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A NOTE ON CONTACT CONFORMAL CURVATURE TENSOR

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ABSTRACT. In this paper we show that every contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form.

1. Introduction

The contact conformal curvature tensor ([3]) is a curvature-like tensor defined on a contact metric manifold $(M; \varphi, \xi, \eta, g)$ which is constructed from the conformal curvature tensor ([6]) by using the Boothby-Wang's fibration ([3]). It seems to play an important role to studying the spectral geometry of compact Sasakian manifolds (cf. [5], [7]).

On the other hand Tanno([10, 11]) proved that every conformally flat K-contact manifold is a space form, and Blair and Koufogiorgos ([2]) improved Tanno's result as follow:

Every conformally flat contact metric manifold with $R\varphi = \varphi R$ is a space form, where R denotes the Ricci operator.

In this paper we shall prove the following theorem which gives a geometric characterization of a contact metric manifold with vanishing contact conformal curvature tensor:

THEOREM. *Every $(2n+1)(n > 2)$-dimensional contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form.*

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2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g). Then we have by definition

$$(2.1) \qquad \begin{cases} \varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0 \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \end{cases}$$

for any vector fields X, Y tangent to M, where I denotes the identity transformation(cf, [1], [9]). Denoting by ∇ the Riemannian connection and by Φ the fundamental 2-form defined by

$$(2.2) \qquad \Phi(X, Y) = g(\varphi X, Y)$$

the almost contact metric structure (φ, ξ, η, g) is called a *contact metric structure* if Φ satisfies

$$(2.3) \qquad \Phi = d\eta.$$

A manifold with a contact metric structure is called a *contact metric manifold.* (cf. [1], [9]). A contact metric manifold for which ξ is Killing is called a *K-contact manifold.* Also, a manifold with a normal contact metric structure is called a *Sasakian manifold.* Thus a Sasakian manifold is K-contact but the converse is not true except in dimension 3. (cf. see [1]). Moreover, the following lemmas are well known (cf. see [1]), which give inclusion relations among those manifolds.

LEMMA 2.1. *On a contact metric manifold, the followings are equivalent to each other:*

(1) *The manifold is a K-contact manifold.*
(2) *The sectional curvature of plane section containing ξ is equal to 1.*
(3) *The Ricci curvature in the direction of ξ is $2n$.*

LEMMA 2.2. *Let M be a $(2n+1)$-dimensional Riemannian manifold admitting a unit Killing vector field ξ such that*

$$K(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

where K denotes the curvature tensor. Then M is a Sasakian manifold.

Finally we recall the definition and fundamental properties of D-homothetic deformation due to Tanno ([11]), where D means the distribution defined by η. *D-homothetic deformation* $g \mapsto {}^*g$ is defined by

$$^*g = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$$

for a positive constant α. The following identities for D-homothetic deformation are well known ([11]):

$$(2.4)\quad \begin{cases} {}^*K(X,Y)Z = K(X,Y)Z + (\alpha - 1) \\ \times\{g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X \\ +2g(\varphi X, Y)\varphi Z\} + (\alpha - 1)^2\{\eta(Y)X \\ -\eta(X)Y\}\eta(Z) + (\alpha - 1)[\eta(X)\{g(Y,Z)\xi \\ -\eta(Z)Y\} - \eta(Y)\{g(X,Z)\xi - \eta(Z)X\} \\ +\eta(Z)\{\eta(Y)X - \eta(X)Y\}], \\ {}^*g({}^*RX, Y) = g(RX, Y) - 2(\alpha - 1)g(X,Y) \\ +2(\alpha - 1)(n\alpha + n + 1)\eta(X)\eta(Y), \\ {}^*s = \alpha^{-1}s - 2n\alpha^{-1}(\alpha - 1). \end{cases}$$

Moreover, if (φ, ξ, η, g) is a Sasakian structure, then $({}^*\varphi, {}^*\xi, {}^*\eta, {}^*g)$ is also a Sasakian structure, where

$$(2.5)\quad \begin{cases} {}^*\varphi = \varphi, \ {}^*\xi = \alpha^{-1}\xi, \ {}^*\eta = \alpha\eta, \\ {}^*g = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta \end{cases}$$

for a positive constant α. In this case it is said that $(M; \varphi, \xi, \eta, g)$ is *D-homothetic* to $(M; {}^*\varphi, {}^*\xi, {}^*\eta, {}^*g)$([11]).

3. Proof of the main theorem

Let $(M; \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional contact metric manifold. Then we can consider the following contact conformal curvature tensor

C_0 of type (1,3) on M, which is defined by ([4])

$$C_0(X,Y)Z = K(X,Y)Z + \frac{1}{2n}\{R_0(Y,Z)X - R_0(X,Z)Y + g(Y,Z)RX - g(X,Z)RY + \eta(Y)R_0(X,Z)\xi - \eta(X)R_0(Y,Z)\xi + \eta(X)\eta(Z)RY - \eta(Y)\eta(Z)RX + S_0(X,Z)\varphi Y - S_0(Y,Z)\varphi X + \Phi(X,Z)SY - \Phi(Y,Z)SX + 2\Phi(X,Y)SZ + 2S_0(X,Y)\varphi Z\} + \frac{1}{2n(n+1)}\{2n^2 - n - 2 + \frac{(n+2)s}{2n}\} \times \{\Phi(Y,Z)\varphi X - \Phi(X,Z)\varphi Y - 2\Phi(X,Y)\varphi Z\} + \frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)s}{2n}\} \times \{g(Y,Z)X - g(X,Z)Y\} - \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)s}{2n}\} \times \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\}, \tag{3.1}$$

where R_0 and s denote the Ricci tensor and the scalar curvature, respectively, i.e., $R_0(X,Y) := g(RX,Y)$, $s := \text{trace}R$, and

$$SX := R(\varphi X), \quad S_0(X,Y) := g(SX,Y). \tag{3.2}$$

Using (2.4), we can easily verify that the tensor C_0 is invariant under the D-homothetic deformation defined by (2.5)([4]).

From now on we assume that the contact conformal curvature tensor C_0 vanishes identically on M, i.e. $C_0 \equiv 0$. Then from (3.1) with $C_0 = 0$ we can easily see that

$$RX = \frac{1}{4n-5}\{-3\varphi R\varphi X - 3\eta(RX)\xi - 2\eta(X)R\xi\} + \frac{2}{n(4n-5)}\{(n-2)s - 2n(n-2)\}X + \frac{2}{n(4n-5)}\{2n(2n^2+n-2) - (n-2)s\}\eta(X)\xi. \tag{3.3}$$

Putting $X = \xi$ in (3.3) and using (2.1), we have

$$R\xi = 2n\xi, \tag{3.4}$$

which and Lemma 2.1 give

LEMMA 3.1. *Every contact metric manifold with $C_0 \equiv 0$ is a K-contact manifold.*

Substituting (3.4) into (3.3), we obtain

$$\begin{aligned} RX = {} & \frac{-3}{4n-5}\varphi R\varphi X + \frac{2(n-2)}{n(4n-5)}(s-2n)X \\ & + \frac{2}{n(4n-5)}\{n(4n^2-3n-4)-(n-2)s\}\eta(X)\xi. \end{aligned} \tag{3.5}$$

Applying the operator φ to the both side of (3.6) and using (2.1) and (3.4), it can be easily verified that

$$g(\varphi RX, Y) = \frac{3}{4n-5}g(R\varphi X, Y) + \frac{2(n-2)(s-2n)}{n(4n-5)}g(\varphi X, Y) \tag{3.6}$$

because of R being a symmetric endomorphism. Since φ is a skew-symmetric endomorphism, (3.6) implies

$$g(R\varphi X, Y) = \frac{3}{4n-5}g(\varphi RX, Y) + \frac{2(n-2)(s-2n)}{n(4n-5)}g(\varphi X, Y),$$

from which together with (3.6), we have $g(\varphi RX, Y) = g(R\varphi X, Y)$ that is,

$$\varphi R = R\varphi. \tag{3.7}$$

Hence it follows from (3.5) and (3.7) that

$$g(RX, Y) = (\frac{s}{2n} - 1)g(X, Y) + (2n + 1 - \frac{s}{2n})\eta(X)\eta(Y), \tag{3.8}$$

provided $n > 2$. Substituting (3.8) into (3.1) with $C_0 = 0$, we can easily see that

$$K(X,Y)Z = \frac{k+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{k-1}{4}\{g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi\}, \tag{3.9}$$

where $k = \frac{1}{n(n+1)}\{s - n(3n+1)\}$. Moreover (3.9) yields

$$K(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

which together with Lemma 2.2 and Lemma 3.1 implies our main theorem stated in section 1 because s is constant, provided $n > 2$.

Combining the theorem with Proposition 4.1 ([13], p.504) and Corollary ([8], p.282), we have

COROLLARY 3.2. *Let M be a $(2n+1)(n>2)$-dimensional complete, simply connected contact metric manifold with vanishing contact conformal curvature tensor.*

(1) *If $s > -2n$, it is D-homothetic to the unit sphere S^{2n+1} ;*
(2) *If $s = -2n$, it is isometric to $E^{2n+1}(-3)$;*
(3) *If $s < -2n$, it is D-homothetic to the universal pseudo-Riemannian covering manifold of S^{2n+1}_{2n}, which is diffeomorphic to $E^{2n} \times S^1$.*

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