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ON THE BRAKED SUBSIMILAR SETS

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ABSTRACT. We define a new form of fractals, called the braked subsimilar set from a self similar set and find the relation between their fractal dimensions.

1. Introduction

Self similar sets are used in many fields of mathematics such as dynamical systems, chaos theory and fractal geometry as well as in various physical phenomena. These self similar sets are defined mathematically as follows. Let $S_1, S_2, \cdots, S_N : X \to X$ be the similarity maps on a complete metric space (X, d) with each contraction ratio $0 < r_i < 1$, i.e.

$$d(S_i(x), S_i(y)) = r_i d(x, y) \text{ for } i = 1, 2, \cdots, N \text{ and } x, y \in X.$$

Then there exists an invariant set F which we call the self similar set. In fact,

$$F = \bigcap_{n>0} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1, 2, \cdots, N\}^n} S_{(i_1, i_2, \cdots, i_n)}(X)$$

in which $S_{(i_1, i_2, \cdots, i_n)}(X)$ means $S_{i_1} \circ S_{i_2} \circ \cdots S_{i_n}(X)$ and

$$\bigcup_{i=1}^{N} S_i(F) = F.$$

In particular, when all contraction ratios are equal, we call above set the monic self similar set.

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In this paper, by weakening the process of generating the self similar set, we define the new fractals, called the braked subsimilar set as a shadow image, and compare them by investigating the relations of their fractal dimensions. Let's now recall that the definitions of dimensions and measures appropriate to the fractal set. For given $E \subset X$ and $\delta > 0$, let $\mathcal{C}_\delta(E)$ be defined such that $\{U_i\} \in \mathcal{C}_\delta(E)$ if and only if $E \subset \cup U_i$ and $\mathrm{diam}(U_i) < \delta$ for all i. Then the α-dimensional Hausdorff measure of E, $H^\alpha(E)$, is defined by

$$H^\alpha(E) = \lim_{\delta \to 0} \inf\{\sum \mathrm{diam}(U_i)^\alpha : \{U_i\} \in \mathcal{C}_\delta(E)\}$$

and the Hausdorff dimension of E, $\dim(E)$, is defined by

$$\dim(E) = \inf\{\alpha : H^\alpha(E) < \infty\}.$$

And let $\mathcal{P}_\delta(E)$ be defined such that $\{U_i\} \in \mathcal{P}_\delta(E)$ if and only if all U_i's are pairwise disjoint balls with each center in E. Then α-dimensional packing measure of E, $P^\alpha(E)$, is defined by

$$P^\alpha(E) = \inf\{\sum p^\alpha(E_i) : \{E_i\} \text{ with } E \subset \cup E_i\}$$

where

$$p^\alpha(E) = \lim_{\delta \to 0} \sup\{\sum \mathrm{diam}(U_i)^\alpha : \{U_i\} \in \mathcal{P}_\delta(E)\}$$

and the packing dimension of E, $\mathrm{Dim}(E)$, is defined by

$$\mathrm{Dim}(E) = \inf\{\alpha : P^\alpha(E) < \infty\}.$$

In general, it is well known that $\dim(E) \leq \mathrm{Dim}(E)$. Moreover if F is self similar set induced by the similarity maps $S_1, S_2, \cdots, S_N$ as above and each S_i satisfies the open set condition, i.e. there exists a non-empty open set U in X such that $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$, then $\dim(E)$ or $\mathrm{Dim}(E)$ is s, in which s is calculated by the equation $\sum_i^N r_i^s = 1$ [1].

From now on, we assume that all S_i's satisfy the open set condition.

2. Main Results

Let the self similar set F be generated by the similar maps $S_1, S_2, \cdots, S_N$ as above and let

$$\Sigma = \{\sigma : \sigma = (i_1, i_2, \cdots),\ i_j = 1, 2, \cdots, N\}.$$

Then for each $\sigma = (i_1, i_2, \cdots) \in \Sigma$, define $\sigma|_n = (i_1, i_2, \cdots, i_n)$ and let

$$\Sigma_n = \{\sigma|_n : \sigma \in \Sigma\}.$$

And for each σ in Σ_n and τ in Σ or Σ_m, say, $\sigma = (i_1, i_2, \cdots, i_n)$ and $\tau = (j_1, j_2, \cdots, j_m)$ (or $\tau = (i_1, i_2, \cdots)$), let

$$\sigma\tau = (i_1, i_2, \cdots, i_n, j_1, j_2, \cdots, j_m) \text{ (or } = (i_1, i_2, \cdots, i_n, j_1, j_2, \cdots)),$$

$$\sigma \wedge \tau = (i_1, i_2, \cdots, i_l) \text{ if } i_k = j_k \text{ for } k \leq l \text{ and } i_{l+1} \neq j_{l+1},$$

and let

$$I[\sigma] = \{\tau \in \Sigma : \tau|_n = \sigma\}.$$

And consider the braking systems, $(\{n_k\}, \{m_k\})$, two increasing sequences of integers such that

$$0 < n_1 < m_1 \leq n_2 < m_2 \leq \cdots.$$

Then define Σ' and Σ'_n by

$$\Sigma' = \{\sigma = (i_1, i_2, \cdots) \in \Sigma : i_j = a(j) \text{ if } n_k \leq i_j < m_k \text{ for some } k\}$$

where $a(j)$ is the fixed value in $\{1, 2, \cdots, N\}$ and let

$$\Sigma'_n = \{\sigma'|_n : \sigma' \in \Sigma'\}.$$

REMARK. When two increasing sequences $\{n_k\}$ and $\{m_k\}$ are finite sequences with l terms such that

$$0 < n_1 < m_1 \leq n_2 < m_2 \leq \cdots \leq n_l < m_l,$$

we may suppose the braking system $(\{n_k\}; \{m_k\})$ by defining $n_i = \infty$ and $m_i = \infty$ for $i > l$.

DEFINITION 2.1. Let F be the self similar set generated by N similar maps S_i, $(i = 1, 2, \cdots, N)$ and $(\{n_k\}, \{m_k\})$ be the braking systems as above. Then the braked subsimilar set $F(n_k; m_k)$ is defined by

$$F(n_k; m_k) = \bigcap_{n>0} \bigcup_{\sigma'|_n \in \Sigma'_n} S_{\sigma'|_n}(X).$$

In general, this braked subsmilar set $F(n_k; m_k)$ is the subset of self similar set F but need not be a self similar set.

Now let us find the relations between their fractal dimensions.

DEFINITION 2.2. For the braking systems $(\{n_k\}; \{m_k\})$ as above, the upper (or lower) braking ratio $\bar{\alpha}$ (or $\underline{\alpha}$) is defined by

$$\bar{\alpha} = \limsup_{k\to\infty} \Big(\sum_{i=1}^{\alpha_k} d_i + e_k\Big)/k \quad \Big(\text{or } \underline{\alpha} = \liminf_{n\to\infty} \Big(\sum_{i=1}^{\alpha_k} d_i + e_k\Big)/k\Big)$$

where $\alpha_k = \max\{i : m_i \le k\}$, $d_i = m_i - n_i$ and $e_k = k - \min\{k, n_{\alpha_{k+1}}\}$.

LEMMA 2.3. ([2], [3]). *Let μ be a mass distribution on E, i.e. a Borel measure on E with $0 < \mu(E) < \infty$ and let $A \subset E$.*

(a) If $\liminf_{r\to 0} \log \mu(B_r(x))/\log r \ge t$ for all $x \in A$, then

$$\dim(A) \ge t.$$

(b) If $\limsup_{r\to 0} \log \mu(B_r(x))/\log r \le t$ for all $x \in A$, then

$$\mathrm{Dim}(A) \le t.$$

THEOREM 2.4. *Let F be a self similar set generated by N similar maps S_i's with each contraction ratio $0 < r_i < 1$, $(i = 1, 2, \cdots, N)$ and let $F(n_k; m_k)$ be the braked subsimilar set of F induced by the braking systems $(\{n_k\}; \{m_k\})$. Then*

$$\begin{aligned}\{1 - \limsup_{n\to\infty} \sup_{\sigma|_n \in \Sigma'_n} (\log \hat{\mathbf{r}}_{\sigma|_n}/\log \mathbf{r}_{\sigma|_n})\}\dim(F) & \\ &\le \dim\{F(n_k; m_k)\} \\ &\le \mathrm{Dim}\{F(n_k; m_k)\} \\ &\le \{1 - \liminf_{n\to\infty} \inf_{\sigma|_n \in \Sigma'_n} (\log \hat{\mathbf{r}}_{\sigma|_n}/\log \mathbf{r}_{\sigma|_n})\}\dim(F),\end{aligned}$$

in which

$$\mathbf{r}_{\sigma|_n} = \prod_{1\le j\le n} r_{i_j}$$

and

$$\hat{\mathbf{r}}_{\sigma|_n} = \prod\{r_{i_j} : n_k \le i_j < m_k \ (1 \le k \le \alpha_n) \text{ or } n_{\alpha_{n+1}} \le i_j \le n\}$$

for $\sigma|_n = (i_1, i_2, \cdots, i_n)$.

PROOF. To apply Lemma 2.3, we first define the mass distribution on $F(n_k; m_k)$. Let $s = \dim(F)$ or $\sum_{i=1}^N r_i^s = 1$ and suppose that $\mathrm{diam}(X) = 1$. And for each $\sigma'|_n \in \Sigma'_n$ define μ by

$$\mu(S_{\sigma'|_n}(X)) = \mu(I(\sigma'|_n) = (\mathbf{r}_{\sigma'|_n}/\hat{\mathbf{r}}_{\sigma'|_n})^s.$$

Then for each $\sigma'|_n \in \Sigma'_n$,

$$\sum_{\substack{j \\ \sigma'|_n(j)\in\Sigma'_{n+1}}} \mu(S_{\sigma'|_n(j)}(X)) = \mu(S_{\sigma'|_n}(X))$$

since

$$\sum_{\substack{j \\ \sigma'|_n(j)\in\Sigma'_{n+1}}} \mu(I(\sigma'|_n(j)) = \begin{cases} \mu(I(\sigma'|_n)) \sum_{i=1}^N r_i^s, & \text{for } j \notin \bigcup [n_k, m_k), \\ \mu(I(\sigma'|_n)) r_j^s / r_j^s, & \text{otherwise.} \end{cases}$$

Thus μ can be extended to the mass distribution on Σ' with

$$\mu(\Sigma') = \begin{cases} \sum_{i=1}^N r_i^s = 1 & \text{for } 1 < n_i, \\ \qquad r_j^s & \text{for } n_1 = 1, \ (j) \in \Sigma'_1, \end{cases}$$

and so it can be viewed as a mass distribution on $F(n_k; m_k)$ by defining

$$\mu(A) = \mu\{\sigma \in \Sigma' : S_\sigma(X) = \lim_{n\to\infty} S_{\sigma|_n}(X) \in A\}$$

for each $A \subset F(n_k; m_k)$. Put $\dot{r} = \max_{1\le i\le N} r_i$ and let $r > 0$. Then for each $x \in F(n_k; m_k)$ there exists n with $\dot{r}^n < r \le \dot{r}^{n-1}$ and $\sigma' \in \Sigma'$ such that $x \in S_{\sigma|_n}(X)$. Since $\mathrm{diam}(S_{\sigma|_n}(X)) < \dot{r}^n < r$, we can apply above Lemma in sequential form.

$$
\begin{aligned}
&\limsup_{r\to 0} \log \mu(B_r(x)/\log r\\
&= \limsup_{n\to\infty} \log(\mathbf{r}_{\sigma'|_n}/\hat{\mathbf{r}}_{\sigma'|_n})^s/\log \mathbf{r}_{\sigma'|_n}\\
&= s\{\limsup_{n\to\infty}(\log \mathbf{r}_{\sigma'|_n} - \log \hat{\mathbf{r}}_{\sigma'|_n})/\log \mathbf{r}_{\sigma'|_n}\}\\
&= s\{\limsup_{n\to\infty}\{1-(\log \hat{\mathbf{r}}_{\sigma'|_n}/\log \mathbf{r}_{\sigma'|_n})\}\}\\
&= s(1-\liminf_{n\to\infty} \log \hat{\mathbf{r}}_{\sigma'|_n}/\log \mathbf{r}_{\sigma'|_n})\\
&\le s(1-\liminf_{n\to\infty} \inf_{\tau'|_n\in\Sigma'_n} \log \hat{\mathbf{r}}_{\tau'|_n}/\log \mathbf{r}_{\tau'|_n})
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\liminf_{r\to 0} \log \mu(B_r(x)/\log r\\
&= \liminf_{n\to\infty} \log(\mathbf{r}_{\sigma'|_n}/\hat{\mathbf{r}}_{\sigma'|_n})^s/\log \mathbf{r}_{\sigma'|_n}\\
&= s(1-\limsup_{n\to\infty} \log \hat{\mathbf{r}}_{\sigma'|_n}/\log \mathbf{r}_{\sigma'|_n})\\
&\ge s(1-\limsup_{n\to\infty} \sup_{\tau'|_n\in\Sigma'_n} \log \hat{\mathbf{r}}_{\tau'|_n}/\log \mathbf{r}_{\tau'|_n}).
\end{aligned}
$$

Thus from Lemma 2.3, we have the Theorem. □

COROLLARY 2.5. *Let F be a monic self similar set generated by N similar maps satisfying open set condition. Then for the braking systems $(\{n_k\};\{m_k\})$ with upper (or lower) braking ratio $\bar{\alpha}$ (or $\underline{\alpha}$), we have*

$$
\begin{aligned}
(1-\bar{\alpha})\dim(F) &\le \dim\{F(n_k; m_k)\}\\
&\le \mathrm{Dim}\{F(n_k; m_k)\} \le (1-\underline{\alpha})\dim(F).
\end{aligned}
$$

In particular, when $\underline{\alpha} > 0$, we have

$$
\dim\{(F \setminus F(n_k; m_k)\} = \mathrm{Dim}\{(F \setminus F(n_k; m_k)\} = \dim(F).
$$

EXAMPLE 2.6. Let $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ for $0 \le x \le 1$. Then these two maps generate the usual symmetric Cantor set F, i.e.

$$F = \{x \in [0,1] : x = \sum \frac{b_n}{3^n}, b_n = 0, 2 \text{ for } n = 1, 2, \cdots\}.$$

Then a braking subsimilar set $F(n_k; m_k)$ generated by a finite braking system $(\{n_k\}_{k=1}^L; \{m_k\}_{k=1}^L)$, for example,

$$\{x \in [0,1] : x = \sum \frac{b_n}{3^n}, b_n = 0, 2 \text{ but } = 0 \text{ for } n_k \le n < m_k\},$$

has fractal dimension $\log 2/\log 3$ since the brake length $\alpha = 0$.

Now consider another braking system $(\{n_k\} = \{3k-1\}; \{m_k\} = \{3k+1\})$. Then the braking subsimilar set $F(n_k; m_k)$ induced by this system, for example,

$$\left\{x \in [0,1] : x = \sum \frac{b_n}{3^n}, \quad b_n = \begin{cases} 0 & \text{for } n = 3k-1, \\ 2 & \text{for } n = 3k, \\ 0, 2 & \text{otherwise.} \end{cases} \right\},$$

has fractal dimension $\frac{1}{3}\log 2/\log 3$, since the braking length $\alpha = 2/3$.

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