Comm. Korean Math. Soc. **13** (1998), No. 2, pp. 217–224

MINIMAL RESOLUTION CONJECTURES AND ITS APPLICATION

Young Hyun Cho

Abstract. In this paper we study the minimal resolution conjecture which is a generalization of the ideal generation conjecture. And we show how the results about this conjecture can make the calculation of minimal resolution in certain cases.

0. Introduction

For the s distinct points in $\mathbb{P}^n$ which are in generic s-position, we state the minimal resolution conjecture proposed in [1]. This conjecture has been solved for $n = 2$ ([3], [4]) and for $n = 3$ ([1]). We prove the minimal resolution conjecture when the condition in Theorem 3.2 holds. From this, we get the minimal resolution as in Theorem 3.4 and Theorem 3.6.

1. Preliminaries

Let k be a field and $\mathbb{P}^n(k)$ (or, simply, $\mathbb{P}^n$) projective n-space over k $(n \geq 2)$.

Let $I \subseteq R = k[X_0, \ldots, X_n]$ be the ideal of s points in $\mathbb{P}^n$ (i.e., the ideal generated by all forms vanishing at the points). Then I is a perfect homogeneous ideal of R of height n and the homogeneous coordinate ring of points, $A = R/I$, is a Cohen-Macaulay graded ring of Krull dimension 1, generated by A_1 as a k-algebra. The grading of R is given naturally by degrees. If $R = \oplus_{i\geq 0} R_i$, and $I = \oplus_{i \geq 0} I_i$ then $A = \oplus_{i\geq 0} R_i/I_i = \oplus_{i\geq 0} A_i$.

Received August 31, 1996. Revised February 27, 1998.

1991 Mathematics Subject Classification: 13C14, 13D02.

Key words and phrases: Hilbert function, Cohen-Macaulay ring, minimal free resolution.

This research was supported in part by Basic Science Research Institute Program, Ministry of Education, BSRI-97-1414, and GARC-KOSEF at Seoul National University.

The Hilbert function of A, which is defined by

$$H(A,i) = \dim_k A_i, \quad \forall\ i$$

strictly increases until it reaches s.

Thus, the difference of the Hilbert function of A, which is

$$\Delta H(A,i) = H(A,i) - H(A,i-1), \quad \forall i,$$

(where $H(A,i-1) = 0$ for $i \le 0$), is eventually 0 (and it adds up to s). Put $d = \min\{t \mid I_t \neq 0\}$ and define

$$\sigma(I) = \min\{t \mid \Delta H(A,t) = 0\}.$$

Then $s \ge \binom{d-1+n}{n}$, and if $F_1, \ldots, F_u$ is a minimal set of generators of I, then

$$\min\{\deg F_i \mid i = 1, \ldots, u\} = d,$$
$$\max\{\deg F_i \mid i = 1, \ldots, u\} \le \sigma(I).$$

(See [4, Proposition 1.1]).

The s points are said to be in generic s-position (or, simply, in generic position) if they have maximal Hilbert function, i.e., for all $i \ge 0$

$$H(A,i) = \min\left\{\binom{i+n}{n},\ s\right\}.$$

This is the fastest possible growth of the Hilbert function of A. From the definitions above it follows that, for points in generic position, d is also the least integer such that $\binom{d+n}{n} > s$, and if $s \neq \binom{d-1+n}{n}$, that $\sigma(I) = d+1$, hence $I = \langle I_d, I_{d+1}\rangle$ (See [4, Corollary 1.6]). If $v(I)$ denotes the minimal number of generators of I, then clearly $v(I) \ge \dim_k I_d$. Thus, the problem of determining $v(I)$ depends completely on knowing how much of I_{d+1} can be obtained from I_d. We are specially interested in the case that the elements of $R_1 I_d$ are as independent as possible in I_{d+1}; explicityly

$$\dim_k(R_1 I_d) = \min\{(n+1)\dim_k I_d,\ \dim_k I_{d+1}\}.$$

This gives, as a conjectured generic value for $v(I)$, that:

$$\begin{aligned} v(I) &= \dim_k I_d + \{\dim_k I_{d+1} - \min[(n+1)\dim_k I_d,\ \dim_k I_{d+1}]\} \\ &= \max\{\dim_k I_{d+1} - n\dim_k I_d,\ \dim_k I_d\} \qquad (1.1) \\ &= \max\{H(I,d+1) - nH(I,d),\ H(I,d)\}. \end{aligned}$$

We refer to (1.1) as the ideal generation conjecture.

2. The minimal resolution conjecture

Since R/I is a Cohen-Macaulay, the homological dimension of I is $n-1$. Let

$$0 \to F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \xrightarrow{d_1} F_0 \to I \to 0$$

be a graded minimal free resolution of I over R.

Put $N_i = \operatorname{Im} d_i$ for $i = 1, \ldots, n-1$, and $N_0 = I$. Then, the assumption of generic position yields

$$F_0 = R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0}$$
$$v(I) = \alpha_0 + \beta_0.$$

By [5, Theorem 2.2], since $\operatorname{Tor}_i^R(I,k)$ vanishes in every degree different from $d+i$, $d+i+1$ (for $i = 1, \ldots, n-1$)

$$F_i = R(-(d+i))^{\alpha_i} \oplus R(-(d+i+1))^{\beta_i}.$$

The numbers $b_i = \alpha_i + \beta_i$ $(i = 0, \cdots, n-1)$ are called the Betti numbers of I. Now, we can extend the ideal generation conjecture to the whole resolution;

(1.2) $$b_i = \max\{\alpha_i = H(N_i, d+i),\ H(N_i, d+i+1) - n\alpha_i\}$$

for each $i = 0, \ldots, n-1$.

We refer to (1.2) as the minimal resolution conjecture. To figure out how a conjectured resolution should look, first for each $i = 0, \ldots, n-1$, define $W(N_i)$ as the vector subspace of $(N_i)_{d+i+1}$ generated by $(N_i)_{d+i}$ under multiplicaiton by $X_0, \ldots, X_n$,

$$W(N_i) = X_0(N_i)_{d+i} + \cdots + X_n(N_i)_{d+i} \subseteq (N_i)_{d+i+1}.$$

THEOREM 2.1. *Minimal resolution conjecture holds if and only if* $\dim_k W(N_i)$ *is maximal, i.e.,*

$$\dim_k W(N_i) = \min\{H(N_i, d+i+1),\ (n+1)\alpha_i\}.$$

PROOF. If $\dim_k W(N_i)$ is maximal, for $i = 0, \dots, n-1$,

$$\begin{aligned} b_i &= \alpha_i + \beta_i \\ &= \alpha_i + H(N_i, d+i+1) - \dim_k W(N_i) \\ &= \alpha_i + H(N_i, d+i+1) - \min\{H(N_i, d+i+1), (n+1)\alpha_i\} \\ &= \max\{\alpha_i, H(N_i, d+i+1) - n\alpha_i\}. \end{aligned}$$

Conversely, if $b_i = \max\{\alpha_i, H(N_i, d+i+1) - n\alpha_i\}$, for $i = 0, \dots, n-1$,

$$\begin{aligned} \beta_i &= b_i - \alpha_i \\ &= \max\{0, H(N_i, d+i+1) - (n+1)\alpha_i\}. \end{aligned}$$

But

$$\begin{aligned} \dim_k W(N_i) &= H(N_i, d+i+1) - \beta_i \\ &= \min\{H(N_i, d+i+1), (n+1)\alpha_i\}. \end{aligned}$$

Now, suppose that each Betti number has the value conjectured above in (1.2) and that i is the least index for which

$$\dim_k W(N_i) = (n+1)\alpha_i.$$

Then from the minimality of i, we obtain

$$\dim_k W(N_j) = H(N_j, d+j+1), \quad \forall j < i,$$

hence $\beta_0 = \beta_1 = \cdots = \beta_{i-1} = 0$. □

THEOREM 2.2. *For any* $i = 0, \dots, n-2, \alpha_{i+1} = \cdots = \alpha_{n-1} = 0$ *if and only if* $\dim_k W(N_i) = (n+1)\alpha_i$.

PROOF. Refer [7]. □

Finally we have

$$\begin{aligned} F_j &= R(-(d+j))^{\alpha_j}, \quad \forall j < i, \\ F_i &= R(-(d+i))^{\alpha_i} \oplus R(-(d+i+1))^{\beta_i}, \\ F_j &= R(-(d+j+1))^{\beta_j}, \quad \forall j > i. \end{aligned}$$

3. Main results

Since $ht(I) = n$, there exists homogeneous regular sequence of length n in I. Hence I_d can have regular sequence of length at most n. Here, we assume $d \geq 2$, since if $d = 1$ and $L \in I_1$ we can reduce to the case $R/(L) = k[X_0, \ldots, X_{n-1}]$.

DEFINITION 3.1. Let $F_1, \ldots, F_\lambda \in k[X_0, \ldots, X_n] = R$. We say that $F_1, \ldots, F_\lambda$ are quasi-regular sequence if the following condition holds for each ν : $G(Y_1, \ldots, Y_\lambda) \in R[Y_1, \ldots, Y_\lambda]$ is homogeneous of degree ν and $G(F_1, \ldots, F_\lambda) \in J^{\nu+1}$ implies that all the coefficients of G are in J, where $J = \langle F_1, \ldots, F_\lambda \rangle$.

THEOREM 3.2. *Suppose $d \geq 2$ and $\alpha_0 = \lambda \leq n$, and $I_d = \langle F_1, \ldots, F_\lambda \rangle$, where $F_1, \ldots, F_\lambda$ are quasi-regular sequence. Then the minimal resolution conjecture holds.*

PROOF. First we claim that $\{X_i F_j \mid i = 0, \ldots, n, \ j = 1, \ldots, \lambda\}$ are linearly independent. Suppose $X_i F_j$'s are linearly dependent. Then we can write $\sum_{j=1}^{\lambda} L_j F_j = 0$, with some $L_j \neq 0$. Suppose $L_\lambda \neq 0$. Consider $G(Y_1, \ldots, Y_\lambda) = \sum_{j=1}^{\lambda} L_j Y_j$, then $G(F_1, \ldots, F_\lambda) = \sum_{j=1}^{\lambda} L_j F_j = 0 \in \langle F_1, \ldots, F_\lambda \rangle^2$. Since $F_1, \ldots, F_\lambda$ are quasi-regular, all L_j's are in $\langle F_1, \ldots, F_\lambda \rangle$. Especially $L_\lambda \in \langle F_1, \ldots, F_\lambda \rangle = I_d$. This cannot happen since $d \geq 2$. Now $\dim_k W(N_0) = (n+1)\alpha_0$. Hence the ideal generation conjecture is proved in this case.

By Theorem 2.2, $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$, and $H(N_i,\ d+i+1) = \beta_i$, $b_i = \beta_i = \max\{\alpha_i,\ H(N_i, d+i+1) - n\alpha_i\}$ for $i = 1, \ldots, n-1$. Therefore the minimal resolution conjecture holds. □

By Theorem 16.2 in [8], if $F_1, \ldots,\ F_\lambda$ are regular sequence then $F_1, \ldots, F_\lambda$ are quasi-regular sequence.

COROLLARY 3.3. *Suppose $d \geq 2$ and $\alpha_0 = \lambda \leq n$. For the regular sequence $F_1, \ldots, F_\lambda$, set $I_d = \langle F_1, \ldots, F_\lambda \rangle$. Then the minimal resolution conjecture holds.*

THEOREM 3.4. *With the conditions in the* Corollary 3.3, *we get the* *minimal free resolution of I;*

$$0 \to R(-(n+d))^{\beta_{n-1}} \to R(-(n+d-1))^{\beta_{n-2}} \to \cdots \to R(-(d+2))^{\beta_1} \to R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \to I \to 0.$$

And $v(I) = \binom{d+n}{n-1} - (n-1)\lambda$.

PROOF. The first part is obvious from section 2. For the second part, note $\lambda = \binom{n+d}{n} - s$ and $\dim I_{d+1} = \binom{n+d+1}{n} - s$.

Then

$$\begin{aligned} v(I) &= \dim_k I_d + \dim_k I_{d+1} - (n+1)\lambda \\ &= \lambda + \binom{n+d+1}{n} - s - (n+1)\lambda \\ &= \lambda + \binom{n+d+1}{n} - \binom{n+d}{n} + \lambda - (n+1)\lambda \\ &= \binom{n+d}{n-1} - (n-1)\lambda \end{aligned}$$

□

Now consider Theorem 3.4 in the following case.

EXAMPLE 3.5. Let C_1, C_2 be curves in $\mathbb{P}^2$ defined by homogeneous polynomials F_1, F_2 with no common component. Suppose $\deg F_1 = 2$ $\deg F_2 = 3$ and $Z = C_1 \cap C_2$. If $H(Z,i)$ denotes the Hilbert function of the homogeneous coordinate ring of Z, then it is well known that $H(Z,i) = H(R,i) - H(R,i-2) - H(R,i-3) + H(R,i-5)$, where $R = k[x_0, x_1, x_2]$.

Hence $H(Z,0) = 1$, $H(Z,1) = 3$, $H(Z,2) = 5$, $H(Z,i) = 6$, for all $i \geq 3$. Therefore Z is not in generic 6-position, and $I(Z) = \langle F_1, F_2 \rangle$. Next choose any 5 points of Z, and say this subset as X. Then by [2], since $\Delta H(Z,i) = \Delta H(X,i) + \Delta H(Z \backslash X, 3-t)$, $H(X,0) = 1$, $H(X,1) = 3$, $H(X,i) = 5$ for all $i \geq 2$. Therefore X is in generic 5-position, and $d = 2$, $\sigma = 3$ and $I(X) = \langle F_1, F_2, F_3 \rangle$, for some $F_3 \neq F_2$ and $\deg F_3 = 3$. Since $I(X)_2 = \langle F_1 \rangle$, by applying Theorem 3.4 we get the minimal free resolution of $I(X)$ as,

$$0 \to R(-4)^{\beta_1} \to R(-2) \oplus R(-3)^2 \to I(X) \to 0.$$

Now tensoring this exact sequence with the quotient field of R, and comparing the dimension, we get $\beta_1 = 2$.

For the general case of Example 3.5, we have the following theorem.

THEROEM 3.6. *Let $d \geq 2$ and $s = \binom{2+d}{d} - 1$, $X = \{P_1, \ldots, P_s\} \subset \mathbb{P}^2$. Suppose d is the least degree of polynomials passing through X, and $\dim_k I(X)_d = 1$. Then the minimal resolution of $I(X)$ is*

$$0 \to R(-(d+2))^d \to R(-d) \oplus R(-(d+1))^d \to I(X) \to 0.$$

PROOF. $H(X,i) = \binom{2+i}{2}$, for $i < d$, $H(X,d) = \binom{2+d}{d} - 1 = s$. Hence $H(X,i) = s$, $i > d$ and X is in generic s-position. Now $I(X) = \langle I_d, I_{d+1} \rangle$ and $\alpha_0 = 1$ since $\dim_k I(X)_d = 1$.

$$\begin{aligned}
\beta_0 &= \binom{d+3}{d+1} - s - \min\{3, \binom{d+3}{d+1} - s\} \\
&= \binom{d+3}{d+1} - s - 3 \quad (\text{since } \binom{d+3}{d+1} - s > 3) \\
&= \binom{d+3}{d+1} - \binom{d+2}{d} - 1 - 3 \\
&= \binom{d+2}{d+1} - 2 = d.
\end{aligned}$$

Next, by applying the Theorem 3.4, we get the minimal resolution of $I(X)$ as

$$0 \to R(-(d+2))^d \to R(-d) \oplus R(-(d+1))^{l} \to I(X) \to 0.$$

□

References

[1] E. Balico and A. V. Geramita, *The minimal free resolution of s general points in $\mathbb{P}^n$* in "Proceedings of Alg. Geom. Conferences" (Vancouver, 1984) **6**, Canad. Math. Soc. Conf. Proc., Amer. Math. Soc., Providence, RI. 1986, 1–10.

[2] E. Davis, A. V. Geramita and F. Orecchia, *Gorenstein algebra and the Cayley Bacharach theorem*, Proc. A.M.S. **93** (1985), no. 4, 593-597.
[3] A. V. Geramita, D. Gregory and L. G. Roberts, *Monomial ideals and points in projective space* **40** (1986), J. of Pure and Appl. Algebra, 33–62.
[4] A. V. Geramita and P. Maroscia, *The ideal of forms vanishing at a finite set of points in* $\mathbb{P}^n$, J. of Alg. **90** (1984), 528–555.
[5] A. Lorenzini, *Betti numbers of perfect homogeneous ideals* **60** (1989), J. of Pure and Appl. Alg., 273–288.
[6] ———, *Betti numbers of points in projective space*, J. of Pure and Appl. Alg. **63** (1990), 181–193.
[7] H. Matsumura, *Commutative ring theory*, vol. 8, Cambridge studies in advanced mathematics, Cambridge university press, 1986.

Department of Mathematics
College of Natural Sciences
Seoul National University
Seoul 151-742, Korea