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DISCRETE PROOF OF EVEN KAKUTANI EQUIVALENCE VIA α- AND β-EQUIVALENCE

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ABSTRACT. It has been known that if T and S are even Kakutani equivalent, then there exists U such that T and U are α-equivalent and S and U are β-equivalent where α and β are irrationally related. In this paper we give a complete discrete proof of this theorem without using R-actions.

1. Introduction

An orbit equivalence ϕ between two probability spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ is defined to be invertible, measure preserving map which sends the set of the orbit of a point x onto the set of the orbit of $\phi(x)$. Dye's orbit equivalence is the most "unrestrictive" orbit equivalence in the sence that ϕ does not keep any of the group structures of an orbit. Isomorphism between two probability space is an orbit equivalence which preserves the group structure of an orbit. Recently a general theory of orbit equivalence, called restricted orbit equivalence has been developed by D. Rudolph [Ru] and thereafter by many others [dJFR] [KR] [Pa2]. Dye's orbit equivalence, Kakutani equivalence and isomorphism can be recasted in the famework of restricted orbit equivalences [ORW]. More recently the theory has been developed for general groups [KR].

Given an irrational α, even α-equivalence have been defined as follows [dJFR] [Pa2]. $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ are α-equivalent if and only if some flow under a function taking values 1 and $1 + \alpha$ with base $(X, \mathcal{F}, \mu, T)$ can be represented as a flow under a function taking values 1 and $1 + \alpha$ with the base $(Y, \mathcal{G}, \nu, S)$. We assume that the two ceiling functions have the same integrals over their respective bases. Notice that

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without the restrictions on the values of ceiling functions, this defines even Kakutani equivalence between $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$. Hence it is clear that even α-equivalence is a finer equivalence relation than even Kakutani equivalence. We may remark that we can define α-equivalence as we define Kakutani equivalence dropping the term "even" [Pa2].

We begin with our notations. Let $dist(a, Z)$ denote the distance from a real number a to the set Z. We write $T(x,y) = n$ if and only if $y = T^n x$. We denote by (a) the fractional part of a real number a. We denote the induced map of T on A by T_A and denote by $\mathcal{F}_A$ the σ-algebra $\mathcal{F}$ restricted to A. We can give the following equivalent definition for α-equivalence [dJFR].

Definition. We say $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ are α-equivalent if there exists an orbit equivalence $\phi : (X, \mathcal{F}, \mu, T) \to (Y, \mathcal{G}, \nu, S)$ such that given $\epsilon > 0$ and for all A of positive measure, there is a subset $B \subset A$ such that

(i) ϕ is an isomorphism between T_B and $S_{\phi(B)}$.
(ii) For all $x, y \in B$ where $y = T^n x$,

$$dist\left(\frac{T(x,y) - S(\phi(x), \phi(y))}{\alpha}, Z\right) < \epsilon.$$

We may mention that there is a definition of even α-equivalence for continuous actions(flows)[Pa1]. Notice that if ϕ satisfies the condition (i), then ϕ is an even Kakutani equivalence. We require an additional condition (ii) for α-equivalence.

It is proven in [dJFR] that a non-Loosely Bernoulli even Kakutani equivalence class has uncountably many different α-equivalence classes.

2. Main Result

The following theorem discribes the relation between even Kakutani equivalence and even α-equivalence [Pa1]. Also we have the analogous theorem for flows. The proof of the theorem in [Pa1] hinges on the original definition which is given in terms of flows. In this paper we present a discrete and simpler proof of the theorem bypassing the flow construction. Although this method can not handle the proof of the theorem of continuous actions, it is hoped to be useful for proving analogous results for more general groups like Z^n.

THEOREM 1. *If $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ are even Kakutani equivalent, then there exists $(Z, \mathcal{H}, \lambda, U)$ such that T and U are α-equivalent and Y and U are β-equivalent where α and β are irrationals and irrationally related.*

Before we start the construction, we need the following facts.

LEMMA 2. *Let m and n be sufficiently large numbers. Given $\epsilon > 0$, there exists $L(\epsilon)$ such that for some $0 \le k \le L(\epsilon)$,*

$$\left(\frac{m+k}{\alpha}\right) < \epsilon \qquad \text{and} \qquad \left(\frac{n+k}{\beta}\right) < \epsilon$$

where (a) denote the fractional part of a.

PROOF. Let $x_0 = (\frac{m}{\alpha})$ and $y_0 = (\frac{n}{\beta})$. Since $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are irrationally related, the map $R_{(\frac{1}{\alpha}, \frac{1}{\beta})}(x, y) = (x + \frac{1}{\alpha}, y + \frac{1}{\beta})(mod\ 1)$ is ergodic on a two torus $\pi^2 = S^1 \times S^1$. Hence there exists $L(\epsilon)$ such that $\{R^i_{(\frac{1}{\alpha}, \frac{1}{\beta})}(x_0, y_0) : 0 \le i < L\}$ is ϵ-dense in π^2. □

Notice that $L(\epsilon)$ can be chosen independent of m and n.

COROLLARY 3. *Given any integer p and any $m \ge n$ sufficiently large, there exists q such that*

$$|p - q| < \frac{L(\epsilon)}{2}$$

$$\left(\frac{m-q}{\alpha}\right) < \epsilon \qquad \text{and} \qquad \left(\frac{n-q}{\alpha}\right) < \epsilon.$$

The construction of $(Z, \mathcal{H}, \lambda, U)$ is through successive approximation method. We will divide it into two parts.

2.1. First step

We let $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ be even Kakutani equivalent. Let ϕ be an isomorphism between T_A and S_B. Let $\epsilon = \sum \epsilon_i \ll \mu A$ be given.

Given $l > 0$, we denote by R_A a Rochlin tower of height l of $(A, \mathcal{F}_A, \mu, T_A)$ with an error set of measure less than $\frac{\epsilon_1}{2}$. We use R_A to build a skyscraper of $(X, \mathcal{F}, \mu, T)$ as follows : If $x \in R_A$ and $T_A(x) \in R_{k+1}$ where R_k denote a level set of R_A for $k = 0, 1, \cdots, l-1$, then there exists $r(x)$ such that $T_A(x) = T^{r(x)}(x)$. We add $\{Tx, T^2x, \cdots, T^{r(x)-1}x\}$ between x and $T_A(x)$. We do this for every point of R_k for $k = 0, 1, \cdots, l-1$. This builds a skyscraper denoted by Γ of $(X, \mathcal{F}, \mu, T)$. Let $R_B = \phi(R_A)$ be a Rochlin tower of $(B, \mathcal{G}_B, S_B, \nu)$ of height l. We build a skyscraper, denoted by $\bar{\Gamma}$, of $(Y, \mathcal{G}, \nu, S)$ using R_B as we did for $(X, \mathcal{F}, \mu, T)$. Notice that Γ(and $\overline{\Gamma}$) does not have a constant height. Also a point x in the bottom level set of $\overline{\Gamma}$ and its corresponding point $\phi(x)$ in $\overline{\Gamma}$ may not have the same height.

We divide the skyscraper into columns so that each column has a constant height. Moreover if x and x' are in the same level of R_A in a column, then $r(x) = r(x')$ and $r(\phi(x)) = r(\phi(x'))$. That is, if x snd x' are in the bottom level set of R_A, then we have $r(T_A^i(x)) = r(T_A^i(x'))$ and $r(T_B^i(\phi(x))) = r(T_B^i(\phi(x')))$ for all $i = 0, 1, \cdots, l-1$. If $\cup_{i=0}^{l-1} T^i A$ denotes a column of $\Gamma^{(1)}$, then we call by a subcolumn a set $\cup_{i=0}^{l-1} T^i B$ where $B \subset A$. By taking a subset of A if necessary, we may assume that $A \cap (X - \Gamma) = B \cap (Y - \bar{\Gamma}) = \emptyset$.

There exists $n_0(\epsilon_1)$ such that the set $E = \{x : |\frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i x) - \mu A| < \epsilon_1, \quad \text{for all} \quad n \geq n_0\}$ has measure greater than $1 - \epsilon_1$. By the standard argument, we may assume that the bottom level set of Γ is contained in E. We may also assume that the bottom level set of $\bar{\Gamma}$ satisfies that

$$|\frac{1}{n} \sum_{i=0}^{n-1} \chi_B(S^i y) - \mu B| < \epsilon_1$$

for all $n \geq n_0$. If $x \in E$, then we say that x satisfies the ergodic theorem within ϵ_1 with respect to the set A.

Using the skyscrapers, we construct $(Z^{(1)}, \mathcal{H}^{(1)}, \lambda^{(1)}, U^{(1)})$ satisfying the following.

(1.1) $\lambda^{(1)}(Z^{(1)}) > 1 - 2\epsilon_1$.

(1.2) There exists a subset $C^{(1)} \subset Z^{(1)}$ such that $\varphi : T_{A^{(1)} \cap \Gamma} \to U^{(1)}_{C^{(1)}}$ and $\psi : S_{B^{(1)} \cap \bar{\Gamma}} \to U^{(1)}_{C^{(1)}}$ are isomorphisms.

(1.3)

$$\left(\frac{T(T_A^i x, T_A^j x) - U^{(1)}(\varphi(T_A^i x), \varphi(T_A^j x))}{\alpha}\right) < \epsilon_1 \text{for all} T_A^i(x) \text{and} T_A^j(x) \in \Gamma \cap A \text{ and}$$

$$\left(\frac{S(S_B^i y, S_B^j y) - U^{(1)}(\psi(S_B^i y), \psi(S_B^j y))}{\beta}\right) < \epsilon_1 \text{for all} S_B^i(y) \text{and} S_B^j(y) \in \bar{\Gamma} \cap B.$$

(1.4) $|r(\varphi(x)) - \frac{r(x)+r(\phi(x))}{2}| < 2L(\epsilon_1)$ for all $x \in \Gamma \cap A$.

We may assume that $\frac{1}{L(\epsilon_1)} < \frac{\epsilon_1}{2}$. It is easy to take a subset $A^{(1)} \subset A$ such that for every $x \in A^{(1)}$ and $y \in \phi(A^{(1)}))$, $r(x)$ and $r(\phi(x)) > L(\epsilon_1)^2$. Without confusion, we assume that the set A satisfies the above condition. Let h_1 denote the minimum of the heights of all columns of Γ and $\overline{\Gamma}$. We may assume that h_1 is large enough so that $\frac{2L(\epsilon_2)}{h_1 \cdot \mu A} < \frac{\epsilon_2}{2}$ and $h_1 > n_0(\epsilon_1)$.

To construct $(Z^{(1)}, \mathcal{H}^{(1)}, \lambda^{(1)}, U^{(1)})$, we first build a new skyscraper using a copy of the Rochlin tower R_A. We denote the copy of R_A by $R_{C^{(1)}}$. We divide $R_{C^{(1)}}$ by columns so that each column corresponds to a column of R_A. Suppose x is in the first level set of a column of Γ which is contained in A. Let $p_1 = \frac{T(x,T_A x)+S(\phi(x),\phi(T_A(x))}{2}$. By our Corollary 3, we find q_1 such that

(1.i) $|p_1 - L(\epsilon_1) - q_1| < \frac{L(\epsilon_1)}{2}$.

(1.ii) $(\frac{|T(x,T_A x)-q_1|}{\alpha}) < \frac{\epsilon_1}{2}$ and $(\frac{|S(\phi(x),\phi(T_A x))-q_1|}{\beta}) < \frac{\epsilon_1}{2.}$

We add q_1-many level sets of equal measure between the bottom level set and the next level set of the column of $R_{C^{(1)}}$.
If $p_i = \frac{T(x,T_A^i x)+S(\varphi(x),\varphi(T_A^i x))}{2}$ for $i = 2, \cdots, l-1$, then we find q_i such that

(1.iii) $|p_i - L(\epsilon_1) - q_i| < \frac{L(\epsilon_1)}{2}$.

(1.iv) $(\frac{|T(x,T_A^i x)-q_i|}{\alpha}) < \frac{\epsilon_1}{2}$ and $(\frac{|S(\phi(x),\phi(T_A^i x))-q_i|}{\beta}) < \frac{\epsilon_1}{2}$.

We add q_i-many level sets of equal measure between the i^{th} level set and the $(i+1)^{st}$ level set of the column of R_C. We repeat this between level sets of each column of R_C. We now have a skyscraper $\widetilde{\Gamma}^{(1)}$. Notice that $\widetilde{\Gamma}^{(1)}$ is made up of columns each of which corresponds to a column of $\Gamma^{(1)}$. We let $Z^{(1)}$ be the union of all level sets of $\widetilde{\Gamma}^{(1)}$. We define $U^{(1)}$

on $\widetilde{\Gamma}^{(1)}$ in the obvious way except on the top level set of $\widetilde{\Gamma}^{(1)}$. Every point in a level set is mapped to the point directly above in the next level set. The σ-algebra and the measure on $Z^{(1)}$ are also defined in the obvious way.

Now we look at the properties of $(Z^{(1)}, \mathcal{H}^{(1)}, \lambda^{(1)}, U^{(1)})$. We compare the measure of the first column of Γ and the first column of $\widetilde{\Gamma}$ of the skyscraper of $Z^{(1)}$. If we denote the first column of Γ, $\bar{\Gamma}$ and $\widetilde{\Gamma}$ by I_1, J_1, and H_1, respectively, then it is easy to see from our construction of $\widetilde{\Gamma}$ that

$$\frac{\mu(I_1)+\nu(J_1)}{2} - \frac{2L(\epsilon_1)}{L(\epsilon_1)^2} < \lambda^{(1)}(K_1) < \frac{\mu(I_1)+\nu(J_1)}{2}.$$

Since this holds for each column, we can say that

$$\frac{\mu(\Gamma)+\nu(\bar{\Gamma})}{2} - \epsilon_1 < \lambda^{(1)}(\widetilde{\Gamma}) < \frac{\mu(\Gamma)+\nu(\bar{\Gamma})}{2}.$$

$$1-\epsilon_1-\epsilon_1 < \lambda^{(1)}(Z^{(1)}) < 1.$$

It is also clear that $T_{A\cap\Gamma}, S_{B\cap\bar{\Gamma}}$ and $U_{C^{(1)}\cap\widetilde{\Gamma}}$ are pairwise isomorphic.

We note that for every $x \in R_A$ except the top level set of each column, we have

$$\begin{aligned}
&\left(\frac{T(T_A^i x, T_A^j x) - U^{(1)}(\varphi(T_A^i x), \varphi(T_A^j x))}{\alpha}\right) \\
&\leq \left(\frac{-T(x, T_A^i x) + T(x, T_A^j x) + U^{(1)}(\varphi(x), \varphi(T_A^i x)) - U^{(1)}(\varphi(x), \varphi(T_A^j x)}{\alpha}\right) \\
&= \left(\frac{-T(x, T_A^i x) + U^{(1)}(\varphi(x), \varphi(T_A^i x))}{\alpha}\right) + \left(\frac{T(x, T_A^j x) - U^{(1)}(\varphi(x), \varphi(T_A^j x))}{\alpha}\right) \\
&< \left(\frac{|-T(x, T_A^i x) + q_i|}{\alpha}\right) + \left(\frac{|T(x, T_A^j x) - q_i|}{\alpha}\right) \\
&< \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1
\end{aligned}$$

Likewise, for every $y \in R_B$ except the top level set of each column, we have

$$\left(\frac{S(S_B^i y, S_B^j y) - U^{(1)}(\varphi(S_B^i y), \varphi(S_B^j y))}{\beta}\right) < \epsilon_1.$$

2.2. Induction step

We want to build $(Z^{(2)}, \mathcal{H}^{(2)}, \lambda^{(2)}, U^{(2)})$ satisfying

(2.1) $\lambda^{(2)}(Z^{(2)}) > 1 - \epsilon_2$,
$\lambda^{(2)}(Z^{(1)} \cap Z^{(2)}) > 1 - 2\epsilon_1$.

(2.2) There exist subsets $A^{(2)} \subset A^{(1)} = A$, $B^{(2)} \subset B^{(1)} = B$ and $C^{(2)} \subset C^{(1)}$ such that $\varphi^{(2)} : T_{A^{(2)} \cap \Gamma} \to U^{(2)}_{C^{(2)}}$ and $\psi^{(2)} : S_{B^{(2)} \cap \bar{\Gamma}} \to U^{(2)}_{C^{(2)}}$ are isomorphisms, where $\mu(A^{(2)}) = \nu(B^{(2)}) = \lambda^{(2)}(C^{(2)}) > (1 - \epsilon_2)\mu(A^{(1)})$.

(2.3) For any i and j, we have

$$\left(\frac{|T(T^i_{A^{(2)}}(x)), T^j_{A(2)}(x)) - U^{(2)}\varphi(T^i_{A^{(2)}}(x)), \varphi(T^j_{A^{(2)}}(x)))|}{\alpha}\right) < 2\sum_{k=1}^{2} \epsilon_k$$

and

$$\left(\frac{|S(S^i_{B^{(2)}}(y)), S^j_{B(2)}(y)) - U^{(2)}\psi(S^i_{B^{(2)}}(y)), \psi(S^j_{B^{(2)}}(y)))|}{\alpha}\right) < 2\sum_{k=1}^{2} \epsilon_k$$

Recall that h_1 is chosen so that $\frac{2L(\epsilon_2)}{h_1 \cdot \mu(A)} < \frac{\epsilon_2}{2}$. We build a skyscraper $\Gamma^{(2)}$ of X and the corresponding skyscraper $\bar{\Gamma}^{(2)}$ of Y as follows. First we build a Rochlin tower $R^{(2)}_A$ of A and use this to build the skyscraper $\Gamma^{(2)}$ of X. We divide the skyscraper into columns so that each column and its correponding column of $\bar{\Gamma}^{(2)}$ has a constant height. Also for every $x \in A$ in a level set of a column, we have $T(x, T_A x)$ and $S(\phi x, \phi(T_A x))$ are constants. They may differ from a level set to a level set. We assume that the minimum height of all columns, denoted by h_2 is large enough to satisfying $\frac{2L(\epsilon_3)}{h_2 \cdot \mu A} < \frac{\epsilon_3}{2}$. Note that each column of $\Gamma^{(2)}$ consists of columns of Γ and level sets from the error set between the columns of Γ.

Clearly $T|_{A \cap \Gamma^{(2)}}$ and $S|_{B \cap \bar{\Gamma}^{(2)}}$ are isomorphic. Let I^2 denote a column of $\Gamma^{(2)}$ and J^2 denote the corresponding column of $\bar{\Gamma}^{(2)}$. Hence without confusion, we may say that $T|_{A \cap I^2}$ and $S|_{B \cap J^2}$ are isomorphic. We assume that each column of $\Gamma^{(2)}$ and $\bar{\Gamma}^{(2)}$ satisfies the ergodic theorem within ϵ_2 with respect to the set A and B respectively. We build the skyscraper $\widetilde{\Gamma}^{(2)}$ using $\widetilde{\Gamma}^{(1)}$ and some more extra set. Suppose $I^2 = I_1^2 \cup E_1^2 \cup I_2^2 \cup E_2^2 \cup I_3^2 \cup E_3^2 \cup \cdots \cup I_{l_1}^2$ where each I_i^2 denotes a subcolumn of $\Gamma^{(1)}$ and E_i^2 denotes the union of level sets from the error set between I_i^2 and I_{i+1}^2. Note that the bottom level set of I_i^2 is in A. We construct a column K^2 corresponding to I^2 which is made up of subcloumns of $\widetilde{\Gamma}^{(1)}$ and some extra set as follows.

We denote by K_i^2 the subcolumn of $\widetilde{\Gamma}^{(1)}$ corresponding to I_i^2. For each i, we add level sets, denoted by G_i, between K_i^2 and K_{i+1}^2 so that the number of level sets in $K_i^2 \cup G_i^2$ is the average of the number of level sets in $I_i^2 \cup E_i^2$ and the number of level sets in $J_i^2 \cup F_i^2$. Hence we have the number of the level sets in $K^2 = (\cup_{i=1}^{l_1} K_i^2) \cup (\cup_{i=1}^{l_1} G_i^2)$ is the average of the number of level sets in I^2 and the number of level sets in J^2. We construct a column corresponding to each column of $\Gamma^{(2)}$. We denote the new skyscraper by $\hat{\Gamma}^{(2)}$ whose column consists of concaternation of subcolumns of $\widetilde{\Gamma}^{(1)}$ and additional level sets between the subcolumns. Note that

$$\lambda^{(2)}(\hat{\Gamma}^{(2)}) = \frac{\mu(\Gamma^{(2)}) + \nu(\bar{\Gamma}^{(2)})}{2} > 1 - \frac{\epsilon_2}{2},$$

where $\lambda^{(2)}$ denote the obvious measure on $\hat{\Gamma}^{(2)}$. If necessary, we take a subset A_0 of A so that we may assume that there are more than $2L(\epsilon_2)$-many level sets between the last level set in A_0 of I_i^2 and the first level set in A_0 of I_{i+1}^2. Taking further subsets $A^{(2)}, B^{(2)}$ and $C^{(2)}$ of $A^{(1)}, B^{(1)}$ and $C^{(1)}$ respectively, if necessary, we may assume that there are more than $2L(\epsilon_2)$-many level sets between the last level set in $B^{(2)}$ of J_i^2 and the first level set in $B^{(2)}$ of J_{i+1}^2. Also we assume that we still have the property $T|_{A^{(2)} \cap I^2}$, $S|_{B^{(2)} \cap I^2}$ and $U|_{C^{(2)} \cap K^2}$ are isomorphic. The number of level sets removed from $A^{(1)}$ is at most $L(\epsilon_2)$ while the total number of level sets in A of I_i^2 is at least $h_1 \cdot \frac{\mu A}{2}$ for each i. By our choices of h_1 and L, we have

$$\frac{2L(\epsilon_2)}{h_1 \cdot \frac{\mu A}{2}} = \frac{2L(\epsilon_2)}{h_1 \cdot \frac{\mu A}{2}} < \frac{4L(\epsilon_2)}{h_1 \cdot \mu A} < \epsilon_2.$$

Since this holds for every subcolumn I_i^2, we have $\mu A^{(2)} > (1 - \epsilon_2)\mu A^{(1)}$. Since this also holds for the set $B^{(2)}$ and we require $T|_{A^{(2)}}$ be isomorphic to $S|_{B^{(2)}}$, we have $\mu A^{(2)} = \mu B^{(2)} > (1-\epsilon_2)\mu A^{(1)}$.

Now we start to construct $\widetilde{\Gamma}^{(2)}$ using $\hat{\Gamma}^{(2)}$. Let p_1^2 denote the height of $K_1^2 \cup G_1^2$. We let a_1^2 and b_1^2 denote the heights of $I_1^2 \cup E_1^2$ and $J_1^2 \cup F_1^2$ respectively. We find q_1^2 such that

(2.i) $|p_1^2 - L(\epsilon_2) - q_1^2| < \frac{L(\epsilon_2)}{2}$,

(2.ii) $\left(\frac{|a_1^2 - q_1^2|}{\alpha}\right) < \frac{\epsilon_2}{2}$ and $\left(\frac{|b_1^2 - q_1^2|}{\beta}\right) < \frac{\epsilon_2}{2}$

We remove $(p_1^2 - q_1^2)$-many level sets from the top of F_1^2. For each successive I_i^2 and I_{i+1}^2, we let by a_i^2 the height of $\cup_{j=1}^i (I_j^2 \cup E_j^2)$ and by b_i^2 the height of $\cup_{j=1}^i (J_j^2 \cup F_j^2)$.

Let p_i^2 denote the height of $\cup_{j=1}^i (K_j^2 \cup G_j^2)$ where $p_i^2 = \frac{a_i^2 + b_i^2}{2}$. For each i, find q_i^2 such that

(2.iii) $|p_i^2 - L(\epsilon_2) - q_i^2| < \frac{L(\epsilon_2)}{2}$.

(2.iv) $\left(\frac{|a_i^2 - q_i^2|}{\alpha}\right) < \frac{\epsilon_2}{2}$ and $\left(\frac{|b_i^2 - q_i^2|}{\beta}\right) < \frac{\epsilon_2}{2}$.

Note that $p_i^2 - 2L(\epsilon_2) < q_i^2 < p_i^2$. We remove $(p_i^2 - q_i^2)$-many level sets from the top of F_i^2. We repeat this for each column and denote the new skyscraper by $\widetilde{\Gamma}^{(2)}$ and denote the union of all level sets of $\widetilde{\Gamma}^{(2)}$ with the measure structure and the transformation by $(Z^{(2)}, \mathcal{F}^{(2)}, \lambda^{(2)}, U^{(2)})$.

We compute

$$\lambda^{(2)}(Z^{(2)}) = \lambda^{(2)}(\widetilde{\Gamma}^{(2)}) \geq \lambda^{(2)}(\hat{\Gamma}^{(2)})(1 - \frac{2L(\epsilon_2)}{h_1}) = \lambda^{(2)}(\hat{\Gamma}^{(2)})(1 - \frac{\epsilon_2}{2})$$
$$\geq (1 - \frac{\epsilon_2}{2})(1 - \frac{\epsilon_2}{2}) \geq 1 - \epsilon_2.$$

Hence we note that

$$\lambda^{(2)}(Z^{(1)} \cap Z^{(2)}) \geq \lambda^{(2)}(Z^{(2)}) - \epsilon_2.$$

Let $x \in A^{(2)}$ be in the bottom level set of a column of $\widetilde{\Gamma}^{(2)}$. We have

$$\left(\frac{|T(T^i_{A^{(2)}}(x), T^j_{A^{(2)}}(x) - U(U^i_{C^{(2)}}(\varphi(x)), U^j_{C^{(2)}}(\varphi(x)))|}{\alpha}\right)$$

$$= \left(\frac{|(T(T^i_{A^{(2)}}(x), x) + T(x, T^j_{A^{(2)}}(x))) - (U(U^i_{C^{(2)}}(\varphi(x)), \varphi(x)) + U(\phi(x), U^j_{C^{(2)}}(\varphi(x)))|}{\alpha}\right)$$

$$= \left(\frac{|(T(T^i_{A^{(2)}}(x), x) - U(U^i_{C^{(2)}}(\varphi(x)), \varphi(x)))|}{\alpha}\right)$$

$$+ \left(\frac{|(T(x, T^i_{A^{(2)}}(x)) - U(\varphi(x), U^j_{C^{(2)}}(\varphi(x))|}{\alpha}\right)$$

for all $T^i_{A^{(2)}}(x)$ and $T^j_{A^{(2)}}(x) \in \widetilde{\Gamma}^{(2)} \cap A^{(2)}$.

If I^2_k denote a subcolumn of $\Gamma^{(1)}$ that contains $T^i_{A^{(2)}}(x)$, then we denote by i_0 the integer such that $T^{i_0}_{A^{(2)}}(x)$ is in the bottom level set of I^2_k. Recall that the bottom level set of $\Gamma^{(1)}$ is contained in $A^{(2)}$. From our construction of $\widetilde{\Gamma}^{(2)}$, we have

$$\left(\frac{|(T(T^i_{A^{(2)}}(x), x) - U(U^i_{C^{(2)}}(\varphi(x)), \varphi(x)))|}{\alpha}\right)$$

$$= \left(\frac{|T(T^{i_0}_{A^{(2)}}(x), x) - U(U^{i_0}_{C^{(2)}}(\varphi(x)), \varphi(x)))|}{\alpha}\right)$$

$$+ \left(\frac{|T(T^{i_0}_{A^{(2)}}(x), T^i_{A^{(2)}}(x)) - U(U^{i_0}_{C^{(2)}}(\varphi(x)), U^i_{C^{(2)}}(\varphi(x)))|}{\alpha}\right)$$

$$< \epsilon_2 + \epsilon_1.$$

Likewise we have

$$\left(\frac{|T(x, T^j_{A^{(2)}}(x)) - U(\phi(x), U^j_{C^{(2)}}(\varphi(x)))|}{\alpha}\right) < \epsilon_2 + \epsilon_1.$$

Hence for any i and j,

$$\left(\frac{|T(T^i_{A^{(2)}}(x)), T^j_{A^{(2)}}(x)) - U(U^i_{C^{(2)}}(\varphi(x)), U^j_{C^{(2)}}(\varphi(x)))|}{\alpha}\right) < 2(\epsilon_2 + \epsilon_1).$$

This also holds between $S_{B(2)}$ and $U_{C(2)}$. That is,

$$\left(\frac{|S(S^i_{B(2)}(y), S^j_{B(2)}(y)) - U(U^i_{C(2)}(\psi(y)), U^j_{C(2)}(\psi(x)))|}{\beta}\right) < 2(\epsilon_2 + \epsilon_1).$$

We successively construct $(Z^{(n)}, \mathcal{H}^{(n)}, \lambda^{(n)}, U^{(n)})$ with the following property.

(n.1) $\lambda^{(n)}(Z^{(n)}) > 1 - \epsilon_n$,

$\lambda^{(n)}(\cap_{i=k}^n Z^{(i)}) > 1 - \sum_{i=k}^n \epsilon_i > 1 - \sum_{i=k}^\infty \epsilon_i$ for all $k = 1, \cdots, n$.

(n.2) There exist subsets $A^{(n)} \subset A^{(n-1)}, B^{(n)} \subset B^{(n-1)}$ and $C^{(n)} \subset C^{(n-1)}$ such that $\varphi : T_{A(n)} \to U_{C(n)}$ and $\psi : S_{B(n)} \to U_{C(n)}$ are isomorphisms, where $\mu(A^{(n)}) = \nu(B^{(n)}) = \lambda^{(n)}(C^{(n)}) > (1 - \sum_{i=1}^n \epsilon_i)\mu(A)$.

(n.3) For any i and j, we have

$$\left(\frac{|T(T^i_{A(n)}(x)), T^j_{A(n)}(x)) - U^{(n)}\varphi(T^i_{A(n)}(x)), \varphi(T^j_{A(n)}(x)))|}{\alpha}\right) < 2\sum_{k=1}^n \epsilon_k$$

and

$$\left(\frac{|S(S^i_{B(n)}(y)), S^j_{B(n)}(y)) - U^{(n)}\psi(S^i_{B(n)}(y)), \psi(S^j_{B(n)}(y)))|}{\alpha}\right) < 2\sum_{k=1}^n \epsilon_k.$$

Let $Z = \lim_{k\to\infty} \cap_{i=k}^\infty Z^{(i)}$. We denote this final system by $(Z, \mathcal{H}, \lambda, U)$. It is easy to check that the system $(Z, \mathcal{H}, \lambda, U)$ has the following properties.

(1) $\lambda(Z) = 1$.

(2) There exist subsets $A^\circ = \lim_{n\to\infty} A^{(n)}, B^\circ = \lim_{n\to\infty} B^{(n)}$ and $C^\circ = \lim_{n\to\infty} C^{(n)}$ such that $\varphi : T_{A^\circ} \to U_{C^\circ}$ and $\psi : S_{B^\circ} \to U_{C^\circ}$ are isomorphisms and $\mu(A^\circ) = \nu(B^\circ) = \lambda(C^\circ) > (1 - \epsilon)\mu A$.

(3) For any i and j, we have

$$\left(\frac{|T(T^i_{A^\circ}(x), T^j_{A^\circ}(x)) - U(U^i_{C^\circ}(\varphi(x)), U^j_{C^\circ}(\varphi(x))|}{\alpha}\right) < 2\epsilon$$

and

$$\left(\frac{|S(S^i_{B^\circ}(y), S^j_{B^\circ}(y)) - U(U^i_{C^\circ}(\psi(y)), U^j_{C^\circ}(\psi(y))|}{\alpha} \right) < 2\epsilon.$$

This completes our construction.

References

[dJFR] A. delJunco, A. Fieldsteel and A. Rudolph, *α-equivalence : refinement of even Kakutani equivalence*, J. of Ergodic Th. and Dyn. Systems **14** (1994), 69-102.

[KR] J. Kammayer and D. Rudolph, *Restricted orbit equivalence for actions of discrete amenable groups*, preprint.

[ORW] D. S. Ornstein, B. Weiss and D. Rudolph, *Equivalence of measure preserving transformation*, Mem. Amer. Math. Soc. **262** (1982).

[Pa1] K. K. Park, *Even Kakutani equivalence via α- and β-equivalences*, J. of Math. Analy. and Appl. **195** (1995), 335-353.

[Pa2] K. K. Park, *On α-equivalence*, to appear in Studia Mathematica.

[Ru] D. Rudolph, *A restricted orbit equivalence*, Mem. Amer. Math. Soc. **323** (1985).

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