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ACTIONS OF FINITE-DIMENSIONAL SEMISIMPLE HOPF ALGEBRAS AND INVARIANT ALGEBRAS

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ABSTRACT. Let H be a finite dimensional Hopf algebra over a field k, and A be an H-module algebra over k which the H-action on A is $\mathcal{D}$-continuous. We show that $Q_{max}(A)$, the maximal ring of quotients of A, is an H-module algebra. This is used to prove that if H is a finite dimensional semisimple Hopf algebra and A is a semiprime right(left) Goldie algebra then $A\#H$ is a semiprime right(left) Goldie algebra. Assume that A is a semiprime H-module algebra. Then A^H is left Artinian if and only if A is left Artinian.

Throughout we let k be a field. Tensor products are assumed to be over k. Let H be a Hopf algebra over k; that is, H is an algebra with 1 and a coalgebra over k with:

(1) comultiplication $\Delta\colon H \to H \otimes H$
(2) counit $\epsilon\colon H \to k$
(3) antipode $S\colon H \to H$
(4) multiplication $m\colon H \otimes H \to H$
(5) unit $u\colon k \to H$,

where Δ and ϵ are algebra homomorphisms and S is an algebra antihomomorphism.

An algebra A is said to be a *left H-module algebra* if

(1) A is a left H-module
(2) $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$
(3) $h \cdot 1_A = \epsilon(h) 1_A$,

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for all $h \in H, a, b \in A$.

A related algebra arising from a left H-module algebra A is the *subalgebra of H-invariants*, $A^H = \{a \in A | h \cdot a = \epsilon(h)a$, for all $h \in H\}$. That is the subalgebra on which H acts trivially.

Let A be a left H-module algebra. Then the *smash product algebra*, $A\#H$ is defined as follows: for all $a, b \in A$ and $g, h \in H$,

(1) $A\#H = A \otimes H$, as k-spaces. We write $a\#h$ for the element $a \otimes h$.
(2) multiplication is given by

$$(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g.$$

It is easy to verify $1\#1$ is the identity of $A\#H$ and that $A \cong A\#1$ and $H \cong 1\#H$; for this reason we frequently abbrevate the element $a\#h$ by ah.

If I is a right ideal of A and $x \in A$, then we define the *residual*, $x^{-1}I$ by $x^{-1}I = \{a \in A | xa \in I\}$. If I is a subset of A, we call the set $l.ann_A(I) = l.ann(I) = \{a \in A | aI = 0\}$ the left annihilator of I in A. If I is any right ideal of A then I is said to be *dense* if and only if $l.ann(x^{-1}I) = 0$ for all $x \in A$.

Let A be an H-module algebra. Let $\mathcal{D} = \mathcal{D}(A)$ denote collection of dense right ideals of A. Then the H-action on A is *$\mathcal{D}$-continuous* if given any $I \in \mathcal{D}$ and $h \in H$, there exists $J \in \mathcal{D}$ such that $h \cdot J \subseteq I$.

EXAMPLE. Let H be a pointed Hopf algebra and let A be an H-module algebra. Let I, J be dense right ideals of A. From [10, Lemma 24.5], $x^{-1}I$ is dense for $x \in A$ and $I \cap J$ is dense. These facts are used to show that the H-action on A is $\mathcal{D}$-continuous as in the proof of [8, Proposition 2.3].

Let $\mathcal{F}$ be the set of all pairs (I, f), where $I \in \mathcal{D}$ and $f: I \to A$ is a right A-module map. Two elements (I, f) and (J, g) are equivalent if $f = g$ on some $K \in \mathcal{D}$, $K \subseteq I \cap J$. In the case we denote $(I, f) \sim (J, g)$. Then $Q_{max}(A) = \mathcal{F}/\sim$. More compactly:

$$Q_{max}(A) = \varinjlim_{I \in \mathcal{D}} Hom_A(I_A, A).$$

$Q_{max}(A)$ becomes a ring as follows: for (I, f) and (J, g), $(I \cap J, f + g)$ determines addition and $(IJ, g \circ f)$ multiplication. We call $Q = Q_{max}(A)$ the *maximal ring of quotients* of A.

Let A be an H-module algebra which the H-action on A is $\mathcal{D}$-continuous. By [10, Theorem 24.8], if $q \in Q$ then there exists $I_q \in \mathcal{D}$ with $qI_q \subseteq A$. We have $q\colon I_q \to A$ is a right A-module homomorphism via $x \mapsto qx$. For $h \in H$, let $\Delta h = \sum h_1 \otimes h_2$. Since H-action on A is $\mathcal{D}$-continuous, there exists $J_{qh} \in \mathcal{D}$ with $Sh_2 \cdot J_{qh} \subseteq I_q$. Define for any $h \in H$, $h \cdot q\colon J_{qh} \to A$ as $(h \cdot q)(x) = \sum h_1 \cdot [q(Sh_2 \cdot x)]$ for all $x \in J_{qh}$. Then $h \cdot q$ is a right A-module homomorphism by the following two lemmas, so determines an element of Q. And the action is well-defined.

LEMMA 1. *Let H be a Hopf algebra and A be an H-module algebra. For any $a, b \in A$ and $h \in H$,*

$$(h \cdot a)b = \sum h_1 \cdot [a(Sh_2 \cdot b)].$$

PROOF.

$$\begin{aligned}\sum h_1 \cdot [a(Sh_2 \cdot b)] &= \sum (h_1 \cdot a)[h_2 \cdot (Sh_3 \cdot b)]\\ &= \sum (h_1 \cdot a)(\epsilon(h_2)b)\\ &= [(\sum h_1\epsilon(h_2)) \cdot a]b\\ &= (h \cdot a)b.\end{aligned}$$

□

LEMMA 2. *$h \cdot q\colon J_{qh} \to A$ via $x \mapsto (h \cdot q)(x) = \sum h_1 \cdot [q(Sh_2 \cdot x)]$ is a right A-module homomorphism.*

PROOF. For any $a \in A$,

$$\begin{aligned}(h \cdot q)(xa) &= \sum h_1 \cdot [q(Sh_2 \cdot (xa))]\\ &= \sum h_1 \cdot [q((Sh_2 \cdot x)(Sh_3 \cdot a))]\\ &= \sum h_1 \cdot [(q(Sh_2 \cdot x))(Sh_3 \cdot a)]\\ &= \sum (h_1 \cdot [q(Sh_2 \cdot x)])a\\ &= (\sum h_1 \cdot [q(Sh_2 \cdot x)])a\\ &= (h \cdot q)(x)a.\end{aligned}$$

where the last equality follows from Lemma 1. □

PROPOSITION 3. *Let H be a finite dimensional Hopf algebra and A be an H-module algebra on which the H–acton is $\mathcal{D}$-continuous. If we define $\gamma\colon H\otimes Q_{max}(A)\to Q_{max}(A)$, $h\otimes q\mapsto h\cdot q$ as above then $Q_{max}(A)$ is an H-module algebra.*

PROOF. For any $q\in Q$, there exists $I_q\in\mathcal{D}$ with $qI_q\subseteq A$. For any $h,l\in H$, there exist $J_{qh},J_{ql}\in\mathcal{D}$ such that $Sh_2\cdot J_{qh}\subseteq I_q$ and $Sl_2\cdot J_{ql}\subseteq J_{qh}$ since the H-action on A is $\mathcal{D}$-continuous. Let $J_q=J_{qh}\cap J_{ql}$. Then $J_q\in\mathcal{D}$ by [10, Lemma 24.5]. For any $x\in J_q$, by Lemma 1,

$$\begin{aligned}\gamma\circ(m\otimes id)(l\otimes h\otimes q)(x) &= \gamma(lh\otimes q)(x)\\ &= ((lh)\cdot q)(x)\\ &= \sum(lh)_1\cdot[q(S(lh)_2\cdot x)]\\ &= \sum l_1h_1\cdot[q((Sh_2)(Sl_2)\cdot x)]\\ &= \sum l_1h_1\cdot[q(Sh_2\cdot(Sl_2\cdot x))]\\ &= \sum l_1\cdot(\Sigma h_1\cdot[q(Sh_2\cdot(Sl_2\cdot x)])\\ &= \sum l_1\cdot[(h\cdot q)(Sl_2\cdot x)]\\ &= (l\cdot(h\cdot q))(x)\\ &= \gamma\circ(id\otimes\gamma)(l\otimes h\otimes q)(x).\end{aligned}$$

For any $x\in I_q$ and $\alpha\in k$,

$$\begin{aligned}\gamma\circ(u\otimes id)(\alpha\otimes q)(x) &= \gamma(u(\alpha)\otimes q)(x)\\ &= ((\alpha\cdot 1_H)\cdot q)(x)\\ &= \alpha 1_H\cdot[q(1_H\cdot x)]\\ &= (\alpha q)(x).\end{aligned}$$

Therefore $Q_{max}(A)$ is a left H-module. For any $p,q\in Q_{max}(A)$, there exist $I_p,I_q\in\mathcal{D}$ such that $pI_p\subseteq A$ and $qI_q\subseteq A$. For any $h\in H$, there exists $J_{ph}\in\mathcal{D}$ with $Sh_2\cdot J_{ph}\subseteq I_p$ and there exists $J_{qh}\in\mathcal{D}$ with

$Sh_2 \cdot J_{qh} \subseteq I_q$. Let $J_h = J_{ph} \cap J_{qh} \in \mathcal{D}$. For any $x \in J_h$,

$$\begin{aligned}(h \cdot (qp))(x) &= \sum h_1 \cdot [(qp)(Sh_2 \cdot x)] \\ &= \sum h_1 \cdot [(qp)(\epsilon(h_2)Sh_3 \cdot x)] \\ &= \sum h_1 \cdot [q(\epsilon(h_2)p)(Sh_3 \cdot x)] \\ &= \sum h_1 \cdot [q(Sh_2)h_3 \cdot p)(Sh_4 \cdot x)] \\ &= \sum h_1 \cdot [q\{Sh_2 \cdot ((h_3 \cdot p)(Sh_4 \cdot x))\}] \\ &= \sum (h_1 \cdot q)(h_2 \cdot p(Sh_3 \cdot x)) \\ &= \sum (h_1 \cdot q)(h_2 \cdot p)(x)\end{aligned}$$

and,

$$\begin{aligned}(h \cdot 1_Q)(x) &= \sum h_1 \cdot [1_Q(Sh_2 \cdot x)] \\ &= \sum h_1 \cdot (Sh_2 \cdot x) \\ &= \epsilon(h)x.\end{aligned}$$

Therefore $Q_{max}(A)$ is a left H-module algebra. □

LEMMA 4. *Let A be an H-module algebra. Then for all $a \in A$ and $h \in H$,*

$$ah = \sum h_2(S^{-1}h_1 \cdot a).$$

PROOF.

$$\begin{aligned}\sum h_2(S^{-1}h_1 \cdot a) &= \sum h_2\#(S^{-1}h_1 \cdot a) \\ &= \sum (1\#h_2)((S^{-1}h_1 \cdot a)\#1) \\ &= \sum [1 \cdot (h_2 \cdot (S^{-1}h_1 \cdot a))]\#(h_3 \cdot 1) \\ &= \sum 1 \cdot ((h_2S^{-1}h_1) \cdot a)\#h_3 \\ &= \sum (\epsilon(h_1)1_H \cdot a)\#h_2 \\ &= a\#h = ah.\end{aligned}$$

□

Since the multiplication on $A\#H$ is given by

$$(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g,$$

we may write $ha = (1\#h)(a\#1) = \sum (h_1 \cdot a)h_2$.

THEOREM 5. *Let us consider a finite dimensional semisimple Hopf algebra H and H-module algebra A which the H-action on A is $\mathcal{D}$-continuous. If A is a semiprime right(left) Goldie algebra then $A\#H$ is a semiprime right(left) Goldie.*

PROOF. Let Q be the right(left) classical algebra of quotients of A. The classical algebra of quotients for a Goldie algebra coincides with the maximal of quotients by [10, p.261]. Furthermore, Q is a semisimple Artinian. By Proposition 3, we can consider the smash product algebra, $Q\#H$. It was proved in [2] that $Q\#H$ is semisimple Artinian. If we prove that $A\#H$ is a right(left) order in $Q\#H$, we are done by the Goldie Theorem [4, Proposition 5.2]. Let T be the set of all nonzero divisors in A. The element of T are invertible in $Q\#H$ which forces them to be nonzero divisors in $A\#H$. It is enough to show that each $x \in Q\#H$ can be represented as rt^{-1} (resp. $t^{-1}r$) where $r \in A\#H$ and $t \in T$.

$$x = \sum_i a_i h_i$$

for all $a_i \in Q, h_i \in H$. To complete the proof in the left case we take a common denominator for all a_i. To complete the right case we use the folllowing identity in $Q\#H$. By Lemma 4,

$$\begin{aligned} qh = (at^{-1})h &= \sum h_2(S^{-1}h_1 \cdot (at^{-1})) \\ &= \sum h_2[((S^{-1}h_1)_1 \cdot a)((S^{-1}h_1)_2 \cdot t^{-1})] \\ &= \sum h_3[(S^{-1}h_2 \cdot a)(S^{-1}h_1 \cdot t^{-1})] \\ &= \sum \{h_3 \cdot (S^{-1}h_2 \cdot a)\}h_4(S^{-1}h_1 \cdot t^{-1}) \end{aligned}$$

for some $a \in A$ and $t \in T$. We conclude that $Q\#H = (A \otimes 1)(1 \otimes H)(Q \otimes 1)$. Now we can use a common denominator arguments as in the left case. □

If H is a finite dimensional Hopf algebra then the *left integral* of H, $\int_H^l = \{t \in H | ht = \epsilon(h)t$, for all $h \in H\}$ is one dimensional [5]. Choose $0 \neq t \in \int_H^l$. Let A be a left H-module algebra and let $A^H = \{a \in A | h \cdot a = \epsilon(h)t$ for all $h \in H\}$. Then the map $\hat{t}\colon A \to A$ given by $\hat{t}(a) = t \cdot a$ is an A^H-bimodule map with values in A^H. If H is finite dimensional Hopf algebra then H is semisimple if and only if $\epsilon(\int_H^l) \neq 0$ [5]. Hence if H is semisimple, we may choose $t \in \int_H^l$ with $\epsilon(t) = 1$. It follows that $\hat{t}(1) = t \cdot 1 = \epsilon(t) \cdot 1 = 1$ and so $\hat{t}\colon A \to A^H$ is surjective.

PROPOSITION 6. *Let H be a finite dimensional Hopf algebra and let A be an H-module algebra. If A is left Noetherian then A is a left Noetherian A^H-module.*

PROOF. Since H is a finite dimensional Hopf algebra, the antipode S of H is bijective. Therefore $A\#H$ is a free right A-module with rank $n = dim_k H$ since S is invertible [1]. The proof is similar to [6, Theorem 4.4.2]. □

THEOREM 7. *Let H be a finite dimensional semisimple Hopf algebra and A be a semiprime H-module algebra which the H-action on A is $\mathcal{D}$-continuous. Then A^H is left Artinian if and only if A is left Artinian.*

PROOF. Assume that A is semiprime and that Krull dimension of A^H, $\mathcal{K}\dim A^H$, exists. Then A is semiprime Goldie by [3, Theorem 2.10(ii)]. The algebra A has a classical algebra of quotients $Q_{cl}(A)$ which is semisimle Artinian. For any algebra with a $Q_{cl}(A)$, we have a maximal quotient algebra of A, $Q_{max}(A)$. By [10, p.261] ,$Q_{cl}(A) = Q_{max}(A)$. Let denote Q instead of $Q_{cl}(A) = Q_{max}(A)$. By Proposition 6, Q is a finite generated as a left module over the fixed algebra Q^H. We can find a finite set of generators $x_1, \cdots, x_n$ for Q as a left Q^H-module. Choose a regular b and a_i both in A such that $x_i = a_i b^{-1}$. Then $Q = \sum_{i=1}^n Q^H x_i = \sum_{i=1}^n Q^H a_i b^{-1}$. Hence $Qb = \sum Q^H a_i$. But $Qb = Q$ since b is invertible. Thus we assume $x_i \in A$. Define $T\colon A \to \oplus_{i=1}^n A^H$ via $a \mapsto [\hat{t}(x_i a)]_{i=1}^n$ where $\hat{t}$ is above. Then T is a right A^H-module map. If $T(a) = 0$ then $\hat{t}(x_i a) = 0$ for all i. But $\hat{t}$ is a left Q^H-module map. Thus $\hat{t}(Qa) = 0$ and Qa is a left H-stable ideal of Q by Proposition 3. Since Q is a semiprime H-module algebra [10, p.260] and Artinian,

$Q\#H$ is semiprime by [2]. By [6, Lemma 4.4.6], $Qa = 0$. Since Q is semiprime, $a = 0$. Hence we deduce that A as a left A^H-module can be embedded in a finte direct sum of copies of A^H. Since the later module has the same Krull-dimension as A^H, we conclude that $\mathcal{K}\dim_{A^H} A$ exists and $\mathcal{K}\dim_{A^H} A \leq \mathcal{K}\dim_{A^H} A^H$. By [9, Proposition 1], $\mathcal{K}\dim_A A = \mathcal{K}\dim_{A^H} A$. Therefore A is left Artinian. Conversely if A is left Artinian then A^H is left Artinian by [9, Theorem 1]. This completes the proof.□

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