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ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM IN TERMS OF JACOBI OPERATORS

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ABSTRACT. We study real hypersurfaces of a complex space form such that the Jacobi operator with respect to the structure vector field and the structure tensor ϕ on the real hypersurface commute.

0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space C_n, or a complex hyperbolic space H_nC according as $c > 0, c = 0$ or $c < 0$.

The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, g, ξ, η). The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha = \eta(A\xi)$.

Typical examples of real hypersurfaces in P_nC are homogeneous one. Takagi ([23]) classified homogeneous real hypersurfaces of a complex projective space P_nC as the following six types.

THEOREM A. *Let M be a homogeneous real hyperspace of P_nC. Then M is locally congruent to one of the followings:*

(A_1) *a geodesic hypersphere (that is, a tube over a hyperplane $P_{n-1}C$),*
(A_2) *a tube over a totally geodesic $P_kC (1 \leq k \leq n-2)$,*
(B) *a tube over a complex quadric Q_{n-1},*

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(C) *a tube over* $P_1C \times P_{(n-1)/2}C$ *and* $n(\geq 5)$ *is odd,*
(D) *a tube over a complex Grassman* $G_{2,5}C$ *and* $n = 9$,
(E) *a tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$.

This result is generalized by many authors ([2], [4], [12], [15], [16], [22] and [23] etc.). One of them, Kimura ([12]) asserts that M has constant principal curvatures and the structure vector field ξ is principal if and only if M is locally congruent to a homogeneous real hypersurface.

On the other hand, real hypersurface of H_nC have been also investigated by many geometers ([9], [10], [18] and [19] etc.) from different points of view. In particular, Berndt ([3]) proved the following:

THEOREM B. *Let* M *be a real hypersurface of* H_nC. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the followings:*

(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a hyperplane* $H_{n-1}C$,
(A_2) *a tube over a totally geodesic* $H_kC(1 \leq k \leq n-2)$,
(B) *a tube over a totally real hyperbolic space* H_nR.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space P_nC or that of type A_0, A_1 or A_2 in a complex hyperbolic space H_nC. Then M is said to be of *type A* for simplicity. By a theorem due to Okumura ([20]) and to Montiel and Romero ([19]) we have (see also Ki [7])

THEOREM C. *If the shape operator* A *and the structure tensor* ϕ *commute to each other, then a real hypersurface of a complex space form* $M_n(c)$, $c \neq 0$ *is locally congruent to be of type* A.

We denote by ∇ the Levi-Civita connection with respect to the metric tensor g. The curvature tensor field R on M is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where X and Y are vector fields on M. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is well-known that the notion of Jacobi vector fields imvolve many important geometric properties. In the preceding work ([6]), we prove the following:

THEOREM D. *Let M be a connected real hypersurface of a complex projective space P_nC. If M satisfies $R_\xi \phi A = A \phi R_\xi$, then M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$);*

(A_2) *a tube of radius r over a totally geodesic $P_kC (0 \leq k \leq n-1)$, where $0 < r < \frac{\pi}{2}$.*

In this paper we study some real hypersurfaces in a complex space form $M_n(c), c \neq 0$ in terms of the Jacobi operator, the structure tensor ϕ and the Jacobi operator R_ξ with respect to the structure vector field ξ, and improve above Theorem D.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be orientable.

1. Preliminaries

Let $M_n(c)$ be a real 2n-dimensional complex space form equipped with parallel almost complex structure J and a Riemannian metric tensor G which is J-Hermitian, and covered by a system of coordinate neighborhoods $\{W; x^A\}$.

Let M be a real (2n-1)-dimensional hypersurface of $M_n(c)$ covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in $M_n(c)$ by the immersion $i : M \to M_n(c)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n; \quad i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from $i(M)$. Thus, for simplicity, a point p in M may be identified with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j{}^A)$ are also (2n-1)-linearly indenpendent local tangent vectors of M, where $B_j{}^A = \partial_j x^A$ and $\partial_j = \partial / \partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G_{BA}B_j{}^B B_i{}^A$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhood:

$$JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^i B_i, \tag{1.1}$$

where we have put $\phi_{ji} = G(JB_j, B_i)$ and $\xi_i = G(JB_i, C), \xi^h$ being components of a vector field ξ associated with ξ_i and $\phi_{ji} = \phi_j{}^r g_{ri}$. By the properties of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By the properties of the almost complex structure J, the following relations are then given:

$$\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \quad \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0, \quad \xi_i \xi^i = 1,$$

that is, the aggregate (ϕ, g, ξ) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerdern-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j{}^h B_h, \tag{1.2}$$

where $H = (A_{ji})$ is a second fundamental form and $A = (A_j{}^h)$, which is related by $A_{ji} = A_j{}^r g_{ri}$ is the shape operator derived from C. By means of (1.1) and (1.2) the covariant derivatives of the structure tensors are yielded:

$$\nabla_j \phi_i{}^h = -A_{ji} \xi^h + A_j{}^h \xi_i, \quad \nabla_j \xi_i = -A_{jr} \phi_i{}^r. \tag{1.3}$$

Since the ambient space is complex space form, equations of the Gauss and Codazzi for M are respectively given by

(1.4)

$$\begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki}, \end{aligned}$$

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}), \tag{1.5}$$

where R_{kjih} are components of the Riemannian curvature tensor R of M.

In what follows, to write our formulas in convention forms, we denote by $A_{ji}{}^2 = A_{jr}A_i{}^r, h = g_{ji}A^{ji}, \alpha = A_{ji}\xi^j\xi^i$ and $\beta = A_{ji}{}^2\xi^j\xi^i$. If we put $U_j = \xi^r\nabla_r\xi_j$, then U is orthogonal to the structure vector field ξ. Because of the properties of the almost contact metric structure and the second equation of (1.3), we can get

$$\phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j, \tag{1.6}$$

which shows that $g(U,U) = \beta - \alpha^2$.

From (1.3), we have

$$\nabla_k\nabla_j\xi_i = (A_{jr}\xi^r)A_{ki} - A_{jk}{}^2\xi_i - (\nabla_k A_{jr})\phi_i{}^r, \tag{1.7}$$

By the definition of U and the second equation of (1.3), we easily see that

$$U^r\nabla_j\xi_r = A_{jr}{}^2\xi^r - \alpha A_{jr}\xi^r. \tag{1.8}$$

On the other hand, differentiating (1.6) covariantly along M and making use of (1.3), we find

$$\xi_j A_{kr}U^r + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} - \alpha_k\xi_j + \alpha A_{kr}\phi_j{}^r, \tag{1.9}$$

which shows that

$$(\nabla_k A_{ji})\xi^j\xi^i = 2A_{kr}U^r + \alpha_k, \tag{1.10}$$

where $\alpha_k = \partial_k\alpha$.

Transforming (1.9) by $\phi_i{}^j$ and making use of (1.3) and (1.8), we find

$$\nabla_k U_i + \xi_i A_{kr}{}^2\xi^r + \xi^r(\nabla_k A_{sr})\phi_i{}^s = (\nabla_k\xi^r)(\nabla_r\xi_i) + \alpha A_{ki}, \tag{1.11}$$

which together with (1.3) and (1.10) yields

$$\xi^r\nabla_r U_j = -3U^s A_{rs}\phi_j{}^r + \alpha A_{jr}\xi^r - \beta\xi_j - \phi_{jr}\alpha^r. \tag{1.12}$$

We put

$$A_{jr}\xi^r = \alpha\xi_j + \mu W_j, \tag{1.13}$$

where W is a unit vector field orthogonal to ξ. Then by (1.6) we see that $U = \mu\phi W$, and W is also orthogonal to U, and we have $\mu^2 = \beta - \alpha^2$. We assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M | \mu(p) \neq 0\}$. Then Ω is an open subset of M and from now on we discuss our argument on Ω.

Since we have $U_j = \mu\phi_{rj}W^r$, we can, using (1.3), verify that

$$\mu\xi^r\nabla_j W_r = A_{jr}U^r \tag{1.14}$$

because ξ and W are mutually orthogonal.

2. Real hypersurfaces satisfying $R_\xi\phi = \phi R_\xi$

Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$. Then from (1.4) we have

$$(R_\xi)_{ji} = \frac{c}{4}(g_{ji} - \xi_j\xi_i) + \alpha A_{ji} - (A_{jr}\xi^r)(A_{is}\xi^s).$$

Suppose that $R_\xi\phi = \phi R_\xi$. Then we obtain

$$\alpha(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + U_i A_{jr}\xi^r + U_j A_{ir}\xi^r = 0. \tag{2.1}$$

Transforming by U^i, we have

$$U^s A_{sr}\phi_j{}^r = A_{jr}{}^2\xi^r - \lambda A_{jr}\xi^r, \tag{2.2}$$

where we have put $\beta = \alpha\lambda$. Thus, because of properties of the almost contact metric structure it is seen that

$$-\xi^r A_{rs}{}^2\phi_j{}^s = A_{jr}U^r + \lambda U_j. \tag{2.3}$$

Applying (2.1) by $\nabla_k\xi^j$ and using the second equation of (1.3) and (1.6), we find

$$\alpha(\nabla_j\xi^r)(\nabla_r\xi_i) + \alpha A_{ji}{}^2 = (A_{jr}{}^2\xi^r)(A_{is}\xi^s) - U_i A_{jr}U^r.$$

Substituting this into (1.11), we get

(2.4)
$$\nabla_k U_j + (A_{kr}{}^2\xi^r)\xi_j = -\xi^r(\nabla_k A_{sr})\phi_j{}^s + \alpha A_{jk} - A_{jk}{}^2 - \frac{1}{\alpha}\{U_j A_{kr} U^r - (A_{kr}{}^2\xi^r)(A_{js}\xi^s)\}$$

since $\alpha \neq 0$ on Ω.

In what follows, we suppose that

$$A_{jr}{}^2\xi^r = \theta A_{jr}\xi^r + \tau\xi_j$$

for some function θ and τ on M. Then we have $\alpha(\lambda - \theta) = \tau$ and hence

$$A_{jr}{}^2\xi^r = \theta A_{jr}\xi^r + \alpha(\lambda - \theta)\xi_j \tag{2.5}$$

From this and (1.13), we see that

$$A_{jr}W^r = \mu\xi_j + (\theta - \alpha)W_j \tag{2.6}$$

because $\mu \neq 0$ on Ω and consequently

$$A_{jr}{}^2W^r = \mu\theta\xi_j + (\lambda\alpha + \theta^2 - 2\alpha\theta)W_j. \tag{2.7}$$

Differentiating (2.6) covariantly along Ω, we find

$$\begin{aligned}(\nabla_k A_{jr})W^r + A_{jr}\nabla_k W^r &= \\ \mu_k\xi_j + \mu\nabla_k\xi_j + (\theta_k - \alpha_k)W_j &+ (\theta - \alpha)\nabla_k W_j,\end{aligned} \tag{2.8}$$

which enables us to obtain

$$(\nabla_k A_{rs})W^r W^s + \mu\xi^r\nabla_k W_r = \mu W^r\nabla_k\xi_r + \theta_k - \alpha_k$$

or using (1.14)

$$(\nabla_j A_{rs})W^r W^s = -2A_{jr}U^r + \theta_j - \alpha_j \tag{2.9}$$

because ξ and W are mutually orthogonal.

Transvecting (2.8) with ξ^j, we also have

$$\mu(\nabla_k A_{rs})W^r\xi^s = (\theta - 2\alpha)A_{kr}U^r + \mu\mu_k, \tag{2.10}$$

which together with the Codazzi equation (1.5)

$$\mu(\nabla_r A_{ks})W^r\xi^s = (\theta - 2\alpha)A_{kr}U^r - \frac{c}{2}U_k + \mu\mu_k. \tag{2.11}$$

From (2.3) and (2.5), we obtain

$$A_{jr}U^r = (\theta - \lambda)U_j. \tag{2.12}$$

Hence, (1.12) turns out to be

$$\xi^r\nabla_r U_j = (3\lambda - 3\theta + \alpha)\mu W_j - \alpha(\lambda - \alpha)\xi_j - \phi_{jr}\alpha^r \tag{2.13}$$

with the aid of (1.13).

Differentiating (2.12) covariantly along Ω, we find

$$(\nabla_k A_{jr})U^r + A_{jr}\nabla_k U^r = (\theta_k - \lambda_k)U_j + (\theta - \lambda)\nabla_k U_j, \tag{2.14}$$

which together with (2.13) implies that

$$\begin{aligned}(\nabla_k A_{jr})\xi^k U^r = {} & \{d\theta(\xi) - d\lambda(\xi)\}U_j \\ & - (3\lambda - 3\theta + \alpha)\mu\{A_{jr}W^r - (\theta - \lambda)W_j\} \\ & + \alpha(\lambda - \alpha)\{A_{jr}\xi^r - (\theta - \lambda)\xi_j\} + A_j{}^r\phi_{rs}\alpha^s - (\theta - \lambda)\phi_{jr}\alpha^r,\end{aligned}$$

where $d\theta(\xi) = \theta_t\xi^t$. If we use (1.5), then above equation can be written as

$$\begin{aligned}(\nabla_r A_{is})U^r\xi^s = {} & \frac{c}{4}\mu W_i + \{d\theta(\xi) - d\lambda(\xi)\}U_i \\ & - (3\lambda - 3\theta + \alpha)\mu\{A_{ir}W^r - (\theta - \lambda)W_i\} \\ & + \alpha(\lambda - \alpha)\{A_{ir}\xi^r - (\theta - \lambda)\xi_i\} + A_i{}^r\phi_{rs}\alpha^s - (\theta - \lambda)\phi_{ir}\alpha^r.\end{aligned}$$

Transforming for $\phi_j{}^i$ and taking account of (1.6), (1.13) and (2.6), we can get

$$\begin{aligned}(\nabla_r A_{is})U^r\xi^s\phi_j{}^i = &-\frac{c}{4}U_j + \frac{3}{\mu}(\lambda-\theta)(\lambda-\alpha)U_j + \mu U_j \\ &- \alpha(\lambda-\alpha)U_j + \{d\theta(\xi) - d\lambda(\xi)\}\mu W_j \\ &- A_i{}^r\phi_{rs}\alpha^s\phi_j{}^i + (\theta-\lambda)\{\alpha_j - d\alpha(\xi)\xi_j\},\end{aligned}$$

or, using (1.6), (1.13) and (2.1),

$$\begin{aligned}(\nabla_r A_{is})U^r\xi^s\phi_j{}^i = &\{-\frac{c}{4} + \frac{3}{\mu}(\lambda-\theta)(\lambda-\alpha) + \mu - \alpha(\lambda-\alpha) - \frac{1}{\alpha}d\alpha(U)\}U_j \\ &+ \mu\{d\theta(\xi) - d\lambda(\xi) + \frac{1}{\alpha}g(A\xi,\nabla\alpha)\}W_j - A_{jr}\alpha^r \\ &+ g(A\xi,\nabla\alpha)\xi_j + (\theta-\lambda)\{\alpha_j - d\alpha(\xi)\xi_j\},\end{aligned}$$

where we have put $g(A\xi,\nabla\alpha) = A_{ji}\xi^j\alpha^i$. From this and (1.11), we can, using (2.12), verify that

(2.15)

$$\begin{aligned}U^k\nabla_k U_j = &\{\frac{c}{4} - \frac{3}{\mu}(\lambda-\theta)(\lambda-\alpha) - \mu + \alpha(\lambda-\alpha) \\ &+ \frac{1}{\alpha}d\alpha(U) + (\theta-\lambda)(2\alpha-\theta)\}U_j \\ &+ A_{jr}\alpha^r - g(A\xi,\nabla\alpha)\xi_j - (\theta-\lambda)\alpha_j + (\theta-\lambda)d\alpha(\xi)\xi_j \\ &- \mu\{d\theta(\xi) - d\lambda(\xi) + \frac{1}{\alpha}g(A\xi,\nabla\alpha)\}W_j.\end{aligned}$$

If we take the skew-symmetric part of (2.14) and make use of (1.5), then we find

(2.16)

$$\begin{aligned}&\frac{c}{4}(\xi_k A_{jr}\xi^r - \xi_j A_{kr}\xi^r) + A_{jr}\nabla_k U^r - A_{kr}\nabla_j U^r \\ &= (\theta_k - \lambda_k)U_j - (\theta_j - \lambda_j)U_k + (\theta-\lambda)(\nabla_k U_j - \nabla_j U_k),\end{aligned}$$

which together with (2.12) yields

$$A_{jr}U^k\nabla_k U^r - (\theta-\lambda)U^k\nabla_k U_j = \{d\theta(U) - d\lambda(U)\}U_j - \alpha(\lambda-\alpha)(\theta_j-\lambda_j).$$

Substituting (2.15) into this and using (1.13) and (2.6), we obtain

(2.17)
$$\begin{aligned} &{A_{jr}}^2\alpha^r - 2(\theta-\lambda)A_{jr}\alpha^r + (\theta-\lambda)^2\alpha_j \\ &+ \{g(A\xi,\nabla\alpha) - (\theta-\alpha)d\alpha(\xi)\}(\frac{\lambda}{\alpha}A_{jr}\xi^r - (\theta-\lambda)\xi_j\} \\ &= \{d\theta(U) - d\lambda(U)\}U_j - \alpha(\lambda-\alpha)(\theta_j-\lambda_j). \end{aligned}$$

3. Lemmas

In this section, we suppose that a real hypersurface of a complex space form is satisfied $R_\xi\phi = \phi R_\xi$ and $A^2\xi = \theta A\xi + \tau\xi$ for some constant τ. Then we have $\alpha(\lambda-\theta) = const.$ Thus it is clear that

$$(\lambda-\theta)\alpha_j = \alpha(\theta_j - \lambda_j). \tag{3.1}$$

Therefore (2.17) is reduced to

(3.2)
$$\begin{aligned} &\alpha{A_{jr}}^2\alpha^r - 2\alpha(\theta-\lambda)A_{jr}\alpha^r + \alpha(\theta-\lambda)(\theta+\alpha-2\lambda)\alpha_j - \alpha(\theta-\lambda)d\alpha(U)U_j \\ &\quad = \{g(A\xi,\nabla\alpha) - (\theta-\lambda)d\alpha(\xi)\}\{\lambda A_{jr}\xi^r - \alpha(\theta-\lambda)\xi_j\}. \end{aligned}$$

REMARK. Kimura [14] constructed a minimal ruled real hypersurface in P_nC, where the vector field $A^2\xi = \mu^2\xi$ but not ξ is principal. On the other hand, Ahn, Lee and Suh [1] constructed also a similar real hypersurface in H_nC. Thus our assumption has a meaning.

LEMMA 1. *$g(A\xi,\nabla\alpha) = \lambda d\alpha(\xi)$ on Ω.*

PROOF. From (1.13) and (2.6) we have respectively

$$\mu d\alpha(W) = g(A\xi,\nabla\alpha) - \alpha d\alpha(\xi),$$
$$\mu g(AW,\nabla\alpha) = (\theta-\alpha)g(A\xi,\nabla\alpha) + \alpha(\lambda-\theta)d\alpha(\xi).$$

We also have from (2.7)

$$\mu g(A^2W,\nabla\alpha) = (\alpha\lambda+\theta^2-2\alpha\theta)g(A\xi,\nabla\alpha) + \alpha(\lambda-\theta)(\theta-\alpha)d\alpha(\xi).$$

Transforming (3.2) for U^j and taking account of the last three equations, we can see that

$$(\theta - \lambda)\{g(A\xi, \nabla\alpha) - \alpha d\alpha(\xi)\} = 0.$$

Let Ω_1 be the set of points in Ω such that $g(A\xi, \nabla\alpha) - \alpha d\alpha(\xi) \neq 0$, and suppose that Ω_1 be nonvoid. Then we have $\theta - \lambda = 0$ and hence $AU = 0$ on Ω_1. Thus (2.5) means that $A\xi$ is principal curvature vector field on Ω_1. Further (2.16) becomes

$$A_{jr}\nabla_k U^r - A_{kr}\nabla_j U^r + \frac{c}{4}(\xi_k A_{jr}\xi^r - \xi_j A_{kr}\xi^r) = 0.$$

From this, we can verify that ξ is a principal curvature vector field (for detail, see [11]), which a contradiction. This completes the proof. □

Since $\alpha(\lambda - \theta)$ is constant, by differentiating (2.5) covariantly along Ω and using the second equation of (1.3), we find

(3.3)
$$\begin{aligned}(\nabla_k A_{jr})A_s{}^r\xi^s + A_j{}^r(\nabla_k A_{rs})\xi^s - A_{jr}{}^2 A_{ks}\phi^{rs} + \theta A_{jr}A_{ks}\phi^{rs} \\ = \theta_k A_{jr}\xi^r + \theta\xi^r\nabla_k A_{jr} - \alpha(\lambda - \theta)A_{kr}\phi_j{}^r.\end{aligned}$$

Transvecting (3.3) with ξ^j and making use of (1.10), we obtain

$$(\nabla_k A_{rs})\xi^r A_t{}^s\xi^t = \theta A_{kr}U^r + \frac{1}{2}(\alpha\theta)_k, \tag{3.4}$$

which together with (1.5) and (2.12) gives

$$(\nabla_t A_{kr})\xi^t A_s{}^r\xi^s = \{\theta(\theta - \lambda) - \frac{c}{4}\}U_k + \frac{1}{2}(\alpha\theta)_k. \tag{3.5}$$

If we transvect (3.3) with ξ^k and using (1.10), (2.12) and (3.5), we find

$$\{(\theta - \lambda)(\theta + \alpha - 3\lambda) - \frac{c}{4}\}U_j = d\theta(\xi)A_{jr}\xi^r - A_{jr}\alpha^r + \frac{1}{2}(\theta\alpha_j - \alpha\theta_j). \tag{3.6}$$

Since we have, $g(U,U)=\alpha(\lambda-\alpha)$ and $g(A\xi,U)=0$, by applying U^j, we have

(3.7) $$(2\lambda-\theta)d\alpha(U)-\alpha d\theta(U)=2\alpha(\lambda-\alpha)\{(\theta-\lambda)(\theta+\alpha-3\lambda)-\frac{c}{4}\}.$$

On the other hand, using (1.10), (2.9), (2.10), (2.11) and (2.12), we get

$$\begin{aligned}&(\nabla_k A_{jr})(\alpha\xi^k+\mu W^k)(\alpha\xi^r+\mu W^r)\\&\qquad=\{2\alpha(\theta-\lambda)^2-\frac{3}{4}c\alpha\}U_j+\alpha\lambda\theta_j+\alpha(\theta-\lambda)\alpha_j.\end{aligned}$$

Transforming for $A_t{}^k\xi^t$ to (3.3) and taking account of (1.5), (1.13), (2.6), (3.4) and the last relationship, we have

$$\begin{aligned}&\{2\alpha(\theta-\lambda)^2-\frac{3}{4}c\alpha\}U_j+\alpha\lambda\theta_j\\&+\alpha(\theta-\lambda)\alpha_j+2\theta A_{jr}{}^2U^r-2\theta^2A_{jr}U^r-\frac{c}{2}A_{jr}U^r\\&+\{\frac{c}{2}-\theta\alpha(\lambda-\theta)\}U_j+\frac{1}{2}A_{jr}(\theta\alpha)^r-\frac{1}{2}\theta(\theta\alpha)_j\\&=g(A\xi,\nabla\theta)A_{jr}\xi^r,\end{aligned}$$

or using (2.12),

(3.8)
$$\begin{aligned}g(A\xi,\nabla\theta)A_{jr}\xi^r=&\frac{1}{2}A_{jr}(\theta\alpha)^r-\frac{1}{2}\theta(\theta\alpha)_j+\alpha\lambda\theta_j+\alpha(\theta-\lambda)\alpha_j\\&+\{(\theta-\lambda)(3\alpha\theta-2\alpha\lambda-2\theta\lambda)+\frac{c}{2}\lambda-\frac{3}{4}c\alpha\}U_j.\end{aligned}$$

Applying by U^j, we obtain

$$\begin{aligned}&\frac{1}{2}\alpha\lambda d\theta(U)+(\alpha\theta-\alpha\lambda-\frac{1}{2}\lambda\theta)d\alpha(U)\\&+\{(\theta-\lambda)(3\alpha\theta-2\alpha\lambda-2\theta\lambda)+\frac{c}{2}\lambda-\frac{3}{4}c\alpha\}\alpha(\lambda-\alpha)=0,\end{aligned}$$

which together with (3.7) implies that

(3.9) $$(\theta-\lambda)d\alpha(U)=3\alpha(\lambda-\alpha)\{\frac{c}{4}-(\theta-\lambda)^2\}.$$

Transvecting (3.6) with ξ^j and using Lemma 1, we have

(3.10) $$\alpha d\theta(\xi)=(2\lambda-\theta)d\alpha(\xi).$$

LEMMA 2. $\mu\alpha d\theta(W) = (\lambda - \alpha)(4\lambda - 3\theta)d\alpha(\xi)$ *on* Ω.

PROOF. From (1.13) and Lemma 1, it is clear that

$$\mu d\alpha(W) = (\lambda - \alpha)d\alpha(\xi). \tag{3.11}$$

Applying (3.8) by U^j and making use of (1.13), we find

$$\begin{aligned} &\mu(\alpha d\theta(W) - \theta d\alpha(W)) \\ &= (3\alpha\theta - 2\alpha\lambda - \theta^2)d\alpha(\xi) + \alpha(2\lambda - \theta - \alpha)d\theta(\xi). \end{aligned}$$

If we substitute (3.10) and (3.11) into this, we can get the required equation. This completes the proof of Lemma 2. □

Because of (3.9) and Lemma 1, (3.2) is reduced to

(3.12)
$$\begin{aligned} &\alpha A_{jr}{}^2\alpha^r - 2\alpha(\theta - \lambda)A_{jr}\alpha^r + \alpha(\theta - \lambda)(\theta + \alpha - 2\lambda)\alpha_j \\ &= d\alpha(\xi)(2\lambda - \theta)\{\lambda A_{jr}\xi^r - \alpha(\theta - \lambda)\xi_j\} + 3\alpha(\lambda - \alpha)\{\frac{c}{4} - (\theta - \lambda)^2\}U_j. \end{aligned}$$

LEMMA 3. $(\theta - \lambda)\alpha\alpha_j = (\theta - \lambda)d\alpha(\xi)A_{jr}\xi^r + 3\alpha\{\frac{c}{4} - (\theta - \lambda)^2\}U_j$ *on* Ω.

PROOF. Because of (3.9) and Lemma 1, (3.2) turns out to be

(3.13)
$$\begin{aligned} &\alpha A_{jr}{}^2\alpha^r - 2\alpha(\theta - \lambda)A_{jr}\alpha^r + \alpha(\theta - \lambda)(\theta + \alpha - 2\lambda)\alpha_j \\ &= (2\lambda - \theta)d\alpha(\xi)\{\lambda A_{jr}\xi^r - \alpha(\theta - \lambda)\xi_j\} + 3\alpha(\lambda - \alpha)\{(\theta - \lambda)^2 - \frac{c}{4}\}U_j. \end{aligned}$$

From (3.6) and (3.10) we have

(3.14)
$$\begin{aligned} &\alpha A_{jr}\alpha^r + \frac{1}{2}\alpha(\alpha\theta_j - \theta\alpha_j) \\ &\quad = (2\lambda - \theta)d\alpha(\xi)A_{jr}\xi^r + \alpha\{(\theta - \lambda)(3\lambda - \alpha - \theta) + \frac{c}{4}\}U_j, \end{aligned}$$

which together with (2.5) and (2.12) implies that

$$\begin{aligned}(3.15)\qquad & \alpha A_{jr}{}^2\alpha^r + \frac{1}{2}\alpha A_{jr}(\theta\alpha)^r - \theta\alpha A_{jr}\alpha^r \\ &= (2\lambda-\theta)d\alpha(\xi)\{\theta A_{jr}\xi^r + \alpha(\lambda-\theta)\xi_j\} \\ &+ \alpha(\theta-\lambda)\{(\theta-\lambda)(3\lambda-\alpha-\theta) + \frac{c}{4}\}U_j.\end{aligned}$$

Making use of (1.13), (3.11) and Lemma 2, we have

$$\alpha g(A\xi, \nabla\theta) = \{\alpha(2\lambda-\theta) + (\lambda-\alpha)(4\lambda-3\theta)\}d\alpha(\xi).$$

Thus (3.8) becomes

$$\begin{aligned}\frac{1}{2}\alpha A_{jr}(\theta\alpha)^r =& \{\alpha(2\lambda-\theta) + (\lambda-\alpha)(4\lambda-3\theta)\}d\alpha(\xi)A_{jr}\xi^r \\ &+ (\frac{1}{2}\theta^2 - \alpha\theta + \alpha\lambda)\alpha\alpha_j - \alpha^2(\lambda - \frac{1}{2}\theta)\theta_j \\ &+ \{(\theta-\lambda)(2\theta\lambda + 2\alpha\lambda - 3\alpha\theta) - \frac{c}{2}\lambda + \frac{3}{4}c\alpha\}U_j.\end{aligned}$$

Substituting (3.13) and this into (3.15), we find

$$\begin{aligned}&\alpha(\theta-2\lambda)A_{jr}\alpha^r \\ &\quad + \{\frac{1}{2}\theta^2 - \alpha\theta + \alpha\lambda - (\theta-\lambda)(\theta-2\lambda+\alpha)\}\alpha\alpha_j - \alpha^2(\lambda-\frac{1}{2}\theta)\theta_j \\ &= \{(2\lambda-\theta)(\theta-\lambda-\alpha) - (\lambda-\alpha)(4\lambda-3\theta)\}d\alpha(\xi)A_{jr}\xi^r \\ &\quad + \alpha(\theta-\lambda)\{(\theta-\lambda)(2\alpha-\theta) + 3\alpha\theta - 2\alpha\lambda - 2\theta\lambda\}U_j \\ &\quad + \frac{c}{4}\alpha(\theta+4\lambda-6\alpha)U_j,\end{aligned}$$

or, using (3.14) we get the required equation. Hence Lemma 3 is proved.□

As in the proof of Lemma 1, we know that $\theta-\lambda$ does not vanish in Ω. Thus, from (3.9) and Lemma 3, we have

$$(3.16)\qquad \alpha\alpha_j = \rho A_{jr}\xi_r + \tau U_j$$

on Ω, where we have put

$$(3.17)\qquad \rho = d\alpha(\xi), \qquad (\lambda-\alpha)\tau = d\alpha(U)$$

LEMMA 4. *$d\alpha(U)d\alpha(\xi) = 0$ on Ω.*

PROOF. Because of properties of the almost contact metric structure, we see, using (3.16), that

$$\alpha\phi_{jr}\alpha^r = -\rho U_j + \tau(A_{jr}\xi^r - \alpha\xi_j). \tag{3.18}$$

On the other hand, from (2.4) we have

(3.19)
$$\begin{aligned}\nabla_k U_j - \nabla_j U_k &+ \theta(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r)\\ &= \xi^r(\nabla_j A_{rs})\phi_k{}^s - \xi^r(\nabla_k A_{rs})\phi_j{}^s - (\lambda - \theta)(\xi_k A_{jr}\xi^r - \xi_j A_{kr}\xi^r),\end{aligned}$$

where we have used (2.5) and (2.12).
Transvecting (3.19) with ξ^k and taking account of (1.6), (1.10) and (3.18), we find

$$\alpha\xi^k(\nabla_k U_j - \nabla_j U_k) = (\lambda - \tau)(A_{jr}\xi^r - \alpha\xi_j) + \rho U_j. \tag{3.20}$$

Differentiating (3.16) covariantly along Ω and using (1.3), we find

$$\alpha_k\alpha_j + \alpha\nabla_k\alpha_j = \rho_k A_{jr}\xi^r + \rho(\nabla_k A_{jr})\xi^r - \rho A_{jr}A_{ks}\phi^{rs} + \tau_k U_j + \tau\nabla_k U_j,$$

from which, taking the skew-symmetric part and making use of (1.5),

$$\begin{aligned}\rho_k A_{jr}\xi^r - \rho_j A_{kr}\xi^r - \frac{c}{2}\rho\phi_{kj} - 2\rho A_{jr}A_{ks}\phi^{rs} + \tau_k U_j - \tau_j U_k&\\ + \tau(\nabla_k U_j - \nabla_j U_k) &= 0.\end{aligned}$$

Applying this by ξ^k and taking account of (2.12) and (3.20), we get

$$\begin{aligned}\alpha^2\rho_j =& \{\alpha d\rho(\xi) + \tau(\lambda - \tau)\}A_{jr}\xi^r - \alpha\tau(\lambda - \tau)\xi_j\\ &+ \{2\alpha\rho(\theta - \lambda) + \rho\tau + \alpha d\tau(\xi)\}U_j.\end{aligned}$$

Thus, the last equation is reduced to

(3.21)
$$\begin{aligned}&\tau(\lambda - \tau)(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r)\\ &+ \{2\rho(\theta - \lambda) - \frac{1}{\alpha}\rho\tau + d\tau(\xi)\}(U_k A_{jr}\xi^r - U_j A_{kr}\xi^r)\\ &- \frac{c}{2}\rho\alpha\phi_{kj} - 2\rho\alpha A_{jr}A_{ks}\phi^{rs}\\ &+ \alpha(\tau_k U_j - \tau_j U_k) + \alpha\tau(\nabla_k U_j - \nabla_j U_k) = 0.\end{aligned}$$

From (3.9) we have $(\theta-\lambda)\tau = 3\alpha\{\frac{c}{4}-(\theta-\lambda)^2\}$. Differentiating this along Ω and using (3.1), we obtain $\alpha\tau_j = 2\tau\alpha_j - 6\alpha\alpha_j$, which together with (3.16) implies

$$\alpha^2\tau_j = 2(\tau-3\alpha)(\rho A_{jr}\xi^r + \tau U_j).$$

Hence (3.21) becomes

(3.22)

$$\tau(\lambda-\tau)(\xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r) + (2\theta - 2\lambda + \frac{\tau}{\alpha})\rho(U_k A_{jr}\xi^r - U_j A_{kr}\xi^r)$$
$$-\frac{c}{2}\rho\alpha\phi_{kj} - 2\rho\alpha A_{jr}A_{ks}\phi^{rs} + \tau\alpha(\nabla_k U_j - \nabla_j U_k) = 0.$$

Transvecting (3.22) with $U^k W^j$ and using (2.5) and (2.12), we find

(3.23)

$$\tau U^k W^j(\nabla_k U_j - \nabla_j U_k) + (2\theta - 2\lambda + \frac{\tau}{\alpha})\rho\mu(\lambda-\alpha)$$
$$+\frac{c}{2}\mu\rho - 2\rho\mu(\theta-\lambda)(\theta-\alpha) = 0.$$

On the other hand, transvecting (2.15) with W^j and making use of (2.6) and (3.1), we find

$$W^j U^k \nabla_k U_j = \mu\tau + (\lambda-\alpha)d\alpha(W) + \frac{\mu}{\alpha}\theta\tau,$$

or using (3.11),

$$\alpha W^j U^k \nabla_k U_j = \mu(\theta-\lambda)\tau.$$

Applying (2.4) for $W^k U^j$ and taking account of (2.9), (2.12) and (3.10), we also have

$$\alpha W^k U^j \nabla_k U_j = \mu(2\lambda-\theta-\alpha)\tau.$$

Combining the last two relationships, we can get

$$\alpha W^j U^k(\nabla_k U_j - \nabla_j U_k) = \mu\tau(2\theta - 3\lambda + \alpha).$$

Substituting this into (3.23), we obtain $\mu\rho\{(\theta-\lambda)\tau - \alpha(\theta-\lambda)^2 + \frac{c}{4}\alpha\} = 0$. From this and (3.9) and (3.17), we see that $(\theta-\lambda)\tau\rho = 0$, which proves $\tau\rho = 0$ on Ω. This completes the proof. □

4. Proof of theorems

First of all, we shall prove that ξ is a principal curvature vector field under the assumptions as those stated in section 3.

From (3.16) and Lemma 4, we easily see that ρ vanishes on Ω and hence $\alpha\alpha_j = \tau U_j$, and (3.20) becomes

$$\alpha\xi^k(\nabla_k U_j - \nabla_j U_k) = (\lambda - \tau)\mu W_j \tag{4.1}$$

because of (1.13).

By (1.12) we have

$$\xi^r \nabla_r U_j = -3(\theta - \lambda)\mu W_j + \alpha A_{jr}\xi^r - \lambda\alpha\xi_j - \frac{\tau}{\alpha}\mu W_j,$$

where we have used (2.5), which together with (1.7) and (2.5) gives

$$\xi^r(\nabla_r U_j - \nabla_j U_r) = (3\lambda - 2\theta - \frac{\tau}{\alpha})\mu W_j.$$

From this and (4.1) we verify that $\theta - \lambda = 0$, which a contradiction. Consequently we have proved that Ω is empty.

Thus we have

PROPOSITION 5. *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$ such that $R_\xi\phi = \phi R_\xi$. If the shape operator A of M satisfies $A^2\xi = \theta A\xi + \tau\xi$ for some constant τ, then ξ is a principal curvature vector field, where θ is a function on M.*

From (1.4) we see that the Ricci tensor S of M is given by

$$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - {A_{ji}}^2,$$

where $h = trA$.

Now, we suppose that $S\xi = \sigma\xi$ for some constant σ on M. Then we have $A^2\xi = hA\xi + \{\frac{c}{2}(n-1) - \sigma\}\xi$. Therefore by virtue of (2.1) we obtain $U = 0$ and hence $\alpha(A\phi - \phi A) = 0$ because of Proposition 5.

According to Theorem C, we have

THEOREM 6. *Let M be a connected real hypersurface of a complex space form $M_n(c), c \neq 0$ satisfying $R_\xi \phi = \phi R_\xi$. If $S\xi = \sigma\xi$ for some constant σ, then M is locally congruent to be of type A provided that $g(A\xi, \xi) \neq 0$.*

REMARK. In the pseudo umbilical real hypersurface of a complex space form, we can reduce that $S\xi = \sigma\xi$ and σ is constant.

In the previous paper [6], [8], we proved that if a real hypersurface of a complex space form is satisfied $R_\xi \phi A = A\phi R_\xi$. Then we have from (1.4)

$$\frac{c}{4}(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) = (A_{jr}\xi^r)(A_{is}U^s) + (A_{ir}\xi^r)(A_{js}U^s)$$

and hence $\alpha AU = -\frac{c}{4}U$. Thus, it follows that

$$\alpha(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + U_j A_{ir}\xi^r + U_i A_{jr}\xi^r = 0,$$

namely, $R_\xi \phi = \phi R_\xi$. From this we can easily see that $A^2\xi = \theta A\xi + \frac{c}{4}\xi$ for some differentiable function θ. Thus, owing to Theorem C and Proposition 5, we have

COROLLARY 7(cf. [8]). *Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$. If M satisfies $R_\xi \phi A = A\phi R_\xi$, then M is locally congruent to be of type A.*

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