some constant $0 < D < \infty$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying $a_n \neq 0, 0 < b_n \uparrow \infty, b_n/|a_n| \uparrow$, and*

$$\frac{b_n^2}{a_n^2} \sum_{j=n}^{\infty} \frac{a_j^2}{b_j^2} = O(n). \tag{2}$$

If

$$\sum_{n=1}^{\infty} P(|a_n X| > D b_n) < \infty \tag{3}$$

and

$$\frac{\sum_{j=1}^{n} |a_j| E|X_j| I(|a_j X_j| \leq D^2 b_j)}{b_n} \longrightarrow 0, \tag{4}$$

then

$$\frac{\sum_{j=1}^{n} a_j X_j I(|a_j X_j| \leq D^2 b_j)}{b_n} \longrightarrow 0 \text{ almost surely.} \tag{5}$$

PROOF. Set $c_0 = 0, c_n = b_n/|a_n|, n \geq 1$. Let $X'_n = X_n I(|X_n| \leq D^2 c_n), n \geq 1$. For each $k \geq 1$ we define $m_k = \inf\{n : b_n \geq 2^k\}$. Observe that for $m_k \leq n < m_{k+1}$

(6)
$$\frac{|\sum_{j=1}^{n} a_j X'_j|}{b_n} \leq \frac{\sum_{j=1}^{m_{k+1}-1}(|a_j X'_j| - |a_j| E|X'_j|)}{b_{m_k}} + \frac{\sum_{j=1}^{m_{k+1}-1} |a_j| E|X'_j|}{b_{m_k}}.$$

To prove (5), it is enough to show that the right-hand side of (6) converges to 0 almost surely. The second term on the right-hand side of (6) is $o(1)$ by (4). Next it will be shown that the first term on the right-hand of (6) is $o(1)$ almost surely. By the Borel-Cantelli lemma, it suffices to show that for every $\epsilon > 0$

$$\sum_{k=1}^{\infty} P\left(\left|\frac{\sum_{j=1}^{m_{k+1}-1}(|a_j X'_j| - |a_j| E|X'_j|)}{b_{m_k}}\right| > \epsilon\right) < \infty. \tag{7}$$

Using Lemma 1 and the pairwise independence of $\{X_n\}$, we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty} E\Big(\frac{\sum_{j=1}^{m_{k+1}-1}(|a_jX_j'|-|a_j|E|X_j'|)}{b_{m_k}}\Big)^2 \\
&\le \sum_{k=1}^{\infty}\frac{1}{b_{m_k}^2}\sum_{j=1}^{m_{k+1}-1} a_j^2E|X_j'|^2 \\
(8)\qquad &= \sum_{j=1}^{\infty} a_j^2E|X_j'|^2 \sum_{\{k:m_{k+1}-1\ge j\}}\frac{1}{b_{m_k}^2} \\
&\le \frac{16}{3}\sum_{j=1}^{\infty}\frac{1}{c_j^2}E|X_j'|^2 \\
&\le \frac{16D^5}{3}\sum_{j=1}^{\infty}P(|X|>Dc_j)+\frac{16D^3}{3}\sum_{j=1}^{\infty}\frac{1}{c_j^2}EX^2I(|X|\le Dc_j) \\
&= \frac{16D^5}{3}\sum_1 + \frac{16D^3}{3}\sum_2 \quad \text{(say)}.
\end{aligned}
$$

The second inequality of (8) follows from the following:

$$
\begin{aligned}
\sum_{\{k:m_{k+1}-1\ge j\}}\frac{1}{b_{m_k}^2} &= \sum_{k=k_0}^{\infty}\frac{1}{b_{m_k}^2} \le \sum_{k=k_0}^{\infty}\frac{1}{2^{2k}} \\
&= \frac{16}{3}\frac{1}{2^{2(k_0+1)}} < \frac{16}{3b_{m_{k_0+1}-1}^2} \le \frac{16}{3b_j^2},
\end{aligned}
$$

where $k_0 = \min\{k : m_{k+1}-1 \ge j\}$. Note that (3) is equivalent to (see, Chow and Teicher[4], p.117)

$$(9)\qquad \sum_{n=1}^{\infty} nP(Dc_{n-1} < |X| \le Dc_n) < \infty.$$

Now, we see by (3) that $\sum_1$ is finite. Further, by virtue of (2) and (9), we have

$$\sum_2 = \sum_{j=1}^{\infty} \frac{1}{c_j^2} \sum_{n=1}^{j} EX^2 I(Dc_{n-1} < |X| \le Dc_n)$$
$$= \sum_{n=1}^{\infty} EX^2 I(Dc_{n-1} < |X| \le Dc_n) \sum_{j=n}^{\infty} \frac{1}{c_j^2}$$
$$\le CD^2 \sum_{n=1}^{\infty} nP(Dc_{n-1} < |X| \le Dc_n) < \infty,$$

and so (7) follows from (8) and Markov's inequality. □

The main result may now be established. It reduces to Theorem 6 of Adler, Rosalsky, and Taylor[3] when $\{X_n, n \ge 1\}$ is a sequence of independent random variables which are stochastically dominated by a random variable X.

THEOREM 3. *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables which are stochastically dominated by a random variable X in the sense that (1) holds for some constant $0 < D < \infty$. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of constants satisfying $a_n \ne 0, 0 < b_n \uparrow \infty$,*

$$\frac{b_n}{|a_n|} \uparrow, \tag{10}$$

$$\frac{b_n}{n|a_n|} \to \infty, \quad \frac{b_n}{n|a_n|} = O(\inf_{j \ge n} \frac{b_j}{j|a_j|}), \tag{11}$$

and

$$\sum_{j=1}^{n} |a_j| = O(n|a_n|). \tag{12}$$

If (3) holds, then

$$\frac{\sum_{j=1}^{n} a_j X_j}{b_n} \to 0 \text{ almost surely.} \tag{13}$$

PROOF. Note that, by (11), for every $n \geq 1$

$$\frac{b_n^2}{a_n^2}\sum_{j=n}^{\infty}\frac{a_j^2}{b_j^2} = O(n).$$

From the proof of Theorem 6 of Adler, Rosalsky, and Taylor[3]

$$\frac{1}{b_n}\sum_{j=1}^{n}|a_j|E|X_j|I(|a_jX_j| \leq D^2 b_j) \to 0.$$

Hence we have by Lemma 2 that

$$\frac{\sum_{j=1}^{n} a_jX_jI(|a_jX_j| \leq D^2b_j)}{b_n} \to 0 \text{ almost surely,}$$

and so if

$$\frac{\sum_{j=1}^{n} a_jX_jI(|a_jX_j| > D^2b_j)}{b_n} \to 0 \text{ almost surely}$$

then (13) follows. By the Borel-Cantelli lemma, it is enough to show that

$$\sum_{n=1}^{\infty} P(|a_nX_n| > D^2b_n) < \infty. \tag{14}$$

However, the series of (14) converges since by (3)

$$\begin{aligned}&\sum_{n=1}^{\infty} P(|a_nX_n| > D^2b_n)\\ \leq &\sum_{n=1}^{\infty} DP(|a_nX| > Db_n) < \infty.\end{aligned}$$

Thus the proof is complete. □

The next Corollary is an immediate consequence of Theorem 3 by letting $a_n = 1$ for $n \geq 1$. If $\{X_n, n \geq 1\}$ is a sequence of identically distributed random variables, then the condition (1) holds with $X = X_1$ and $D = 1$. Thus, Corollary 4 is an extension of Theorem 2 of Feller[5].

COROLLARY 4. *Let* $\{X_n, n \geq 1\}$ *be as in* Theorem 3. *Let* $\{b_n, n \geq 1\}$ *be a sequence of constants such that* $0 < b_n/n \uparrow \infty$. *If* $\sum_{n=1}^{\infty} P(|X| > Db_n) < \infty$, *then*

$$\frac{\sum_{j=1}^{n} X_j}{b_n} \to 0 \text{ almost surely.}$$

The following theorem shows that the condition (3) in Theorem 3 can be replaced by

$$\sum_{n=1}^{\infty} P(|a_n X| > D_1 b_n) < \infty \tag{15}$$

for some constant $D_1 > 0$. Note that (15) is weaker than (3) when $D < D_1$.

THEOREM 5. *Let* $\{X_n, n \geq 1\}, \{a_n, n \geq 1\}$, *and* $\{b_n, n \geq 1\}$ *be as in* Theorem 3 *except that* (3) *is replaced by* (15). *Then* (13) *holds, that is,*

$$\frac{\sum_{j=1}^{n} a_j X_j}{b_n} \to 0 \text{ almost surely.}$$

PROOF. By Theorem 3, it suffices to show that (3) holds. Let $c_n = b_n/|a_n|$ and $\alpha_n = \inf_{j \geq n} c_j/j$ for $n \geq 1$. It follows from (10) and (11) that $0 < c_n \uparrow \infty, 0 < \alpha_n \uparrow \infty$, and

$$\frac{c_n}{n} \leq C\alpha_n \leq C\frac{c_n}{n} \tag{16}$$

for some constant $C > 0$. Set $m = [\frac{CD_1}{D}] + 1$. Then we have by (16) that

$$D_1 c_n \leq C D_1 n \alpha_n \leq D m n \alpha_n \leq D m n \alpha_{mn} \leq D c_{mn}.$$

Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} P(|X| > D_1 c_n) \\ \geq &\sum_{n=1}^{\infty} P(|X| > D c_{mn}) \\ \geq &\sum_{n=1}^{\infty} P(|X| > D c_{mn+j}) \end{aligned}$$

for $j = 0, 1, \cdots, m-1$. Thus we get by (15) that

$$\begin{aligned}\infty > & m \sum_{n=1}^{\infty} P(|X| > D_1 c_n) \\ & \geq \sum_{j=0}^{m-1} \sum_{n=1}^{\infty} P(|X| > D c_{mn+j}) \\ & = \sum_{n=m}^{\infty} P(|X| > D c_n),\end{aligned}$$

and so (3) holds. □

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