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NOTES ON THE FACTORS OVER THE HILBERT SPACE $L^2(\mathcal{R}_G, \mu_l)$

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Abstract. We construct factors of type III and of type II_∞ over the Hilbert spaces on the equivalence relations.

1. Introduction

In this paper, we show that the relations between the Hilbert spaces on the equivalence relations over the product space with the σ-finite measure and the Hilbert spaces on the equivalence relations. For this we construct the von Neumann algebras on the equivalence relation.

In §2, we define the equivalence relations and group actions and introduce some properties. Also we construct a measure on the equivalence relation(is called the left counting measure). For given relations, we define ergodicity and invariance and give some assumptions for the measure theoretic construction. The properties of ergodicity and invariance are introduced and we define the factors in the measure theoretic sense.

In §3, we define the Hilbert space $L^2(\mathcal{R}_G, \mu_l)$ on the equivalence relations $\mathcal{R}_G$ and prove $L^2(\mathcal{R}_G, \mu_l) \cong L^2(X) \otimes l^2(G)$ and $L^2(\widetilde{\mathcal{R}}_{\widetilde{G}}, \widetilde{\mu}_l) \cong L^2(\mathcal{R}_G, \mu_l) \otimes L^2(\mathbb{R}, m)$. Also we construct factors $\mathcal{W}^*(\mathcal{R}_G)$ of type III and $\mathcal{W}^*(\widetilde{\mathcal{R}}_{\widetilde{G}})$ of type II_∞.

2. The Hilbert Space $L^2(X, \mu)$ on $(X, \mathfrak{B}, \mu)$ and von Neumann Algebra on $L^2(X, \mu)$

Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and $\mathcal{R} \subset X \times X$ be an equivalence relation. We write $x \sim y$ for $(x, y) \in \mathcal{R}$, and define $\pi_l(x, y) = x$, *the*

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left projection of $\mathcal{R}$, and $\pi_r(x,y) = y$, *the right projection* of $\mathcal{R}$. For any $x \in X, \mathcal{R}(x) = \{y|(x,y) \in \mathcal{R}\}$ is the *equivalence class* of x, and for a subset $A \subset X$, $\mathcal{R}(A) = \cup\{\mathcal{R}(x)|x \in A\}$ is called the *saturation* of X. If $\mathcal{R}(x)$ is countable (finite) for each x, then the relation $\mathcal{R}$ is called *countable* (finite). Now if X is a standard Borel space with σ-algebra $\mathfrak{B}$, then we say that $\mathcal{R}$ is standard if $\mathcal{R}$ is a Borel subset of $X \times X$; that is, $\mathcal{R}$ is in $\mathfrak{B} \times \mathfrak{B}$. We suppose $\mathcal{R}$ is countable. It is important that π_l and π_r send Borel sets (in $\mathcal{R}$) to Borel set in X, since these maps are countable to one. It follows that if A is a Borel set in X, then $\mathcal{R}(A)$ is also a Borel set in X. If μ is a σ-finite measure on X with the property that $\mu(\mathcal{R}(A)) = 0$ if $\mu(A) = 0$, then μ will be called $quasi-invariant$ for $\mathcal{R}$, and $\mathcal{R}$ will be called *nonsingular* with respect to μ.

Let G be a countable group acting on X as automorphisms. Let $\mathcal{R}_G = \{(x, gx)|x \in X, g \in G\}$. Then $\mathcal{R}_G$ is a countable equivalence relation. If μ is a σ-finite measure on $(X, \mathfrak{B})$, μ is quasi-invariant for $\mathcal{R}_G$ if and only if μ is quasi-invariant for G in the usual sense.

THEOREM 2.1 [5]. *Let $(\mathcal{R}, \mathfrak{C})$ be an equivalence relation. The function $y \mapsto |{\pi_l}^{-1}(y) \cap C|$ is Borel and the measure μ_l defined by*

$$\mu_l(C) = \int_X |{\pi_l}^{-1}(y) \cap C| d\mu(y)$$

is σ - finite; it will be called the left counting measure of μ.

DEFINITION 2.2. Given $\mathcal{R}$ on $(X, \mathfrak{B}, \mu)$, a set $A \in \mathfrak{B}$ is said to be *invariant* if $\mathcal{R}(A) = A$ up to null sets. The invariant sets, denoted by $\mathcal{J}(\mathcal{R})$, form a σ - subalgebra of $\mathfrak{B}$. $\mathcal{R}$ is called *ergodic* if $\mathcal{J}(\mathcal{R})$ consists only of null or conull sets. A Borel function f is called invariant if $f(x) = f(y)$ for a.a. $(x,y) \in \mathcal{R}$.

DEFINITION 2.3. We say that G *acts ergodically* on $(X, \mathfrak{B}, \mu)$ if we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ whenever $A \in \mathfrak{B}$ and $\mu(g(A) \setminus A) = 0$ for each $g \in G$.

LEMMA 2.4 [4]. *G acts ergodically on X if and only if the following condition is satisfied: if f is a bounded measurable complex-valued function on X, and $f(g(x)) = f(x)$ a.e. on X, for each $g \in G$, then there is a complex number c such that $f(x) = c$, a.e. on X.*

LEMMA 2.5 [4]. *For each g in G, there is a non-negative real-valued measurable function ϕ_g on X such that*

$$\int_X \chi(g(x))d\mu(x) = \int_X \chi(x)\phi_g(x)d\mu(x)$$

for every non-negative measurable function χ on X. Moreover, for each g and h in G,

$$\phi_g(x) > 0, \phi_{gh}(x) = \phi_g(x)\phi_h(g^{-1}(x)), \phi_e(x) = 1 \text{ a.e. } on\ X.$$

Let $\mathcal{H}$ be the Hilbert space $L^2(X, \mathfrak{B}, \mu)$. For $u \in L^\infty(X, \mathfrak{B}, \mu)$, we associate the operator L_u in $\mathcal{B}(\mathcal{H})$ of left multiplication by u. Let $\mathcal{A} = \{L_u | u \in L^\infty\}$. $\mathcal{A}$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. With f in $\mathcal{H}$ and g in G, we can define a measurable function $U_g f$ on X by

$$(U_g f)(x) = [\phi_g(x)]^{1/2} f(g^{-1}(x)),$$

where ϕ_g is the function introduced in Lemma 2.5. Then $U : g \to U_g$ is a unitary representation of G on $\mathcal{H}$ which satisfies

(a) $U_g \mathcal{A} U_g^* = \mathcal{A}$ for each $g \in G$.

(b) $\mathcal{A} \cap U_g \mathcal{A} = 0$ for all $g(\neq e)$ in G.

We say that G *acts ergodically on* $\mathcal{A}$ if A is a scalar multiple of 1 when $U_g A U_g^* = A$ for each $g \in G$ and $A \in \mathcal{A}$.

PROPOSITION 2.6 [4]. *G acts ergodically on $\mathcal{A}$ if and only if G acts ergodically on X.*

DEFINITION 2.7. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. M is called a *factor* if the center of M is a scalar multiple of 1.

PROPOSITION 2.8 [4]. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann subalgebra generated by $\mathcal{A}$ and U. If G acts ergodically on X, then M is a factor, and its type is determined as follows.*

(1) *M is of type I if and only if $\mu(x_0) > 0$ for some x_0 in X. It is then type I_n where n is the (finite or countably infinite) cardinality of G.*

(2) *M is of type II if and only if $\mu(x) = 0$ for each x in X, and there is a non-zero σ-finite measure μ_0, defined on the σ-algebra $\mathfrak{B}$, invariant*

under G. ($\mu_0(g(A)) = \mu_0(A)$ when $A \in \mathfrak{B}$ and $g \in G$) and absolutely continuous with respect to μ. ($\mu_0(A) = 0$ when $A \in \mathfrak{B}$ and $\mu(A) = 0$)

If $\mu_0(X) < \infty, M$ is of type II_1 and if $\mu_0(X) = \infty, M$ is of type II_∞.

(3) M is of type III if and only if there is no measure satifying (2).

3. Main Results

Let $(\mathbb{R}, \mu)$ be a Lebesgue space and G be a countable group acting on $\mathbb{R}$ as automorphisms. An equivalence relation $\mathcal{R}_G$ is a subset $\{(x, y) \in \mathbb{R} \times \mathbb{R} | \exists g \in G,\ y = gx\}$ of $\mathbb{R} \times \mathbb{R}$. Then $(\mathcal{R}_G, \mu_l)$ becomes a Lebesgue space, where μ_l is a left counting measure. Hence we define a Hilbert space $L^2(\mathcal{R}_G, \mu_l)$ on $(\mathcal{R}_G, \mu_l)$. If f is integrable on $(\mathcal{R}_G, \mu_l)$,

$$\int_{\mathcal{R}_G} f(x,y)d\mu_l(x,y) = \int_{\mathbb{R}} \sum_{x \sim y} f(x,y)d\mu(y)$$

For $f, g \in L^2(\mathcal{R}_G, \mu_l)$, the operation of f and g on $L^2(\mathcal{R}_G, \mu_l)$ is the convolution product of f and g as

$$(f * g)(x,y) = \sum_{z \sim x} f(x,z)g(z,y).$$

We shall construct the von Neumann algebra $\mathcal{W}^*(\mathcal{R}_G)$ on $L^2(\mathcal{R}_G, \mu_l)$. Let $\mathcal{R}_G$ be a countable relation on $(\mathbb{R}, \mathfrak{B}, \mu)$. For $f \in L^2(\mathcal{R}_G, \mu_l)$ with *supp f small* (i.e, $f(x, gx) = 0$ a.e. except for finitely many g), $L_f : L^2(\mathcal{R}_G, \mu_l) \to L^2(\mathcal{R}_G, \mu_l)$ is defined by $L_f g \equiv f * g$ for $g \in L^2(\mathcal{R}_G, \mu_l)$. Then we have $L_f L_g = L_{f*g}$, and $L_f{}^* = L_{f^*}$ with $f^*(x,y) = \overline{f(y,x)}$.

REMARK 3.1. A set of operators L_f defined by above is a unital *-subalgebra of $\mathcal{B}(L^2(\mathcal{R}_G, \mu_l))$ with unit. Its weak closure will be denoted by $\mathcal{W}^*(\mathcal{R}_G)$.

For $f \in L^2(\mathcal{R}_G, \mu_l)$ with *supp* $f \subseteq D = \{(x,x) | x \in X\}$ the diagonal, we define $F \in L^\infty(\mathbb{R}, \mu)$ by

$$f(x,y) = \begin{cases} F(x) & x = y \\ 0 & x \neq y. \end{cases}$$

Hence we have correspondence between $\{L_f | suppf \subseteq D\}$ and $L^\infty(\mathbb{R}, \mu)$. For $f,\ f' \in L^2(\mathcal{R}_G)$ with $supp\, f$ in D, if $f(x,y) = \delta_{xy}F(x)$ and $f'(x,y) = \delta_{xy}F'(x)$, then

$$(f * f')(x,y) = \sum_{z \sim x} f(x,z)f'(z,y) = \delta_{xy}F(x)F'(x).$$

Therefore $\mathcal{A} = \{L_f | suppf \subseteq D\}$ is a von Neumann subalgebra of $\mathcal{W}^*(\mathcal{R}_G)$, which is isomorphic to $L^\infty(\mathbb{R})$.

PROPOSITION 3.2. *The abelian algebra $\mathcal{A}$ is a MASA(maximal abelian subalgebra) in $\mathcal{W}^*(\mathcal{R}_G)$. i.e, $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$, where $\mathcal{A}'$ is a commutant of $\mathcal{A}$.*

PROOF. $\supseteq$. Trivial.

$\subseteq$. Assume that $L_f \in \mathcal{W}^*(\mathcal{R}_G)$ commutes with L_h such that $h(x,y) = \delta_{xy}H(x)$ for $H \in L^\infty(\mathbb{R})$. Then

$$(f * h)(x,y) = f(x,y)H(y), \quad (h * f)(x,y) = H(x)f(x,y).$$

Since H is arbitrary, when $x \neq y$, $f(x,y) = 0$. Hence $L_f \in \mathcal{A}$. □

Let $(\mathbb{R}, \mathfrak{B}, \mu)$ be a Lebesgue space and G be a countable group of all mappings $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ of $\mathbb{R}$ with $x \mapsto ax + b$, where $a, b \in \mathbb{Q}$, $a > 0$. Let $G_0 = \{g = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} | g \in G\}$ be a subgroup of G. Then G_0 acts ergodically on $\mathbb{R}$. Hence G acts ergodically on $\mathbb{R}$.

Let $\widetilde{X} = \mathbb{R} \times \mathbb{R}$ and $\widetilde{G} = \{\widetilde{g} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a, b \in \mathbb{Q}, a > 0\}$. $\widetilde{G}$ acts on $\widetilde{X}$ such that $\widetilde{g}\widetilde{x} = (ax + b, u + \ln a)$ for $\widetilde{g} \in \widetilde{G}$ and $\widetilde{x} = (x, u) \in \widetilde{X}$.

LEMMA 3.3. *Let m be a measure on $\mathbb{R}$ such that $dm(u) = e^{-u}d\mu(u)$. Then the measure $\widetilde{\mu} = \mu \otimes m$ is $\widetilde{G}$-invariant on $\widetilde{X}$.*

PROOF.

$$\begin{aligned}\int \tilde{f}(\tilde{g}\tilde{x})d\tilde{\mu}(\tilde{x}) &= \int (\int \tilde{f}(ax,v)e^{-v+\ln a}d\mu(v))d\mu(x) \\ &= \int (a\int \tilde{f}(ax,v)e^{-v}d\mu(v))d\mu(x) \\ &= \int (\int \tilde{f}(y,v)e^{-v}d\mu(v))d\mu(y) \\ &= \int \tilde{f}(\tilde{y})d\tilde{\mu}(\tilde{y}).\end{aligned}$$

□

PROPOSITION 3.4. *Let $\mathcal{R}_G$ be an equivalence relation on $(\mathbb{R}, \mathfrak{B}, \mu)$ and let $\widetilde{R}_{\widetilde{G}} \subseteq \widetilde{X} \times \widetilde{X}$ be an equivalence relation on $(\widetilde{X}, \mathfrak{B} \times \mathfrak{B}, \widetilde{\mu})$. Then $L^2(\mathbb{R}) \otimes l^2(G) \cong L^2(\mathcal{R}_G)$ and $L^2(\mathbb{R}) \otimes l^2(G) \otimes L^2(\mathbb{R}) \cong L^2(\widetilde{X} \otimes l^2(G)) \cong L^2(\widetilde{\mathcal{R}}_{\widetilde{G}})$.*

PROOF. Suppose that $h \in L^2(\mathbb{R}) \otimes l^2(G) = L^2(X, l^2(G))$ and $(x,y) \in \mathcal{R}_G$. The equation $h'(x,gx) = h(g^{-1},gx)$ defines an element h' of $L^2(\mathcal{R}_G)$, and

$$\begin{aligned}||h'||^2_{L^2(\mathcal{R}_G)} &= \int_{\mathcal{R}_G} |h'(x,y)|^2 d\mu_l(x,y) = \int_X \sum_{x\sim y} |h'(x,y)|^2 d\mu(y) \\ &= \int_X \sum_{g\in G} |h'(g^{-1}y,y)|^2 d\mu(y) = \int_X \sum_{g\in G} |h(g^{-1},y)|^2 d\mu(y) \\ &= ||h||^2_{L^2(X)\otimes l^2(G)}.\end{aligned}$$

Accordingly, there is an isometric isomorphism V, from $L^2(\mathbb{R}) \otimes l^2(G)$ onto $L^2(\mathcal{R}_G)$, such that $(Vh)(x,gx) = h(g^{-1},gx)$. Similarly, suppose that $\tilde{h} \in L^2(\mathbb{R}) \otimes l^2(G) \otimes L^2(\mathbb{R}) = L^2(\widetilde{X}, l^2(G))$ and $\tilde{x} = (x,u) \in \widetilde{X}$. Then the equation $\tilde{h}'((x,u),\tilde{g}(x,u)) = \tilde{h}(\tilde{g}^{-1},\tilde{g}(x,u))$ defines an element $\tilde{h}'$ of $L^2(\widetilde{\mathcal{R}}_{\widetilde{G}})$ for $\tilde{g} \in \widetilde{G}$ and $(\tilde{x},\tilde{g}\tilde{x}) \in \widetilde{R}_{\widetilde{G}}$. Hence we have an isometric isomorphism V', from $L^2(\mathbb{R}) \otimes l^2(G) \otimes L^2(\mathbb{R})$ onto $L^2(\widetilde{\mathcal{R}}_{\widetilde{G}})$, such that $(V'\tilde{h})(\tilde{x},\tilde{g}\tilde{x}) = \tilde{h}(\tilde{g}^{-1},\tilde{g}\tilde{x})$.

□

THEOREM 3.5. *For* $\mathcal{A} = L^\infty(\mathbb{R})$ *and a unitary* V *as in* Proposition 3.4,

$$V(\mathcal{A} \rtimes_\alpha G)V^{-1} = \mathcal{W}^*(\mathcal{R}_G).$$

PROOF. By the Proposition 3.4, $L^2(\mathbb{R}) \otimes l^2(G) \cong L^2(\mathcal{R}_G)$.
For $\sum_{i=1}^n \pi_\alpha(L_{f_i})\lambda_{g_i} \in \mathcal{A} \rtimes_\alpha G, h' \in L^2(\mathcal{R}_G)$ and $(x, y) \in \mathcal{R}_G$ with $y = gx$,

$$\begin{aligned}\{(V(\sum_{i=1}^n \pi_\alpha(L_{f_i})\lambda_{g_i})V^{-1})h'\}(x,y) &= \sum_{i=1}^n \pi_\alpha(L_{f_i})\lambda_{g_i}(V^{-1}h')(g^{-1},y) \\ &= \sum_{i=1}^n f_i(g^{-1}y)(V^{-1}h')(g_i^{-1}g^{-1},y) \\ &= \sum_{i=1}^n f_i(x)h'(g_i^{-1}x, gx) \\ &= \sum_{k \in G} f(x,kx)h'(kx,y) \\ &= (L_f h')(x,y),\end{aligned}$$

where $f(x, g'x) = \delta_{g'g_i^{-1}} f_i(x)$. Since the elements considered above are weakly dense in $\mathcal{A} \rtimes_\alpha G$ and $\mathcal{W}^*(\mathcal{R}_G)$, we have $V(\mathcal{A} \rtimes_\alpha G)V^{-1} = \mathcal{W}^*(\mathcal{R}_G)$. □

THEOREM 3.6. $\mathcal{W}^*(\mathcal{R}_G)$ *is a factor of type* III.

PROOF. Let $\Gamma(g) = \{(x, gx) | x \in \mathbb{R}\}$ be the graph of g in $\mathcal{R}_G$. If $L_f \in \mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{W}^*(\mathcal{R}_G)' \subseteq \mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{A}' = \mathcal{A}$, then L_f commutes with $L_{\chi_{\Gamma(g)}}$ where $\mathcal{A}$ is as in Proposition 3.2 and $\chi_{\Gamma(g)}$ is the characteristic function on $\Gamma(g)$. Therefore

$$L_{\chi_{\Gamma(g)}} L_f h(x, gx) = L_f L_{\chi_{\Gamma(g)}} h(x, gx) \quad \text{for all} \quad h \in L^2(\mathcal{R}_G, \mu_l).$$

Thus

$$\begin{aligned} &L_{\chi_{\Gamma(g)}} L_f h(x, gx) \\ &\quad = \chi_{\Gamma(g)} * f * h(x, gx) = \sum_{z \sim x} \sum_{z \sim w} \chi_{\Gamma(g)}(x,z) f(z,w) h(w,gx) \\ &\quad = \chi_{\Gamma(g)}(x,gx) f(gx,gx) h(gx,gx) = \chi_{\Gamma(g)}(x,gx) F(gx) h(gx,gx),\end{aligned}$$

since $L_f \in \mathcal{A}$ and $f(x,y) = \delta_{xy}F(x)$. Also,

$$\begin{aligned}
&L_f L_{\chi_{\Gamma(g)}} h(x,gx) \\
&\qquad = f * \chi_{\Gamma(g)} * h(x,gx) = \sum_{x\sim z}\sum_{z\sim w} f(x,z)\chi_{\Gamma(g)}(z,w)h(w,gx) \\
&\qquad = f(x,x)\chi_{\Gamma(g)}(x,gx)h(gx,gx) = F(x)\chi_{\Gamma(g)}(x,gx)h(gx,gx)
\end{aligned}$$

Hence $F(gx) = F(x)$. The function F on $\mathbb{R}$ is G-invariant. Since G is ergodic, by Lemma 2.4, there exists a constant c with $F(x) = c$. Hence $\mathcal{W}^*(\mathcal{R}_G) \cap \mathcal{W}^*(\mathcal{R}_G)' = \mathbb{C}$. By a simple observation, there is no G-invariant σ-finite measure on $\mathbb{R}$. By Proposition 2.8 (3), $\mathcal{A} \rtimes_\alpha G$ is of type III. The conclusion is obtained by Theorem 3.5. □

THEOREM 3.7. *$V'(L^\infty(\widetilde{X}) \rtimes_{\widetilde{\alpha}} \widetilde{G})V'^* = \mathcal{W}^*(\widetilde{\mathcal{R}}_{\widetilde{G}})$ and $\mathcal{W}^*(\widetilde{\mathcal{R}}_{\widetilde{G}})$ is of type II_∞.*

The proof is the same as the proof of Theorem 3.5, via the unitary V' in Proposition 3.4.

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