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IDEAL BOUNDARY OF CAT(0) SPACES

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ABSTRACT. In this paper we prove the Hopf-Rinow theorem for CAT (0) spaces and show that the ideal boundaries of complete CAT(0) manifolds of dimension 2 or 3 with some additional conditions are homeomorphic to the circle or 2-sphere by the characterization of the local shadows around the branch points.

1. Introduction

CAT(0) space is a simply connected geodesic space of nonpositive curvature in the sense of A. D. Aleksandrov and M. Gromov ([9], [5], [1]). For a CAT(0) Riemannian manifold M of dimension n, which is called the Hadamard manifold, the ideal boundary $M(\infty)$ with the cone topology is always homeomorphic to the standard sphere S^{n-1} ([3]). But if M is a non-Riemannian CAT(0) space, the ideal boundary $M(\infty)$ of M may be complicated topologically even when M is a topological manifold ([7]).

The simplicity of the ideal boundary of a CAT(0) Riemannian manifold comes from the simplicity of the local structure of Riemannian manifolds. A basic difference of the local structure of non-Riemannian spaces of nonpositive curvature compared to Riemannian manifolds is the branching phenomena of geodesics, that is, the geodesic extension of a geodesic segment is not always unique even locally.

The purpose of this paper is to answer the naturally arising question:

To what extent can we ensure that the ideal boundary of a CAT(0) space is homeomorphic to the standard sphere ?

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M. Davis and T. Januskiewicz [7] gave a partially positive answer when M is a CAT(0) piecewise constant, piecewise linear n-manifold. They also gave some examples of CAT(0) manifolds of dimension≥ 4 with the ideal boundaries not homeomorphic to the standard sphere. Concerning this question we will prove that for a complete CAT(0) topological manifold of dimension 2 and 3 with some conditions on the local structure, the ideal boundary is a homeomorphic sphere. But it is still open whether the ideal boundary of a CAT(0) 3-manifold is homeomorphic to the standard sphere or not. This fact is not trivial since the ideal boundary is not a quasi-isometric invariant for CAT(0) spaces ([9]).

In section 1, we introduce basic concepts and prove the Hopf-Rinow theorem for CAT(0) spaces. In Section 2, we define the concept of shadow which measure the amount of branching near a point and study some topological properties of a shadow near a manifold point. In section 3, we prove the theorem about the ideal boundary of a CAT(0) manifold of dimension 2 and 3 with some conditions.

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§1. Hopf-Rinow theorem for CAT(0) spaces

A locally compact metric space (X, d) is said to be a *geodesic space* if every pair of points p, q in X can be joined by a distance realizing continuous curve, which we call a *geodesic segment* $[p, q]$ between p and q if it is parameterized by the arclength. A geodesic can be regarded as an isometric(distance realizing) embedding of a closed interval in the real line $\mathbb{R}$. So a *ray* and a *line* is defined as an isometric embedding of the infinite intervals $[0, \infty)$, $(-\infty, \infty)$ in X, respectively.

A *geodesic triangle* $\triangle(p, q, r)$ in X is a triple of points (vertices) $p, q, r \in X$ together with three geodesic segments (sides) joining each pair of vertices. A *comparison triangle* of a triangle $\triangle pqr$ is a triangle $\bar{\triangle}\bar{p}\bar{q}\bar{r}$ in the Euclidean plane $\mathbb{E}^2$ with $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$, $d(p, r) = d(\bar{p}, \bar{r})$. Given a side on $\triangle$ and a point x on it, there is a unique *comparison point* $\bar{x}$ on the corresponding side of $\bar{\triangle}$ with $d(x, e) = d(\bar{x}, \bar{e})$ for each of the end points e of the given side.

A geodesic triangle $\triangle$ is said to satisfy the *CAT(0) inequality* if for every $x,\ y \in \triangle$ and their comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$, we have

$$d(x,y) \le d(\bar{x},\bar{y}).$$

A geodesic space X is said be a *space of nonpositive curvature* (NPC) if for each $p \in X$ there is a neighborhood U of p such that every geodesic triangle contained in U satisfies the CAT(0) inequality. A geodesic space X is said to be a *CAT*(0) *space* if every triangle in X satisfies the CAT(0) inequality. The Cartan-Hadamard Theorem ([9], [1]) says that a simply connected geodesic space of nonpositive curvature is a CAT(0) space. A CAT(0) space is a straight space, so is contractible. NPC is locally convex and the CAT(0) space is globally convex in the sense that for any geodesics $\gamma, \sigma : [0,1] \to X$ parameterize proportionally to the arclength, we have

$$d(\gamma(t),\sigma(t)) \le (1-t)d(\gamma(0),\sigma(0)) + td(\gamma(1),\sigma(1))$$

for all $t \in [0,1]$.

There are several kinds of angles in geodesic spaces. Given two curves $c,\ c' : [0,1] \to X$ with $c(0) = c'(0) = p$, the *upper angle* $\bar{\alpha}$ between c and c' is defined by

$$\bar{\alpha}(c,c') = \overline{\lim_{x,y\to p}} \cos^{-1}\left(\frac{d(p,x)^2 + d(p,y)^2 - d(x,y)^2}{2d(p,x)d(p,y)}\right)$$

where $x,\ y \to p$ along $c,\ c'$, respectively. The *lower angle* $\underline{\alpha}$ between c and c' is defined by

$$\underline{\alpha}(c,c') = \underline{\lim_{x,y\to p}} \cos^{-1}\left(\frac{d(p,x)^2 + d(p,y)^2 - d(x,y)^2}{2d(p,x)d(p,y)}\right).$$

We say that there is an *angle* $\measuredangle$ between c and c' if $\bar{\alpha} = \underline{\alpha}$ and define the angle by $\measuredangle = \bar{\alpha}$.

It is well known that between two geodesics γ and σ starting at a point in a nonpositively curved geodesic space, there is an angle $\measuredangle(\gamma,\sigma)$ and the angle between two geodesics starting at p is a continuous function of the end points of the geodesics in a convex neighborhood of p.

We denote by $[x,y]$ a geodesic segment from x to y parameterize by the arclength, or the image set of the geodesic segment. Generally a geodesic between two points is not unique. But in a convex subset of a space, it is unique by definition.

We introduce some concepts about geodesics.

DEFINITION 1.1. Let X be a geodesic space and $x, y, p \in X$.

(1) A geodesic segment $[x, y]$ is called a *geodesic extension* of $[x, p]$ if $p \in [x, y]$, and we say that $[x, p]$ is extended through p by $[x, y]$.

(2) A point $p \in X$ is called an *interior point* of X if there is an extension, which passes through p, of the geodesic segment $[x, p]$ for each x near p. (This means that geodesics in every direction can be extended through p locally.)

(3) A point $p \in X$ is called a *boundary point* of X if it is not an interior point, that is, there is a geodesic $[x, p]$ which cannot be extended through p.

(4) The set of all boundary points in X is called the *geodesic boundary* ∂X of X.

For a topological manifold with curvature bounded above, the geodesic boundary coincides with the topological concept of the boundary of manifolds. If the curvature is not bounded above then some manifold point may be a geodesic boundary point. For example, let X be a Euclidean cone over a circle of radius < 1 then the cone point is a geodesic boundary point. The following lemma will characterize the geodesic extensions.(cf. [5])

LEMMA 1.2. *Let X be a space of curvature (locally) bounded above, then a geodesic segment $[x, y]$ in X is a geodesic extension of $[x, p]$ if and only if $\measuredangle([p, x], [p, y]) = \pi$.*

A space is called *geodesically complete* if every geodesic is a part of a geodesic line. We prove the Hopf-Rinow theorem for CAT(0) spaces. (For a different version of this theorem see [1].)

PROPOSITION 1.3. *A complete CAT(0) space without geodesic boundary is geodesically complete.*

PROOF. Let X be a complete CAT(0)-space without geodesic boundary and let $[x, p]$ be a geodesic segment in X, then there is an $\varepsilon_1 > 0$ such that $[p_{-\varepsilon_1}, p]$ is extended to a segment $[p_{-\varepsilon_1}, p_2]$ for some p_2 where $p_{-\varepsilon_1} \in [x, p]$ and $d(p_{-\varepsilon_1}, p) = \varepsilon_1$. By the angle comparison of the comparison triangle in a CAT(0)-space, we can see that the curve $[x, p] \cup [p, p_2]$ is a (minimal) geodesic $[x, p_2]$.

By the same argument, we can extend the segment $[x, p_2]$ to a segment $[x, p_3]$ through p_2 for some $p_3 \in X$. Continuing this process we get a sequence of points $\{p_n\}$ such that $[x, p_{n+1}]$ is an extension of $[x, p_n]$.

We claim that the limit of the sequence of segments $\{[x, p_n]\}$ can be a ray. Suppose the limit can not be a ray, then the sequence $\{[x, p_n]\}$ must be bounded. So

$$\sum_{n=1}^{\infty} d(p_n, p_{n+1}) < c < \infty$$

Hence $\{p_n\}$ is a Cauchy sequence. By the completeness of X, $\{p_n\}$ converges to some point $q \in X$. This contradicts with the extendability of geodesic at q. □

A *direction* $[\gamma]$ at a point p in a CAT(0) space X is an equivalence class of curves starting at p which has zero upper angle with γ. The following is easy to prove from the above proposition.

COROLLARY 1.4. *In a complete CAT(0) space without geodesic boundary, for each point there is a ray in each direction. In particular, every geodesic segment can be extended to a ray.*

§2. Shadows near a point in metric spheres

In this section X will denote a CAT(0) space if not otherwise stated. A basic difference between Riemannian and non-Riemannian CAT(0) space is that geodesic extension of a geodesic segment may not be unique. So we define the following concept.

DEFINITION 2.1. A point $p \in \gamma$ is called a *branch point* of a geodesic γ if there are $x, y \in \gamma$ and $z \notin \gamma$ near p such that $[x, y] \cap [x, z] = [x, p]$. $[p, y]$, $[p, z]$ are called *branches of* the geodesic $[x, p]$.

A branch point p is called *isolated* if there are no branch points near p in $X$and *isolated along a geodesic* γ if there are no branch points near p on γ.

If $p \in X$ is a branch point then the curvature is not bounded below near p. So a geodesic space of curvature bounded below contains no branch points.

We introduce a concept that measure the amount of branching.

DEFINITION 2.2. Let X be a locally convex geodesic space without boundary and let the metric ball $B_p(r)$ be contained in a convex subset of X for $p \in X$ and $r > 0$. The (*local*) *shadow* $\Sigma_p^r(x)$ of $x \in S_p(r)$ in a metric sphere $S_p(r)$ consists of the points at which the geodesic extension of $[x, p]$ and $S_p(r)$ intersects.

Note that if the closed metric ball $\bar{B}_p(r)$ is contained in a convex subset, then we have

$$\Sigma_p^r(x) = \{y \in S_p(r) | d(x, y) = 2r\} = S_p(r) \cap S_x(2r).$$

So the shadows $\Sigma_p^r(x)$ are compact for all $x \in S_p(r)$ because $\Sigma_p^r(x) = S_p(r) \cap S_x(2r)$ is a closed subset of a compact set $S_p(r)$.

DEFINITION 2.3. Given a point $p \in X$ and a subset $S \subset X$, the following subset is called the *geodesic cone* on p over S;

$$C_p(S) := \bigcup_{x \in S} [p, x]$$

For a point in a topological manifold, we can see more about the shadows.

PROPOSITION 2.4. *Let $p \in M$ be a manifold point of a complete geodesic space M. Assume $\bar{B}_p(r)$, $r > 0$, be a convex closed metric ball homeomorphic to the standard closed ball $\bar{B}^{n+1}$ and $n \geq 1$. Then the local shadows have the following properties:*

(1) *$\Sigma_p^r(x)$ does not separate $S_p(r)$ for each $x \in S_p(r)$. In particular,*

$$\pi_{n-1}(\Sigma_p^r(x)) = 0.$$

(2) *If there are no branch points in $B_p(r) \setminus \{p\}$ and $n \geq 1$, then $\Sigma_p^r(x)$ is path connected for each $x \in S_p(r)$.*

PROOF. (1) By the invariance of domain in Euclidean spaces, $S_p(r)$ is homeomorphic to the sphere. Assume $\Sigma_p^r(x)$ separates $S_p(r)$ for some $x \in S_p(r)$. Let U, V be two component of $S_p(r) \setminus \Sigma_p^r(x)$ such that $x \in U$ and let $y \in V$.

Now consider the family of geodesics from p;

$$\mathcal{G} = \{[p,z] | z \in \Sigma_p^r(x)\}$$

Since the geodesics in a convex domain behaves continuously with respect to the end points and $\bar{B}_p(r)$ is homeomorphic to the standard ball $\bar{B}^{n+1}$, the family $\mathcal{G}$ corresponds to a continuous family of curves in $\bar{B}^{n+1}$ from an interior point to a subset in the boundary ∂B^{n+1}. So the image set of the family $\mathcal{G}$, the geodesic cone over $\Sigma_p^r(x)$,

$$C_p(\Sigma_p^r(x)) = \bigcup_{z \in \Sigma_p^r(x)} [p,z]$$

separates $\bar{B}_p(r)$ into at least two parts, one of which contains x and another y.

Since $\bar{B}_p(r)$ is convex, the segment $[x,y]$ must pass through $C_p(\Sigma_p^r(x))$ at some point in $B_p(r)$. This leads to a contradiction with the uniqueness of geodesics in $\bar{B}_p(r)$. Hence $\Sigma_p^r(x)$ cannot separate $S_p(r)$.

(2) Assume $\Sigma_p^r(x)$ is not path connected and let $S \subset S_p(r)$ be a connected component of $\Sigma_p^r(x)$. Then S and $\Sigma_p^r(x) \setminus S$ are compact. So there is a connected open neighborhood U of S in $S_p(r)$ such that $U \cap \Sigma_p^r(x) = S$ and the boundary $\partial U \subset S_p(r)$ does not meet the shadow $\Sigma_p^r(x)$. Note that the set ∂U separates $S_p(r)$ into at least two parts, say, U and $V = S_p(r) \setminus \bar{U}$.

Now consider the geodesic cone $C_x(\partial U)$. Since the geodesics from x to ∂U moves continuously with respect to the end points (this follows from the convexity of geodesics), the cone $C_x(\partial U)$ separates the closed metric ball $\bar{B}_p(r)$ into at least two parts. Since $C_x(S) \subset C_x(U)$, $p \in C_x(U)$. For a point $q \in \Sigma_p^r(x) \setminus S$ $(= \Sigma_p^r(x) \setminus U)$ the geodesic segment $[p,q]$ must meet the cone $C_x(\partial U)$ at some point q_t; i.e., $q_t \in C_x(\partial U) \cap |[p,q]|$. This means that there are two geodesics from x to q_t, one in $C_x(\partial U)$ and another through p, which is a contradiction. Hence $\Sigma_p^r(x)$ cannot be disconnected. □

The simplicities of the topology of S^1 and S^2 imply the following Corollary.

COROLLARY 2.5. *If* $\dim M = 2$ *or* 3 *in the above proposition, then* $\Sigma_p^r(x)$ *is contractible.*

§3. The ideal boundary of CAT(0) spaces

There are two kind of ideal boundaries of Hadamard manifold, one is the set $Bd(X)$ of horofunctions up to additive constants and the other is the set $X(\infty)$ of the asymptotic classes of rays. These concepts may be applied to complete CAT(0) spaces.

Let (X, d) be a complete geodesic space and $C(X)$ be the space of continuous functions on X with the topology of uniform convergence on compact sets. The map $x \mapsto d_x (= d(x, \cdot))$ defines an embedding of X into $C(X)$. We consider the space $C_*(X) := C(X)/(constant\ functions)$ and for $f \in C(X)$, $\bar{f}$ denotes the class in $C_*(X)$ containing f. Then the embedding $X \hookrightarrow C(X)$ induces an embedding $\imath : X \hookrightarrow C_*(X)$, $x \mapsto \bar{d}_x$. The ideal boundary $Bd(X)$ is defined as $Cl(X) - \imath(X)$, where $Cl(X)$ is the closure of $\imath(X)$ in $C_*(X)$. A point in $Bd(X)$ is an equivalence class of functions called *horofunctions*, which are well-defined up to an additive constant.

The another way of defining a boundary at infinity, which can be applied only for CAT(0)-spaces, is the following.(ref. [2]) Let X be a complete CAT(0) space. Two rays γ, $\sigma : [0, \infty) \to X$ are called *asymptotic* if there is a constant $a \in \mathbb{R}$ such that $d(\gamma(t), \sigma(t)) \leq a$ for all $t \geq 0$. The equivalence classes of this relation are called *points at infinity* and denote by $X(\infty)$ the set of all points at infinity. For a ray $\gamma : [0, \infty) \to X$, $\gamma(\infty) \in X(\infty)$ denote the corresponding equivalence class of rays asymptotic to γ. For $z \in \bar{X} = X \cup X(\infty)$ and $p \in X$ we also denote by $\sigma_{p,z}$ the uniquely defined geodesic(or ray) from p to z(if $z \in X(\infty)$ then $\sigma_{p,z} \in z$).

For $p \in X$, $\xi \in X(\infty)$ and $R > 0$, $\epsilon > 0$, let

$$U(p, \xi, R, \epsilon) := \{z \in \bar{X} | z \notin \bar{B}_p(R), d(\sigma_{p,z}(R), \sigma_{p,\xi}(R)) < \epsilon\}$$

This cone is the union of the rays starting at p that pass through the ϵ-neighborhood of $\sigma_{p,\xi}(R)$ minus the closed ball $\bar{B}_p(R)$. The *cone topology* on $\bar{X}$ is the topology generated by the open sets in X and these cones. The induced topology on $X(\infty)$ is also called the *sphere topology*. In case X is a n-dimensional Hardmard manifold, the ideal boundary $X(\infty)$ with this topology is homeomorphic to the sphere S^{n-1}. But if X is not a Riemannian manifold then this not so. For example, let Y be a tree then the ideal boundary of $Y \times \mathbb{R}$ is not homeomorphic to a sphere if there is a branch point in Y.

It is known that two kinds of ideal boundaries are homeomorphic by the correspondence(cf. [2]);

$$\gamma \ (a\ ray) \mapsto \ the\ Busemann\ function\ of\ \gamma$$
$$h \ (a\ horofunction) \mapsto \ the\ gradient\ of\ h$$

On the other hand, If we fix a point $p \in X$, there is some kind of foliation in a CAT(0) space X by metric spheres and there are canonical projections between the leaves of this foliation:

$$\mu_r^s : S_p(s) \to S_p(r), \quad r < s,$$

given by the unique geodesic from a point $x \in S_p(s)$ to p, that is,

$$\mu_r^s(x) = the\ unique\ point\ in\ S_p(r) \cap [x,p].$$

This foliation has two singularities, one is at the origin and the other is at infinity. The singularity at the origin is called the *infinitesimal sphere* and the singularity at infinity is the *visual sphere*. In fact, the followings are the direct and inverse systems, respectively;

$$I = \{S_p(r) : r \leq r_0\}$$
$$V = \{S_p(r) : r \geq r_0\}$$

and the limits

$$S_p(0) := Dir \lim_{r \to 0} S_p(r)$$
$$S_p(\infty) := Inv \lim_{r \to \infty} S_p(r)$$

are the *infinitesimal sphere* and the *visual sphere*, respectively.

If X is a complete CAT(0)-space, the convergence in $S_p(\infty)$ is the pointwise convergence of rays. So the ideal boundary $X(\infty)$ and the visual sphere $S_p(\infty)$ are homeomorphic for any $p \in X$.

Now we focus on the complete CAT(0) manifold. For a complete CAT(0) Riemannian manifold, which is also called a Hadamard manifold, the ideal boundary is homeomorphic to the standard sphere. ([3]) This is also true for piecewise constant CAT(0) PL-manifold. But it may happen that the ideal boundary of some CAT(0) manifold of dimension$\geq$ 4 is not homeomorphic to the standard sphere. ([7]) We are interested in the remaining case of dimension 2 and 3.

To know the topology of the ideal boundary of a CAT(0) manifold, a topological manifold which is also a CAT(0) space, we will use the homotopy characterization of cell-like maps. A compact metric space C is called *cell – like* if the space C have the following property of a cell; there is an embedding of C into the Hilbert cube I^∞ such that for any neighborhood U of C in I^∞, C is null-homotopic in U. A map is called *proper* if the preimage of each compact subset is compact. A proper surjection is called a *cell-like map* if each point-inverse is cell-like.

We state a property of cell-like maps between absolute neighborhood retracts(ANR).

LEMMA 3.1. [8] (Homotopy characterization of cell-like maps) *Suppose $f : X \to Y$ is a proper surjection of ANR's. Then f is cell-like if and only if for each open subset U of Y, the restriction $f : f^{-1}(U) \to U$ is a homotopy equivalence.*

To use the above lemma in the proof of the theorem 3.3, we need the following.

LEMMA 3.2. *In a complete proper CAT(0)-manifold M in which every branch point p is isolated, each metric sphere is an ANR.*

PROOF. For each $p \in M$ and $R > 0$, since $S_p(R)$ is compact and every branch point is isolated, there is an annular neighborhood U of $S_p(R)$ such that $U \setminus S_p(R)$ contains no branch points. Then the projection $\rho : U \to S_p(R)$ along the geodesics from p through $S_p(R)$ is a retraction. Since $\bar{B}_p(R+1)$ can be embedded in some Euclidean space $\mathbb{R}^k$ of large dimension, the retraction ρ in M can be extended to a retraction of a neighborhood of $S_p(R)$ to $S_p(R)$ in $\mathbb{R}^k$. Hence $S_p(R)$ is an ANR. (In fact, $S_p(R)$ is a Euclidean neighborhood retract.) □

It is not clear whether there is a metric sphere for each point in a CAT(0) manifold which is homeomorphic to the standard sphere. So we have to assume some conditions on the spaces.

THEOREM 3.3. *Let M be a complete proper CAT(0) manifold such that every point $p \in M$ has a radius $r > 0$ for which the closed metric ball $\bar{B}_p(r)$ is homeomorphic to the standard closed ball of the same dimension. If M satisfies one of the followings;*

(1) $\dim M = 2$,

(2) $\dim M = 3$ *and every branch point is isolated,*

then the ideal boundary $M(\infty)$ is homeomorphic to the standard sphere.

PROOF. Fix $p \in M$ and we will show that the visual sphere $S_p(\infty)$ at p is homeomorphic to the standard sphere.

If the closed metric ball $\bar{B}_p(r)$ is homeomorphic to the standard closed ball then by the invariance of domain the (open) metric ball $B_p(r)$ is homeomorphic to the standard open ball and the boundary $S_p(r)$ is also homeomorphic to the standard sphere.

Since M satisfies one of (1) or (2), if $B_p(r)$ is homeomorphic to the standard ball then $B_p(\delta)$ is homeomorphic to the standard ball for all $\delta \leq r$.

We will show that $S_p(R)$ is homeomorphic to $S_p(r)$ for each $R > r$. Consider the annular region $A_{r,R} = \bar{B}_p(R) \setminus B_p(r)$ Since M is proper, that is, every closed bounded subset is compact, $A_{r,R}$ is compact. Let $\{B_q(r_q) | q \in A_{r,R}\}$ be the open covering of $A_{r,R}$ consists of open balls homeomorphic to standard balls, then there is a Lebesgue number $\delta > 0$ of this covering such that each $B_q(\delta)$ is homeomorphic to standard balls.

We claim that;

(*) If $S_p(s)$ is homeomorphic to the standard sphere and $r \leq s < s + \lambda \leq R$, then $S_p(s+\lambda)$ is homeomorphic to $S_p(s)$ for each $\lambda < \delta$.

If (*) holds, then by induction we can show that $S_p(R)$ is also homeomorphic to $S_p(r)$ for each $R > r$. Since every metric sphere is homeomorphic to the standard sphere, so is the inverse limit $S_p(\infty)$ of the metric spheres.

Now it remains to prove (*). For each $q \in S_p(s)$, $B_q(\lambda)$ is homeomorphic to the standard ball and $S_q(\lambda) \cong$ standard sphere. Define a projection $\phi : S_p(s+\lambda) \to S_p(s)$ as follows;

for each $y \in S_p(s+\lambda)$, let $\phi(y)$ be the unique point in $S_p(s) \cap [p, q]$. Then ϕ is a continuous surjetion due to the convexity of M, and $\phi^{-1}(q) = \Sigma_q^s(p)$ for each $q \in S_p(s)$.

Using the above map ϕ we will show that $S_p(s+\lambda)$ is homeomorphic to $S_p(s)$ for each $\lambda < \delta$ in each of the following cases.

(1) In case $\dim M = 2$;

By the Proposition 2.4 and Corollary 2.5, $\phi^{-1}(q)$is a 1-cell or a 0-cell. Considering the correspondence between $q \in S_p(s)$ and $\phi^{-1}(q)$, $S_p(s+\lambda)$ can be constructed by attaching a 1-cell or a 0-cell at each point $q \in S_p(s)$. So $S_p(s+\lambda)$ is homeomorphic to $S_p(s)$.

(2) In case $\dim M = 3$ and the set of branch points is discrete;

By the Proposition 2.4 and Corollary 2.5, the projection $\phi : S_p(s+\lambda) \to S_p(s)$ is a cell-like map between ANR's. By the Lemma 3.1, ϕ is a homotopy equivalence and so $S_p(s+\lambda)$ and $S_p(s)$ are homeomorphic since $\dim S_p(s) \neq 3$ and $S_p(s)$ is a topological sphere. □

A CAT(0) piecewise constant PL-manifold M satisfies the conditions of the above theorem if $\dim M \leq 3$. So the above theorem is a generalization of the result in [7].

We end this paper with an example of a CAT(0) space which is not a topological manifold with the ideal boundary homeomorphic to the standard sphere.

EXAMPLE 3.4. Let

$$A = B = \{re^{i\theta} \in \mathbb{C} : r \geq 1, 0 \leq \theta \leq 2\pi\}$$
$$D = \{re^{i\theta} \in \mathbb{C} : r \leq 1\}$$

(here we consider A and B as cut along the x-axis). Each boundary of A and B consists of three parts.

$$A_0 = B_0 = \{re^{i\theta} \in \mathbb{C} : r = 1\}$$
$$A_1 = B_1 = \{re^{i\theta} \in \mathbb{C} : r \geq 1,\ \theta = 0\}$$
$$A_2 = B_2 = \{re^{i\theta} \in \mathbb{C} : r \geq 1,\ \theta = 2\pi\}$$

Glue the boundaries of A, B, D along the pairs:

(a) A_0, B_0, ∂D,

(b) A_1, B_2,

(c) A_2, B_1.

The resulting space X is a CAT(0) space but not a manifold. Since every metric sphere $S_o(r)$ centered at the origin o of radius>1 is homeomorphic to the standard sphere and there are no branch points in $X - B_o(2)$, we can easily see that the ideal boundary $X(\infty)$ is homeomorphic to the circle.

References

[1] S. B. Alexander and R. L. Bishop, *The Hadamard-Cartan Theorem in Locally convex Metric Space*, Enseign. Math., vol. **36**, 1990, pp. 309–320.

[2] W. Ballmann, *Lectures on space of nonpositive curvature*, DMW seminar band, No. 25, Birkhauser, 1995.

[3] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature*, Progress in Math., vol. **61**, Birkhauser, Boston, 1985.

[4] V. N. Berestovskij and I. G. Nikolaev, *Multidimensional generalized Riemannian spaces*, Geometry 4 (Non-regular Riemannian Geometry) (Yu. G. Reshetnyak, eds.), Encyclopedia of Mathematical Sciences, vol. **70**, Springer-Verlag, 1993.

[5] Martin R. Bridson, *On the existence of flat planes in spaces of nonpositive curvature*, preprint (1994).

[6] H. Busemann, *The geometry of geodesics*, Pure and Applied Mathematics, Academic Press, New York, 1955.

[7] M. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. of Diff. Geo. **34** (1991), 347–388.

[8] R. D. Edwards, *The topology of manifolds and cell-like maps*, Proc. of ICM, Helsinki (1978), 111–127.

[9] M. Gromov, *Hyperbolic Manifolds, Groups and Actions, Riemann Surfaces and Related Topics*, Ann. of Math. Studies **97** (1981), Princeton University Press.

[10] ———, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, eds.), MSRI Publ., vol. **8**, Springer Verlag, New York, 1985.

[11] K. Shiohama, *An Introduction to the Geometry of Alexandrov Spaces*, Proc. Workshop in Pure Math.(Daewoo), Seoul Korea, vol. **12**, Part 3, 1992.

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