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ON THE MINIMAL HEDGING PORTFOLIOS OF INTEGRAL OPTION

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ABSTRACT. In this paper, we present the close solution for minimal hedging portfolios Π^* when payment f for American option admits the integral option.

1. Introduction

We consider the classical Blach/Scholes model of security market. We suppose that the investor may invest in two assets: bonds and stocks. The price process of a bond $B = (B_t)_{t\geq 0}$ represents time value of money and appreciates at a constant rate $r \geq 0$ called interest rate:

$$dB_t = rB_t dt, \quad B_0 > 0.$$

The price process $S = (S_t)_{t\geq 0}$ of a stock is random in nature. For the simplest model we assume that the stock price is modeled as a geometric Brownian motion with a constant drift $\mu \in \mathbf{R}$ referred to as appreciation rate, and a constant variance $\sigma > 0$ termed volatility coefficient:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0.$$

Here $W = (W_t)_{t\geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0})$. We suppose that filtration $\mathbf{F}$ is generated by the price process of the stock. Moreover, we assume that the stock continuously pays a constant dividend rate $q \in \mathbf{R}$ called *dividend yield*, i.e.,

$$D_t = q \int_0^t S_u du$$

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is the value of dividends gained by the stock holder up to time t. We remind that wealth and consumption portfolio Π is a triplet (β, γ, C) where $\beta = (\beta_t)_{t\geq 0}$ and $\gamma = (\gamma_t)_{t\geq 0}$ are the predictable processes of numbers of bonds and stocks, and $C = (C_t)_{t\geq 0}$ is an adapted increasing right-continuous consumption process. The wealth process $V = (V_t)_{t\geq 0}$ of portfolio Π is equal to

$$V_t = \beta_t B_t + \gamma_t S_t, \quad t \geq 0$$

which evolves with time by the following stochastic differential equation:

$$dV_t = \beta_t dB_t + \gamma_t dS_t + \gamma_t dD_t - dC_t,$$

where $dD_t = qS_t dt$ is the value of dividends received by the stock holder during the interval $(t, t+dt)$. Let $f = (f_t)_{t\geq 0}$ be a nonnegative progressively measurable process on $(\Omega, \mathcal{F}, \mathbf{F})$. We interpret f_t as the payment or the reward of an American option, or as the prescribed lower value for portfolio wealth at time t. The strategy Π is called a *hedging portfolio* for f, if

$$V_t \geq f_t \quad P-a.s., \quad t \geq 0.$$

Moreover, the portfolio Π^* with wealth V^* is called a *minimal* or *optimal hedge* for f, if, whatever $t \geq 0$ and hedging strategies Π,

$$V_t \geq V_t^* \geq f_t \quad P-a.s.$$

Notice that the initial wealth of minimal hedging portfolio gives fair price or premium for corresponding American option.([2])

In this paper, we present the exact formulas for minimal hedging portfolios Π^* when reward process f admits the integral option.

2. The Main Results

Let $\widetilde{P}$ be the dual martingale measure such that

(1) $\widetilde{P}$ is locally equivalent to P

(2) the process $(e^{(r-q)t}/S_t)_{t\geq 0}$ is a $\widetilde{P}$-martingale.

The measure $\widetilde{P}$ is unique and its density process $\widetilde{Z} = (\widetilde{Z}_t)_{t\geq 0}$ with respect to P equals

$$\widetilde{Z}_t = \exp(\frac{r-q-\mu+\sigma^2}{\sigma}W_t - \frac{(r-q-\mu+\sigma^2)^2}{2\sigma^2}t).$$

By Girsanov Theorem ([3]), the process $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$ with

$$\widetilde{W}_t = W_t - (r+\sigma^2-q-\mu)/\sigma$$

is a Wiener process with respect to $\widetilde{P}$.
Let $\widetilde{E}(\cdot)$ be the associated expectation operator with respect to $\widetilde{P}$ and $\mathcal{M}_t$ be the set of stopping times.

We begin with:

LEMMA 1. *Let $f = (f_t)_{t\geq 0}$ be a nonnegative progressively measurable reward process. The wealth process $V^* = (V_t^*)_{t\geq 0}$ of minimal hedging portfolio $\Pi^* = (\beta^*, \gamma^*, C^*)$ is equal to*

$$V_t^* = e^{qt} S_t \, ess \sup_{\tau\in\mathcal{M}_t} \widetilde{E}[e^{-q\tau}\frac{f_\tau}{S_\tau}|\mathcal{F}_t].$$

and the process $\gamma^* = (\gamma_t^*)_{t\geq 0}$ and $C^* = (C_t^*)_{t\geq 0}$ are uniquely defined from

$$e^{-qt}\frac{V_t^*}{S_t} = \frac{V_0^*}{S_0} + B_0\int_0^t \beta_u^* d(\frac{e^{(r-q)u}}{S_u}) - \int_0^t \frac{e^{-qu}}{S_u} dC_u^*,$$
$$\gamma_t^* = \frac{1}{S_t}(V_t^* - \beta_t^* B_t).$$

PROOF. See [4]. □

To prove main result we need the following statement.

LEMMA 2. *Let $g=(g_t)_{t\geq 0}$ and $U=(U_t)_{t\geq 0}$ be nonnegative continuous adapted processes defined on a filtered probability space $(\Omega,\mathcal{F},Q)$. For $s\geq 0$ we denote by*

$$\tau_s(\omega)=\inf\{t\geq s: U_t(\omega)=g_t(\omega)\},\ \omega\in\Omega,$$

the stopping time on $(\Omega,\mathcal{F})$ and assume that, for each $s\geq 0$, the following conditions are satisfied:

(1) $U_t\geq g_t,\ t\geq 0$
(2) *U is a supermartingale*
(3) *the stopped process $U^{\tau_s}=(U_{t\wedge\tau_s})_{s\leq t<+\infty}$ is a martingale on $[s,+\infty]$*
(4) $\liminf_{t\to\infty}U_t=\limsup_{t\to\infty}g_t$ *(P-a.s.) on $\{\omega:\tau_s(\omega)=+\infty\}$.*

Then the process $U=(U_t)_{t\geq 0}$ is the Snell envelope of the process $g=(g_t)_{t\geq 0}$.

PROOF. Let us denote $U_t^*=ess\sup_{\tau\in\mathcal{M}_t}E[g_\tau|\mathcal{F}_t]$ for $t\geq 0$. Then condition (1) and (2) imply $U_t^*\leq U_t$, $t\geq 0$.
Conversely, for each $T\geq 0$ condition (3) implies

$$\begin{aligned}U_t&=E[U_{\tau_s\wedge T}|\mathcal{F}_t]=E[g_{\tau_s}|\mathcal{F}_t]+E[U_{\tau_s}-g_{\tau_s}|\mathcal{F}_t]I(\tau_s>T)\\&\leq U_t^*+E[U_{\tau_s}-g_{\tau_s}|\mathcal{F}_t]I(\tau_s>T),\end{aligned}$$

and application of the Fatou's Lemma together with condition (4) results in the inequality $U_t\leq U_t^*$, $t\geq 0$. Thus the proof of Lemma is completed. □

We start with some auxiliary facts and notations.
Let γ_1 and γ_2 be the roots of the following quadratic equation:

$$\sigma^2\gamma^2/2-(\sigma^2/2+r-q)\gamma-(\lambda+q)=0,$$

i.e.,

$$\gamma_k=\frac{(\sigma^2/2+r-q)+(-1)^k\sqrt{(\sigma^2/2+r-q)^2+2(\lambda+q)\sigma^2}}{\sigma^2},\quad k=1,2.$$

Then it follows that

$$\begin{aligned}\gamma_1<0&\Leftrightarrow q+\lambda>0,\\ \gamma_2>1&\Leftrightarrow r+\lambda>0.\end{aligned}$$

We now meet:

THEOREM 3. *Let the reward process f be equal to*

$$f_t = e^{-\lambda t}(\int_0^t S_u du + \psi_0 S_0), \quad t \geq 0,$$

where $\lambda \in R$ and $\psi_0 \geq 0$ are some constants. Assume that $r + \lambda > 0$, $q + \lambda > 0$. Then for $t \geq 0$

$$\begin{aligned} V_t^* &= e^{-\lambda t} v(\psi_t) S_t, \\ \beta_t^* &= e^{-\lambda t} v'(\psi_t) S_t / B_t, \\ \gamma_t^* &= e^{-\lambda t}[v(\psi_t) - v'(\psi_t)\psi_t] \end{aligned}$$

where

$$\psi_t = \frac{\int_0^t S_u du + \psi_0 S_0}{S_t}, \tag{1}$$
$$v(\psi) = \begin{cases} \psi & \psi \geq \psi^* \\ v(\psi)\frac{\psi^*}{u(\psi^*)} & \psi < \psi^*, \end{cases}$$

the function $u(\psi)$, $\psi > 0$, equals

$$u(\psi) = \int_0^\infty \exp(-\frac{2y}{\sigma^2}) y^{-\gamma_1 - 1}(1 + \psi y)^{\gamma_2} dy \tag{2}$$

and the level ψ^* is a root of the following equation

$$U(\psi) = \psi u'(\psi), \quad \psi > 0.$$

The consumption process C^* is equal to

$$C_t^* = \int_0^t e^{-\lambda u}[(r + \lambda)\psi_u - 1] S_u I(\psi_u \geq \psi^*) du.$$

Proof. Let us verify the conditions of Lemma 2 for $Q = \widetilde{P}$ and the processes

$$g_t = e^{-(\lambda+q)t}\psi_t, \ \ U_t = e^{-(\lambda+q)t}v(\psi_t), \ \ t \geq 0,.$$

(1) Condition (1) of the Lemma 2 is equivalent to the inequality

$$v(\psi) \geq \psi, \ \ \psi \geq 0. \tag{3}$$

From (1), (2) we obtain that $v = v(\psi)$ is a convex continuously differentiable function such that $v(\psi) = \psi$, $\psi \geq \psi^*$. This implies (3).
(2) From the Ito formula we obtain that the process $\psi = (\psi_t)_{t\geq 0}$ satisfies the following stochastic differential equation

$$d\psi_t = [1 - (r-q)\psi_t]dt - \sigma\psi_t d\widetilde{W}_t.$$

Now applying the Ito's formula, we obtain

$$\begin{aligned} e^{-(\lambda+q)t}v(\psi_t) = v(\psi_0) &+ \int_0^t e^{-(\lambda+q)u}(\mathcal{L}v(\psi_u) - (\lambda+q)v(\psi_u))du \\ &- \int_0^t e^{-(\lambda+q)u}v'(\psi_u)\sigma\psi_u d\widetilde{W}_u \end{aligned} \tag{4}$$

where

$$\mathcal{L} = [1-(r-q)x]\frac{d}{dx} + \frac{\sigma^2}{2}x^2\frac{d^2}{dx^2}$$

is the infinitesimal operator of the Markov process $\psi = (\psi_t)_{t\geq 0}$. Note that, if $u = u(\psi)$ is given by (2), then this function is a solution of

$$\frac{\sigma^2}{2}\psi^2 u''(\psi) + [1-(r-q)\psi]u'(\psi) - (\lambda+q)u(\psi) = 0,$$

where we use the fact $\gamma_1 + \gamma_2 = 1 + 2(r-q)/\sigma^2$. Now, equation (1) implies

$$\mathcal{L}v(\psi) - (\lambda+q)v(\psi) = \begin{cases} 0, & \psi < \psi^* \\ -(\lambda+r)\psi + 1, & \psi \geq \psi^*. \end{cases} \tag{5}$$

Further, for $\psi \geq \psi^*$ we have

$$-(\lambda+r)\psi+1 \leq -(\lambda+r)\psi^*+1$$

(6)
$$= (\mathcal{L}u(\psi)-(\lambda+q)u(\psi))\frac{\psi^*}{u(\psi^*)}|_{\psi=\psi^*} - \frac{\sigma^2}{2}u''(\psi^*)\frac{\psi^*}{u(\psi^*)}\psi^{*2}$$
$$= -\frac{\sigma^2}{2}u''(\psi^*)\frac{\psi^*}{u(\psi^*)}\psi^{*2} < 0.$$

Equation (4), (5) and (6) imply that the process $(e^{-(\lambda+q)t}v(\psi_t))_{t\geq 0}$ is a local supermartingale and being positive, it is also a supermartingale.
(3) Let us fix $s \geq 0$ and denote $\tau_s(\omega) = \inf\{t \geq s : \psi_t(\omega) \geq \psi^*\}$ From (4) and (5) we obtain that the "stopped process" $(e^{-(\lambda+q)t\wedge\tau_s}v(\psi_{t\wedge\tau_s}))_{t\geq s}$ is a $\widetilde{P}$-local martingale and, as by (1) $0 \leq v(\psi_{t\wedge\tau_s}) \leq max(\psi_s, \psi^*)$ and $\widetilde{E}\psi_s < +\infty$, this process is also a $\widetilde{P}$-martingale.
(4) We need to prove that

(7)
$$\liminf_{t\to\infty} v(\psi_t) = \limsup_{t\to\infty} \psi_t$$

To simplify the notations we suppose that $s = 0$, $\psi_0 = 0$. By Ito formula, the process $\psi = (\psi_t)_{t\geq 0}$ satisfies the following stochastic differential equation

$$d\psi_t = [1-(r-q)\psi_t]dt - \sigma\psi_t d\widetilde{W}_t.$$

Let $Y = (Y_t)_{t\geq 0}$ be a solution of stochastic differential equation

$$dY_t = -(r-q)Y_t dt + \sigma Y_t d\widetilde{W}_t, \;\; Y_0 := \psi_0.$$

Then, by the Ito formula

$$Y_t = Y_0 exp[-(r-q+\sigma^2/2)t + \sigma\widetilde{W}_t], \;\; t \geq 0,$$

and by the theorem on comparison of the solutions of one-dimensional Ito equations([1]), $\psi_t \geq Y_t$. If $r-q+\sigma^2/2 < 0$, then $Y_t \to +\infty$ as $t \to +\infty$. So, on $\{\omega : \tau_s(\omega) = +\infty\}$ we have $\psi_t \geq Y_t \to +\infty$ and (7) results from the fact that $v(\psi) \to 0$ as $\psi \to +\infty$. If $r-q+\sigma^2/2 \geq 0$,

(8)
$$\widetilde{P}(\tau_s(\omega) < +\infty) = 1.$$

To prove this, let us introduce the harmonic function $h = h(\psi)$, $\psi > 0$, which satisfies the equation

$$\mathcal{L}h(\psi) = 0, \quad h(1) = 0. \tag{9}$$

The solution of above equation is strictly increasing function

$$h(x) = \int_1^x e^{\frac{2}{y\sigma^2}} y^{\frac{2(r-q)}{\sigma^2}} dy$$

where, as usually, $\int_1^x = -\int_x^1$ for $0 < x \leq 1$.([5]) From (9) and (4)(with $\lambda + q = 0$) we have

$$h(\psi_t) = h(\psi_0) - \int_0^t \sigma\psi_s h'(\psi_s) d\widetilde{W}_s.$$

Since $\tau_0 = \{t \geq 0 : \psi_t \leq \psi^*\} = \inf\{t \geq 0 : h(\psi_t) \leq h(\psi^*)\}$, we must show only that the process $(h(\psi_t))_{t\geq 0}$ reaches any level A almost surely:

$$\widetilde{P}_\psi(\inf_t h(\psi_t) < A) = 0, \quad \psi \geq 0.$$

The process $(h(\psi_t))_{t\geq 0}$ is a diffusion Markov process such that

$$dh(\psi_t) = -\sigma(\psi_t) d\widetilde{W}_t$$

where $\sigma(\psi) = \sigma\psi h'(\psi)$. Because

$$\begin{aligned} c = \inf_{\psi>0} \sigma(\psi) &= \inf_{\psi>0} (\sigma\psi^{\frac{2(r-q)}{\sigma^2+1}} exp(\frac{2}{\psi\sigma^2})) \\ &= \sigma(\frac{2}{2(r-q)+\sigma^2})^{\frac{2(r-q)}{\sigma^2+1}} exp(\frac{2(r-q)}{\sigma^2}) > 0 \end{aligned}$$

(if $2r + \sigma^2 = 0$, then we have $c = \sigma > 0$), the quadratic variation $(\langle h(\psi)\rangle_t)_{t\geq 0}$ of the continuous martingale $(h(\psi_t))_{t\geq 0}$ satisfies $\langle h(\psi)\rangle_t = \int_0^t \sigma^2(\psi_s) ds \geq c^2 t$. Hence $\widetilde{P}_\psi(\langle h(\psi)\rangle_t \to \infty) = 1$ and the stopping time $\kappa(t) = \inf\{s \geq 0 : \langle h(\psi)\rangle_s \geq t\}$ is finite with $\widetilde{P}_\psi$-probability one. Notice that the process $h(\psi_{\kappa(t)})$ is a continuous martingale with the quadratic

variation $\langle h(\psi_{\kappa(t)})\rangle = t$. Therefore, by the Levy Theorem ([3]), this process is a Wiener process and hence with probability one it reaches any level. This implies that the process $(h(\psi_t))_{t\geq 0}$ also possesses such a property, proving (8).
Therefore, on $\{\omega : \tau_s(\omega) = +\infty\}$ we have $\psi_t \to 0$, and

$$\lim\inf_{t\to\infty} v(\psi_t) = \lim_{x\to 0} v(x) = 0, \quad \lim\sup_{t\to\infty} \psi_t = \lim_{x\to 0} x = 0.$$

Thus, the conditions of Lemma 2 are satisfied. So, by Lemma 1 the value of V^* is given. Finally, the standard arguments based on Lemma 1, representation (4) and formula (5) lead to the expressions decribed above for β^*, γ^* and C^*. Theorem 3 is proved. □

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