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SOME PROPERTIES OF THE SEQUENTIAL CLOSURE OPERATOR ON A GENERALIZED TOPOLOGICAL SPACE

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ABSTRACT. We give two sufficient conditions that the space (X, c_*) be a Fréchet-Urysohn space such that $x_n \to x$ in (X, c) if and only if $x_n \to x$ in (X, c_*), where c_* is the sequential closure operator on a generalized topological space (X, c).

1. Introduction and preliminaries

Let us recall definitions. Let X be a non-empty set and $P(X)$ the power set of X. A single-valued function c of $P(X)$ into $P(X)$ is called *a semi-closure operator on* X if it satisfies the following conditions:

(C1): $c(\emptyset) = \emptyset$,

(C2): $A \subset c(A)$ for each $A \in P(X)$,

(C3): for each $A, B \in P(X)$, $A \subset B$ implies $c(A) \subset c(B)$, and

(C4): $c(A) = c(c(A))$ for each $A \in P(X)$.

If a semi-closure operator on X is given, then the pair (X, c) is called *a semi-closure space* (see [3]). Clearly, a semi-closure space is a generalized topological space. We denote $\leqq$ by the partial order on the set of all semi-closure operators on a set X defined as follows: $c_1 \leqq c_2$ if and only if $c_1(A) \subset c_2(A)$ for each $A \in P(X)$. Let $\mathbb{N}^+$ denote the set of all positive integers. Let (X, c) be a semi-closure space, $x \in X$ and $(x_n | n \in \mathbb{N}^+)$(briefly (x_n)) a sequence of points in a subset of X. A subset N of X is called *a neighborhood of* x in (X, c) if $x \in X - c(X - N)$. The sequence (x_n) *converges to* x(written $x_n \to x$) in (X, c) if and only if (x_n) is eventually in each neighborhood of x in (X, c). Let $L_c =$

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$\{((x_n), x) | (x_n)$ is a sequence of points in X, $x \in X$ and $x_n \to x$ in $(X, c)\}$. A closure operator c_* on X defined by for each $A \in P(X)$, $c_*(A) = \{x \in X | ((x_n), x) \in L_c$ for some sequence (x_n) of points in $A\}$ is called *the sequential closure operator* on the semi-closure space (X, c). A topological space (X, c) endowed with the topological closure operator c is called *a sequential space*(see [1]) if it satisfies the following property: *for each set $A \subset X$, if $A = c_*(A)$ then $A = c(A)$*. A topological space (X, c) is called *a Fréchet-Urysohn space*(see [1])(also simply called *a Fréchet space*) if it satisfies the following property(called *the Fréchet-Urysohn property*(see [2])): *for each $A \in P(X), c(A) = c_*(A)$ where c_* is the sequential closure operator on the space (X, c); $c = c_*$*. Of course, every Fréchet-Urysohn space is sequential, but the converse is not true. Topological spaces that satisfy the first axiom of countability form a special group in the class of Fréchet-Urysohn spaces and the metric spaces are distinguished in the former.

It is well known that the sequential closure operator c_* on a topological space (X, c) possesses the following conditions:

(C1): $c_*(\emptyset) = \emptyset$,

(C2$_*$): $A \subset c_*(A) \subset c(A)$ for each $A \in P(X)$, and

(C5): $c_*(A \cup B) = c_*(A) \cup c_*(B)$ for each $A, B \in P(X)$.

However, the sequential closure operator c_* does not satisfy the condition (C4); c_* is not a topological closure operator on X. It is also known that the sequential closure operator c_* on a Fréchet-Urysohn space (X, c) satisfies the condition (C4), but the sequential closure operator c_* on a sequential space (X, c) does not satisfy the condition (C4). For the sequential closure operator on a topological space and related topics, we refer to the reader [1, 2, 5-7].

We can see easily that in a topological space (X, c), the Fréchet-Urysohn property; $c = c_*$ is surely a very strong condition for the sequential closure operator c_* to satisfy the condition (C4). Hence, a question naturally arises in connection with the above facts.

Question: In a semi-closure space (X, c), is there a sufficient property on the space X, which is a generalization of the Fréchet-Urysohn property, for the sequential closure operator c_* on the space X to satisfy the condition $(C4)$?

The purpose of this paper is to show that the answer to *Question* is affrimative. We show that if the sequential closure operator c_* on a semi-closure space (X, c) satisfies the condition (C4), then the topological space (X, c_*) is a Fréchet-Urysohn space with $L_c = L_{c_*}$, where $L_{c_*} = \{((x_n), x) | (x_n)$ is a sequence of points in X, $x \in X$ and $x_n \to x$ in $(X, c_*)\}$. And we give two sufficient properties, which are generalizations of the Fréchet-Urysohn property, for the sequential closure operator c_* on a semi-closure space (X, c) to satisfy the condition (C4).

2. Results

We begin with elementary facts of semi-closure spaces.

LEMMA 2.1[3]. *Let (X, c) be a semi-closure space and $A \in P(X)$. Then, $x \in c(A)$ if and only if $A \cap N \neq \emptyset$ for each neighborhood N of x in (X, c).*

LEMMA 2.2. *Let (X, c) be a semi-closure space and c_* the sequential closure operator on the space (X, c). Then, $c_* \leqq c$; that is, for each $A \in P(X)$, if $((x_n), x) \in L_c$ for some sequence (x_n) of points in A, then $x \in c(A)$.*

PROOF. It is straightforward. □

Note that in a semi-closure space (X, c), the Fréchet-Urysohn property is equivalent to the property $c \leqq c_*$.

The following theorem is the main property of the sequential closure operator on a semi-closure space.

THEOREM 2.3. *If the sequential closure operator c_* on a semi-closure space (X, c) satisfies the condition $(C4)$, then the space (X, c_*) is a Fréchet-Urysohn space with $L_c = L_{c_*}$.*

PROOF. First, we show that the sequential closure operator c_* is a topological closure operator on X. It is sufficient to prove that c_* satisfies the condition (C5). By the condition (C3), it is obvious that $c_*(A) \cup c_*(B) \subset c_*(A \cup B)$ for each $A, B \in P(X)$. Hence we show that $c_*(A \cup B) \subset c_*(A) \cup c_*(B)$ for each $A, B \in P(X)$. Let $x \in c_*(A \cup B)$. Then, by the definition of c_*, $((x_n), x) \in L_c$ for some sequence (x_n) of points in $A \cup B$. Clearly, $\{n \in \mathbb{N}^+ | x_n \in A\}$ or $\{n \in \mathbb{N}^+ | x_n \in B\}$ is infinite. Without loss of generality assume that $\{n \in \mathbb{N}^+ | x_n \in A\}$ is infinite. Then, it is obvious that there exists a subsequence $(x_{\phi(n)})$ of (x_n) such that $\{x_{\phi(n)} | n \in \mathbb{N}^+\} \subset A$, where $\{x_{\phi(n)} | n \in \mathbb{N}^+\}$ is the range of $(x_{\phi(n)})$. It is clear that if $((x_n), x) \in L_c$, then $((x_{\phi(n)}), x) \in L_c$ for each subsequence $(x_{\phi(n)})$ of (x_n). Hence, we have that $((x_{\phi(n)}), x) \in L_c$ and so, by the definition of c_*, $x \in c_*(A) \subset c_*(A) \cup c_*(B)$. Thus, the space (X, c_*) is a topological space.

Next, we show that $L_c = L_{c_*}$. It is not difficult to verify that for each semi-closure operators u and v on X, if $u \leq v$ then $x_n \to x$ in (X, u) implies $x_n \to x$ in (X, v); $L_u \subset L_v$. Hence, by Lemma 2.2, we have $L_{c_*} \subset L_c$. Conversely, if $((x_n), x) \notin L_{c_*}$, then (x_n) is not eventually in some neighborhood N of x in (X, c_*). Obviously, there exists a subsequence $(x_{\phi(n)})$ of (x_n) such that the range $\{x_{\phi(n)} | n \in \mathbb{N}^+\}$ of the sequence $(x_{\phi(n)})$ and the neighborhood N of x in (X, c_*) are disjoint. By Lemma 2.1, we have that $x \notin c_*(\{x_{\phi(n)} | n \in \mathbb{N}^+\})$ and so $((x_{\phi(n)}), x) \notin L_c$ by the definition of c_*. Note that if $((x_n), x) \in L_c$, then $((x_{\phi(n)}), x) \in L_c$ for each subsequence $(x_{\phi(n)})$ of (x_n). Hence, by the contraposition of above fact, $((x_n), x) \notin L_c$. Thus, we have $L_c = L_{c_*}$.

Finally, we show that (X, c_*) is a Fréchet-Urysohn space. Let $A \in P(X)$ and $x \in c_*(A)$. Then, by the definition of c_*, $((x_n), x) \in L_c$ for some sequence (x_n) of points in A. Since $L_c = L_{c_*}$, $((x_n), x) \in L_{c_*}$ and thus it holds.

Therefore, (X, c_*) is a Fréchet-Urysohn space with $L_c = L_{c_*}$. This completes the proof. □

REMARK. Let (X, c) be a semi-closure space. Then, by Theorem 2.3, we immediately have that the sequential closure operator c_* satisfies the condition (C4) if and only if the space (X, c_*) is a Fréchet-Urysohn space.

In order to give the answer to *Question*, we consider the following two properties in a semi-closure space (X, c):

$(*)$: *For each countable subset A of X, $c(A) \subset c_*(A)$.*

$(**)$: *For each double-sequence $(x_{nm} | n \in \mathbb{N}^+, m \in \mathbb{N}^+)$ of points in X such that $((x_{nm} | m \in \mathbb{N}^+), x_n) \in L_c$ for each $n \in \mathbb{N}^+$ and $((x_n), x) \in L_c$, $((y_n), x) \in L_c$ for some sequence (y_n) of points in the set $\{x_{nm} | n \in \mathbb{N}^+, m \in \mathbb{N}^+\}$.*

From the definitions and the following examples, we have easily that the following implications hold

the Fréchet-Urysohn property $\Rightarrow (*) \Rightarrow (**)$

but the converses do not hold and the definitions of sequential and $(*)$, as well as $(**)$, are independent in topological spaces. On other words $(*)$ is a generalization of the property $c \leqq c_*$. Note that in fact, the property $(**)$ is equivalent to the condition (SC 3) of sequential convergence strucrures(see [4]). Many authors(see [2],[4],[6] and [7]) have used some similar properties to $(**)$ to study Fréchet-Urysohn spaces and a sufficient condition that a topological space be a Fréchet-Urysohn space.

EXAMPLE 2.4. (1) Let X be the set consisting of pairwise distinct objects of the following three types: points x_{mn} where $m \in \mathbb{N}^+$ and $n \in \mathbb{N}^+$, points y_n where $n \in \mathbb{N}^+$, and a point z. We set $V_k(y_n) = \{y_n\} \cup \{x_{mn} | m \geqq k\}$ and let γ denote the set of subsets $W \subset X$ such that $z \in W$ and there exists a positive integer p such that $V_1(y_n) - W$ is finite and $y_n \in W$ for all $n \geqq p$. The collection $\mathcal{B} = \{\{x_{mm}\} | m \in \mathbb{N}^+, n \in \mathbb{N}^+\} \cup \gamma \cup \{V_k(y_n) | n \in \mathbb{N}^+, k \in \mathbb{N}^+\}$ is a base of a topology on X. In the space X, for each $n \in \mathbb{N}^+$, the sequence $(x_{mn} | m \in \mathbb{N}^+)$ converges to the point y_n and the sequence (y_n) converges to the point z. However, for the set $A = \{x_{mn} | m \in \mathbb{N}^+, n \in \mathbb{N}^+\}$, we have that A is countable and $z \in c(A)$, but $c_*(c_*(A)) \ni z \notin c_*(A)$, where c is the topological closure operator on the space X and c_* is the sequential closure operator on the space (X, c). Hence, it follows that the space X does not satisfy $(*)$, $(**)$ and the Fréchet-Urysohn property. And we have that X is a sequential space, but the sequential closure operator c_* on this space X does not satisfy the condition (C4)(see [1], p.13).

(2) The space of ordinals $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, is a compact Hausdorff space all of whose countable subsets are metrizable. Note that the point ω_1 is not a cluster point of each countable subset of X not containing ω_1(see [5], p.76). Hence we see that the space X satisfies $(*)$ and $(**)$, but X is surely not sequential and not Fréchet-Urysohn.

(3) The space $\mathbb{N}^* = \beta(\mathbb{N}^+) - \mathbb{N}^+$, the Stone-Čech growth on $\mathbb{N}^+$, is nondiscrete and does not have any convergent regular sequence. Hence,

the space $\mathbb{N}^*$ satisfies $(**)$, but it is not sequential(see [2], p.187) and does not satisfy $(*)$.

Now we show that the answer of *Question* is affrimative.

THEOREM 2.5. *Let (X, c) be a semi-closure space satisfying $(**)$. Then, the sequential closure operator c_* of the space (X, c) satisfies the condition (C4).*

PROOF. By Theorem 2.3, c_* satisfies the condition (C2) and hence it is sufficient to prove that $c_*(c_*(A)) \subset c_*(A)$ for each $A \in P(X)$. Let $A \in P(X)$ and $x \in c_*(c_*(A))$. Then, by the definition of c_*, $((x_n), x) \in L_c$ for some sequence (x_n) of points in $c_*(A)$. Since $x_n \in c_*(A)$ for each $n \in \mathbb{N}^+$, we have that for each $n \in \mathbb{N}^+$, there exists a sequence $(x_{nm} | m \in \mathbb{N}^+)$ of points in A such that $((x_{nm}), x_n) \in L_c$. By the property $(**)$, we have that there is a sequence (y_n) of points in the set $\{x_{nm} | n \in \mathbb{N}^+, m \in \mathbb{N}^+\}$ such that $((y_n), x) \in L_c$. It follows that, by the definition of c_* and the condition (C3), $x \in c_*(\{x_{nm} | n \in \mathbb{N}^+, m \in \mathbb{N}^+\}) \subset c_*(A)$. □

Immediately, we have the following corollary.

COROLLARY 2.6. *Let (X, c) be a semi-closure space satisfying $(*)$. Then, the sequential closure operator c_* of the space (X, c) satisfies the condition $(C4)$.*

According to Theorem 2.3 and Theorem 2.5, we have consequentially the following corollary.

COROLLARY 2.7. *Let (X, c) be a semi-closure space (and hence a topological space) satisfying $(**)$. Then, (X, c_*) is a Fréchet-Urysohn space with $L_c = L_{c_*}$.*

REMARK. It is an interesting and important fact that $L_c = L_{c_*}$, even though $c_* \lneqq c$. From this fact, we have naturally the following:

(1) The properties $(*)$ and $(**)$ are sufficient conditions for a non Fréchet-Urysohn space (X, c) to have the Fréchet-Urysohn expansion (X, c_*)(that is, the space (X, c_*) is a Fréchet-Urysohn space and $c_* \leqq c$) satisfying $L_c = L_{c_*}$. In fact, the space (X, c_*) is the smallest Fréchet-Urysohn expansion of (X, c) satisfying $L_c = L_{c_*}$.

(2) There are close corelations between some topological properties of the two spaces (X, c) and (X, c_*). For examples, (a) the separation

properties of (X,c) transfer to the space (X,c_*), (b) if (X,c_*) is compact(connected or separable), then (X,c) is compact(resp. connected or separable), and (c) (X,c) is sequentially compact if and only if (X,c_*) is sequentially compact, etc.

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