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THE RIEMANN PROBLEM FOR A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE (II)

CHOON-HO LEE

ABSTRACT. We prove that solutions u^ϵ for the mixed hyperbolic-elliptic system of conservation laws with the viscosity term are total variation bounded uniformly in ϵ and that the solution u^ϵ converges to the solution for the mixed hyperbolic-elliptic Riemann problem as $\epsilon \to 0$.

1. Introduction

In [5] We had studied the existence of solutions for nonlinear hyperbolic-elliptic system of conservation laws of the form

$$\begin{aligned} u_t - f(v)_x &= 0, \\ v_t - g(u)_x &= 0 \end{aligned} \tag{1.1}$$

with initial condition

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-) & \text{if} \quad x < 0, \\ (u_+, v_+) & \text{if} \quad x > 0 \end{cases} \tag{1.1}$$

Here, $f \in C^2(\mathbb{R})$ is a strictly increasing convex function, $g \in C^2(\mathbb{R})$ and there exist α, β, η with $\alpha < \eta < \beta$ such that

$$g'(u) \geq 0 \text{ if } u \notin (\alpha, \beta) \text{ and } g'(u) < 0 \text{ for } u \in (\alpha, \beta),$$

$$g''(u) < 0 \text{ if } u < \eta \text{ and } g''(u) > 0 \text{ if } u > \eta.$$

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Under the same hypotheses, the equation (1.1) of change type has been studied by Fan [2], James [3], Shrear [6], Slemrod [7]. In case, the initial data u_- and u_+ was assumed to lie on the metastable state:

$$u_- < \alpha < \beta < u_+.$$

Their method for solving (1.1) was originated by Kalashnikov [4] and Tupciev [8], [9], where the vanishing viscosity method were used. Their idea is to replace (1.1) by the system

$$\begin{aligned} u_t - f(v)_x &= \epsilon t u_{xx}, \\ v_t - g(u)_x &= \epsilon t v_{xx} \end{aligned} \tag{1.3}$$

for $x \in \mathbb{R}$, $t > 0$ and construct solutions as the limit of the solutions of (1.3) and (1.2) as $\epsilon \to 0+$. The solution of the equation (1.1) is preserved under the dilation $(x,t) \to (ax, at)$, $a > 0$ so that (1.3), (1.2) admit solutions of the form $(u(\xi), v(\xi))$, where $\xi = \frac{x}{t}$. A simple calculation shows that $(u(\xi), v(\xi))$ is a solution of (1.3), (1.2) if it satisfies

$$\begin{aligned} \epsilon t u_\epsilon'' &= -\xi u_\epsilon' - f(v_\epsilon)', \\ \epsilon t v_\epsilon'' &= -\xi v_\epsilon' - g(u_\epsilon)' \end{aligned} \tag{1.4a}$$

$$(u_\epsilon(\pm\infty), v_\epsilon(\pm\infty)) = (u_\pm, v_\pm) \tag{1.4b}$$

We will establish an existence of solutions of (1.4a) and (1.4b), and prove that, for some sequence $\epsilon_n \to 0+$, $(u_{\epsilon_n}(\xi), v_\epsilon(\xi))$ converges to a weak solution of

$$\begin{aligned} -\xi u' - f(v)' &= 0, \\ -\xi v' - g(u)' &= 0, \end{aligned} \tag{1.5a}$$

$$(u(\pm\infty), v(\pm\infty)) = (u_\pm, v_\pm), \tag{1.5b}$$

which induces a solution $(u(x/t), v(x/t))$ of (1.1). In order to prove this we will use Helley's Theorem on the uniform boundedness of the total variation.

In case $f(u) = u$, Slemrod [7] and Fan [2] showed the existence of solution of (1.1). Following their ideas we will adopt the following two assumptions:

Assumption 1. *$g(u) \to \pm\infty$ as $u \to \pm\infty$.*

Assumption 2. *$f(v)$ is a strictly increasing convex C^2 function on v.*

In Section 2, we review some results which will be used in the next sections. In Section 3, we prove the main result of this paper:

Theorem 1.1. *Under the* Assumption 1 and 2,

(a) *The solutions of* (1.4) *have total variation bounded uniformly in ϵ.*

(b) *There exist solutions for the mixed hyperbolic-elliptic Riemann problem* (1.1).

In Section 4, we show that the solutions $(u(\xi), v(\xi))$ jump over the spinodal region. In Section 5, we prove that solutions lie on a continuous curve in the (u, v)-plane. The solutions of (1.1) consist of the constant states separated by shocks contact discontinuities, rarefaction waves, and phase boundaries.

2. Preliminaries

In this section we recall some results from [5]. The following lemma is from Lemma 2.1 of [5]:

Lemma 2.1. *Assume that f and g satisfy the* Assumption 1 and 2. *Let $(u_\epsilon(\xi), v_\epsilon(\xi))$ be the solution of* (1.4). *Then one of the following holds on any subinterval (a, b) for which $g'(u_\epsilon(\xi)) > 0$.*

1. *Both $u_\epsilon(\xi)$ and $v_\epsilon(\xi)$ are monotone on (a, b).*
2. *One of the $u_\epsilon(\xi)$ and $v_\epsilon(\xi)$ is a strictly increasing or decreasing function with no critical point on (a, b) while the other has at most one critical point that is respectively maximum or minimum.*

The following lemma was originated by Slemrod [6] in case $f(v) = v$ and this lemma holds also in the general case.

Lemma 2.2. *Assume that f and g satisfy* Assumptions 1 and 2. *Let $(u_\epsilon(\xi), v_\epsilon(\xi))$ be the solution of* (1.4). *If $u_\epsilon(\xi) \notin (\alpha, \beta)$, then $u'_\epsilon(\xi) > 0$ and $v_\epsilon(\xi_\epsilon(u))$ is a convex function of $u \notin (\alpha, \beta)$, where $\xi_\epsilon(u)$ is the inverse function of $u_\epsilon(\xi)$ in the region $u \notin (\alpha, \beta)$.*

3. Uniform Boundedness of $(u_\epsilon(\xi), v_\epsilon(\xi))$

In this section, we shall prove $u_\epsilon(\xi)$ and $v_\epsilon(\xi)$ are uniformly bounded independently of ϵ.

THEOREM 3.1. *Under the* Assumption 1 and 2, *the* $v_\epsilon(\xi)$ *are bounded from above, uniformly in* ϵ*:*

$$v_\epsilon(\xi) \leq \max(v_+, v_-) + \max(u_+ - \beta, \alpha - u_-) \max_{u \in [u_-, \alpha] \cup [\beta, u_+]} \sqrt{\frac{g'(u)}{f'(\bar{v})}}. \tag{3.1}$$

where $\bar{v} = \min\{v_-, v_+\}$.

PROOF. We may assume that each $v_\epsilon(\xi)$ has a local maximum point $\xi = \theta_\epsilon$ with $u_\epsilon(\theta_\epsilon) \geq \beta$. From (1.4) and the chain rule, we have

$$\epsilon \frac{d}{d\xi}\left(\frac{dv_\epsilon}{du_\epsilon}(\xi)\right) = f'(v)\left(\frac{dv_\epsilon}{du_\epsilon}\right)^2 - g'(u) \tag{3.2}$$

This implies that as ξ increases, $\frac{dv_\epsilon}{du_\epsilon}(\xi)$is increasing(resp. decreasing) if $|\frac{dv_\epsilon}{du_\epsilon}(\xi)| \geq \sqrt{\frac{g'(u)}{f'(v)}}$ (resp. $|\frac{dv_\epsilon}{du_\epsilon}(\xi)| \leq \sqrt{\frac{g'(u)}{f'(v)}}$).

Thus the initial condition

$$\left.\frac{dv_\epsilon}{du_\epsilon}(\xi)\right|_{\xi=\theta_\epsilon} = 0 \tag{3.3}$$

leads to

$$\left|\frac{dv_\epsilon}{du_\epsilon}(\xi)\right| \leq \max_{\beta \leq u \leq u_+} \sqrt{\frac{g'(u)}{f'(v)}}$$

as long as $u_+ \geq u_\epsilon(\xi) \geq \beta$ and $\bar{v} = \min\{v_-, v_+\}$. By Lemma 2.1, $u_\epsilon(\xi)$ is increasing when $u_\epsilon(\xi) \geq \beta$. Thus (3.1) follows. □

THEOREM 3.2. *Under* assumptions 1 and 2, *the* $v_\epsilon(\xi)$ *are bounded from below, uniformly in* ϵ.

PROOF. Assume there exists a sequence $\{\epsilon_n\}$ such that each $v_{\epsilon_n}(\xi)$ has a local minimum point τ_n with

$$v_{\epsilon_n}(\tau_n) \to -\infty \quad \text{as } n \to \infty, \tag{3.4}$$

$$u_{\epsilon_n}(\tau_n) \in (\alpha, \beta).$$

We may assume that

$$\tau_n \geq 0 \quad \text{for } n = 1, 2, 3, \ldots \tag{3.5}$$

Let ζ_n be the maximum point of $v_{\epsilon_n}(\xi)$ in the region $u_{\epsilon_n}(\xi) \geq \beta$. By Lemma 2.1 we have

$$v_\epsilon'(\xi) > 0 \quad \text{for } \xi \in (\tau_n, \zeta_n). \tag{3.6}$$

By integrating (1.4) on (τ_n, θ) where $\theta \in (\tau_n, \zeta_n)$, we obtain

$$0 \leq \epsilon v_{\epsilon_n}'(\theta) = \int_{\tau_n}^{\theta} -\xi v_{\epsilon_n}'(\xi)\, d\xi - g(u_{\epsilon_n}(\theta)) + g(u_{\epsilon_n}(\tau_n)).$$

It follows from (3.5) and (3.5) that $-\xi v_{\epsilon_n}'(\xi) < 0$ for $\xi \in (\tau_n, \zeta_n)$. Thus we have

$$\begin{aligned} 0 \leq \epsilon v_{\epsilon_n}'(\theta) &\leq g(u_{\epsilon_n}(\tau_n)) - g(u_{\epsilon_n}(\theta)) \\ &\leq g(\alpha) - g(u_\epsilon(\theta)). \end{aligned} \tag{3.7}$$

Therefore

$$\beta \leq u_{\epsilon_n}(\theta) < \delta \quad \text{for} \quad \theta \in (\tau_n, \zeta_n]$$

Equation (3.7) also gives

$$0 \leq \epsilon v_\epsilon'(\theta) \leq g(\alpha) - g(\beta) \quad \text{for } \theta \in [\tau_n, \zeta_n]. \tag{3.8}$$

We claim that there exists an $\eta_n \in (\tau_n, \zeta_n]$ such that
(3.9)

$$v_{\epsilon_n}(\eta_n) \geq v_+ - \frac{2}{\sqrt{f'(\bar{v})}} \left((\delta - \gamma) \max_{u \in [\gamma, \delta]} \sqrt{|g'(u)|} - \frac{g(\alpha) - g(\beta)}{\max_{u \in [\gamma, \delta]} \sqrt{|g'(u)|}} \right)$$

and

$$\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\bigg|_{\xi=\eta_n} \geq \frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}. \tag{3.10}$$

From the assumption (3.4) we can choose $\xi_n \in (\tau_n, \zeta_n]$ such that

$$\begin{aligned} v_{\epsilon_n}(\xi_n) &= v_{\epsilon_n}(\zeta_n) - \frac{2(\delta-\gamma)\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}{\sqrt{f'(\bar{v})}} \\ &\leq v_+ - \frac{2(\delta-\gamma)\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}{\sqrt{f'(\bar{v})}}. \end{aligned} \tag{3.11}$$

For each n, there exists a $\theta \in [\xi_n, \zeta_n]$ such that $v_{\epsilon_n}(\theta) \geq v_{\epsilon_n}(\xi_n)$ and

$$\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\bigg|_{\xi=\theta} = \frac{u_{\epsilon_n}(\zeta_n) - u_{\epsilon_n}(\xi_n)}{v_{\epsilon_n}(\zeta_n) - v_{\epsilon_n}(\xi_n)}. \tag{3.12}$$

Substituting the denominator of (3.12) by (3.11) and noticing that $|u_{\epsilon_n}(\zeta_n) - u_{\epsilon_n}(\xi_n)| \leq \delta - \gamma$, we have

$$\left|\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\bigg|_{\xi=\theta}\right| \leq \frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}.$$

Thus the set A defined by

$$A = \left\{\eta \in [\tau_n, \theta]\,;\ \left|\frac{du}{dv}(\xi)\right| \leq \frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}, \quad \xi \in [\eta, \theta]\right\}.$$

is nonempty. Since

$$\frac{d^2u_{\epsilon_n}}{dv^2_{\epsilon_n}}(\xi) = \frac{1}{\epsilon v'_{\epsilon_n}}(\xi)\left(-f'(v_{\epsilon_n}) + g'(u_{\epsilon_n})\left(\frac{du}{dv}\right)^2\right), \tag{3.13}$$

we have

$$\left|\frac{d^2u_{\epsilon_n}}{dv^2_{\epsilon_n}}(\xi)\right| \geq \frac{1}{2\epsilon v'_{\epsilon_n}(\xi)} \geq \frac{1}{2(g(\alpha) - g(\beta))}$$

if $\xi \in A$. We must show that (3.9) and (3.10) hold at $\eta_n = \inf A$. Indeed, by the definition of the set A,

$$\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\Big|_{\xi=\inf A} = \frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\begin{aligned}
\frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}} &= \frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\Big|_{\xi=\inf A} \\
&= \frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\Big|_{\xi=\theta} + \int_{v(\theta)}^{v(\inf A)} \frac{d^2u_{\epsilon_n}}{dv_{\epsilon_n}^2}(\xi)\,dv_\epsilon \\
&\geq -\frac{\sqrt{f'(\bar{v})}}{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}} + \frac{|v_{\epsilon_n}(\theta) - v_{\epsilon_n}(\inf A)|}{2(g(\alpha)-g(\beta))}.
\end{aligned}$$

By (3.6), the above inequality implies

$$\begin{aligned}
v_{\epsilon_n}(\inf A) &\geq v_{\epsilon_n}(\theta) - \frac{2\sqrt{f'(\bar{v})}}{\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}(g(\alpha)-g(\beta)) \\
&\geq v_\epsilon(\xi_n) - \frac{2\sqrt{f'(\bar{v})}}{\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}(g(\alpha)-g(\beta)) \\
&\geq v_+ - \frac{2(\delta-\gamma)\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}{\sqrt{f'(\bar{v})}} \\
&\quad - \frac{2\sqrt{f'(\bar{v})}}{\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}(g(\alpha)-g(\beta)).
\end{aligned} \tag{3.16}$$

Similar calculation as (3.13) yields that

$$\frac{dv_{\epsilon_n}}{du_{\epsilon_n}}(\xi) \geq \frac{2\max_{u\in[\gamma,\delta]}\sqrt{|g'(u)|}}{\sqrt{f'(\bar{v})}} \qquad \text{for } \xi \in [\tau_n, \eta_n]$$

Thus the equation

$$v_{\epsilon_n}(\eta_n) - v_{\epsilon_n}(\tau_n) = \int_{u_{\epsilon_n}(\tau_n)}^{v_{\epsilon_n}(\eta_n)} \frac{dv_{\epsilon_n}}{du_{\epsilon_n}}(\xi)\,du_\epsilon$$

implies

$$
\begin{aligned}
v_{\epsilon_n}(\tau_n) &\geq v_{\epsilon_n}(\eta_n) - \frac{2(\delta-\gamma)}{\sqrt{f'(\bar{v})}} \max_{u\in[\gamma,\delta]} \sqrt{|g'(u)|} \\
&\geq v_+ - \frac{4(\delta-\gamma)}{\sqrt{f'(\bar{v})}} \max_{u\in[\gamma,\delta]} \sqrt{|g'(u)|} \\
&\quad - \frac{2\sqrt{f'(\bar{v})}}{\max_{u\in[\gamma,\delta]} \sqrt{|g'(u)|}} (g(\alpha) - g(\beta)),
\end{aligned}
$$

which is a contradiction to (3.4). □

We will denote by

$$
\begin{aligned}
u^* &= \sup\{u_\epsilon(\xi) \,|\, \xi \in \mathbb{R}, 0 < \epsilon < 1\} \\
u_* &= \inf\{u_\epsilon(\xi) \,|\, \xi \in \mathbb{R}, 0 < \epsilon < 1\}.
\end{aligned}
$$

We also define similarly for v.

THEOREM 3.3. *If f and g satisfy* Assumptions 1 and 2, *then $u_\epsilon(\xi)$ are bounded uniformly in ϵ.*

PROOF. We only prove that the $u_\epsilon(\xi)$ are bounded from below uniformly in ϵ. The uniform boundedness of $u_\epsilon(\xi)$ from above can be proved similarly. Assume the contrary. Then there is a sequence $\{\epsilon_n\}$ such that each $u_{\epsilon_n}(\xi)$ has a local minimum point at $\xi = \tau_n$ with

$$
u_{\epsilon_n}(\tau_n) \to -\infty \quad \text{as} \quad n \to \infty. \tag{3.11}
$$

We may assume that

$$
\tau_n \leq 0. \tag{3.12}
$$

From Lemma , $u_\epsilon(\xi)$ and $v_\epsilon(\xi)$ are decreasing on $(-\infty, \tau_n)$. By integrating (1.4) we obtain

$$
0 \leq -\epsilon u'_{\epsilon_n}(-\infty) = -\int_{-\infty}^{\tau_n} \xi u'_{\epsilon_n}(\xi)\, d\xi - f(v_{\epsilon_n}(\tau_n)) + f(v_-).
$$

From (3.12) it follows that $\xi u'_{\epsilon_n}(\xi) \leq 0$ on $(-\infty, \tau_n)$ and hence

$$(3.13)\qquad 0 \leq \int_{-\infty}^{\tau_n} \xi u'_{\epsilon_n}(\xi)\, d\xi \leq f(v_-) - f(v_{\epsilon_n}(\tau_n)) \leq f(v_-) - f(v_*).$$

For any $\theta \leq \min\{-1, \tau_n\}$, we have

$$\int_{-\infty}^{\theta} \xi u'_\epsilon(\xi)\, d\xi \geq -\int_{-\infty}^{\theta} u'_{\epsilon_n}(\xi)\, d\xi = u_- - u_{\epsilon_n}(\theta).$$

Thus (3.13) gives

$$(3.14)\qquad u_{\epsilon_n}(\theta) \geq u_- + f(v_*) - f(v_-).$$

It remain to consider the case that $-1 < \tau_n < 0$. Then for each n we can choose a $\theta \in (-2, -1)$ such that $v'_{\epsilon_n}(\theta) \leq f(v^*) - f(v_*)$. By integrating (1.4) on $[\theta, \tau_n]$, we obtain

$$(3.15)\qquad \begin{aligned} g(u_{\epsilon_n}(\tau_n)) &= -\epsilon v'_\epsilon(\tau_n) + \epsilon v'_{\epsilon_n}(\theta) + g(u_{\epsilon_n}(\theta)) - \int_\theta^{\tau_n} \xi v'_{\epsilon_n}(\xi)\, d\xi \\ &\geq \epsilon v'_{\epsilon_n}(\theta) + g(u_{\epsilon_n}(\theta)) - \int_\theta^{\tau_n} \xi v'_{\epsilon_n}(\xi)\, d\xi \end{aligned}$$

In view of (3.14) and the uniform boundedness of $v_\epsilon(\xi)$, it follows easily from (3.15) that the right-hand side of (3.15) is bounded uniformly in ϵ. Thus, by virtue of assumption 1, the $u_{\epsilon_n}(\tau_n)$ are bounded from below uniformly in ϵ, in contradiction to (3.11). □

THEOREM 3.4. *Under the* assumption 1 and 2, *there exist solutions of* (1.1).

PROOF. Since $(u_\epsilon(\xi), v_\epsilon(\xi))$ has total variation bounded independent of ϵ, Helley's theorem shows that $(u_\epsilon(\xi), v_\epsilon(\xi))$ possesses a subsequence converging almost everywhere on $(-\infty, \infty)$ to a function $(u(\xi), v(\xi))$ of bounded variation. Thus $(u(\frac{x}{t}), v(\frac{x}{t}))$ is a weak solution of (1.1). □

4. The shock jumps over the spinodal region

In this section we assume that $(u_\epsilon(\xi), v_\epsilon(\xi))$ is bounded uniformly in ϵ. We shall prove that $u(\xi)$ jumps over the spinodal region (α, β) and also establish some useful lemmas for later use.

Consider $u_{\epsilon_n}(\xi)$ (or $v_{\epsilon_n}(\xi)$) as a multivalued function of v_{ϵ_n} (or u_{ϵ_n}). Denote these functions by $U_{\epsilon_n}(v)$(or $V_{\epsilon_n}(u)$).

LEMMA 4.1. (a) $\{\frac{dv}{du}(\xi)|u_\epsilon(\xi) \in F\}$ *is uniformly bounded in* ϵ_n *on any compact subset* F *of* (u_-, u_+).
(b) *If*

$$\left\{\frac{dv}{du}(\xi) \mid \xi \in \mathbb{R} \text{ such that } u_\epsilon(\xi) < \alpha\right\}$$

$$\left(\left\{\frac{dv}{du}(\xi) \mid \xi \in \mathbb{R} \text{ such that } u_\epsilon(\xi) > \beta\right\}\right)$$

is not bounded uniformly in ϵ_n, *then there is a subsequence of* $\{\epsilon_n\}$, *again denoted by* $\{\epsilon_n\}$, *such that*

$$\left\{\frac{du}{dv}(\xi) \mid u_{\epsilon_n}(\xi) \in F\right\}$$

is bounded independent of ϵ_n *on any compact subset* F *of* $(-\infty, \alpha)$*(or* (β, ∞)*).*
(c) $\{V_{\epsilon_n}(u)\}$ *has a subsequence which converges to a continuous curve. Furthermore,* $(u(\xi), v(\xi))$ *lies on this curve for every* $\xi \in \mathbb{R}$.

PROOF. (a) Without loss of generality, we can assume that F is closed interval $[u_- + \delta, u_+ - \delta]$ for some small $\delta > 0$. We assert that

$$\left|\frac{dv}{du}(\xi)\bigg|_{u(\xi)\in F}\right| \le \max\left\{\max_{u\in[u_*,u^*]}\sqrt{\frac{|g'(u)|+1}{|f'(v)|}}, \frac{2}{\delta}(v^* - v_* + 1)\right\} \tag{4.1}$$

To prove this assertion, we assume that (4.1) does not hold at some point $\xi = \xi_\epsilon \in \mathbb{R}$ such that $u_\epsilon(\xi_\epsilon) \in F$. Without loss of generality, we assume that $\frac{dv_\epsilon}{du_\epsilon}(\xi)|_{\xi=\xi_\epsilon} < 0$. Then

$$\epsilon\frac{d}{d\xi}\left(\frac{dv}{du}(\xi)\right) = f'(v)\left(\frac{dv}{du}(\xi)\right)^2 - g'(u(\xi)) \tag{4.2}$$

implies that $\frac{dv}{du}(\xi)$ is decreasing as ξ decreases from ξ_ϵ until it reaches the minimum point τ_ϵ of $u_\epsilon(\xi)$. Since $u_\epsilon(\tau_\epsilon) \le u_-$, it follows that

$$\frac{dv_\epsilon}{du_\epsilon}(\xi) \le \left.\frac{dv_\epsilon}{du_\epsilon}(\xi)\right|_{\xi=\xi_\epsilon} \tag{4.3}$$

if $u_\epsilon(\xi) \in [u_- + \frac{\delta}{2}, u_\epsilon(\xi)]$. Thus

$$\begin{aligned} v_\epsilon(\xi_\epsilon) - V_\epsilon(u_- + \frac{\delta}{2}) &= \int_{u_-+\frac{\delta}{2}}^{u_\epsilon(\xi_\epsilon)} \frac{dv_\epsilon}{du_\epsilon}(\xi)\, du \\ &\le -\frac{2}{\delta}(v^* - v_* + 1)\left(u_\epsilon(\xi_\epsilon) - u_- - \frac{\delta}{2}\right). \end{aligned} \tag{4.4}$$

Since $u_\epsilon(\xi) \in F$, we have

$$\begin{aligned} v_* - v^* &\le v_\epsilon(\xi_\epsilon) - V_\epsilon\left(u_- + \frac{\delta}{2}\right) \\ &\le -\frac{2}{\delta}(v^* - v_* + 1)\left(u_\epsilon(\xi) - u_- - \frac{\delta}{2}\right) \\ &\le -(v^* - v_* + 1) \end{aligned}$$

which is impossible.

(b) Since the $u_\epsilon(\xi)$ are bounded uniformly in ϵ, it follows that

$$\sqrt{\frac{g'(u_\epsilon(\xi))}{f'(v_\epsilon(\xi))}} \le C_1$$

for some constant $C_1 > 0$. If

$$\left\{\frac{dv_\epsilon}{du_\epsilon}(\xi) \mid \xi \in \mathbb{R} \text{ such that } u_\epsilon(\xi) < \alpha\right\}$$

is not bounded uniformly in ϵ, there are subsequences $\{\epsilon_n\}$ and $\{\xi_n\}$ such that $u_{\epsilon_n}(\xi_n) < \alpha$ and

$$\left|\left.\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi)\right|_{\xi=\xi_n}\right| < \frac{1}{C_1}. \tag{4.5}$$

From (1.4), we derive that

$$\epsilon\frac{d}{d\xi}\left(\frac{du_\epsilon}{dv_\epsilon}(\xi)\right)=g'(u_\epsilon)\left(\frac{du_\epsilon}{dv_\epsilon}\right)^2-f'(v_\epsilon). \tag{4.6}$$

From (4.5) and (4.6), we conclude that

$$\left|\frac{du_\epsilon}{dv_\epsilon}(\xi)\Big|_{u_{\epsilon_n}(\xi)}\in F\right|\le\max_{u\in F}\sqrt{\frac{g'(u)}{f'(\bar{v})}}<\infty$$

for any compact subset of $(-\infty,\alpha)$.

(c) For simplicity, we assume that $\{\frac{dv_\epsilon}{du_\epsilon}(\xi)\}$ is not uniformly bounded only when $u_{\epsilon_n}(\xi)<\alpha$. The proofs in other cases are similar. Then, by (a) and (b), we can extract a subsequence of $\{\epsilon_n\}$, denoted again by $\{\epsilon_n\}$ such that, as $n\to\infty$, $V_{\epsilon_n}(u)\to V(u)$ for $u\in[\frac{1}{2}(u_-+\alpha),u_+]$ and $U^{(1)}_{\epsilon_n}(v)\to U(v)$ for

$$v\in\{v\in\mathbb{R}\,|\,v=v_{\epsilon_n}(\xi),u_{\epsilon_n}(\xi)\le\frac{1}{4}(u_-+3\alpha)\ \text{for large } n \text{ and for some } \xi\in\mathbb{R}\},$$

where $U^{(1)}_{\epsilon_n}(v)$ is the branch of $U(v)$ with $U_{\epsilon_n}(v)\le\alpha$. $u=U(v)$ and $v=V(u)$ coincide when $u\in[\frac{1}{2}(u_-+\alpha),\frac{1}{4}(u_-+3\alpha)]$, since both $\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}$ and $\frac{dv_{\epsilon_n}}{du_{\epsilon_n}}$ are bounded uniformly in ϵ there. There

$$(u,v)=\begin{cases}(U(v),v)\ \text{for } u=U(v)\le\frac{1}{2}(u_-+\alpha)\\ (u,V(u))\ \text{for } u\ge\frac{1}{2}(u_-+\alpha)\end{cases}$$

is the desired curve. That $(u(\xi),v(\xi))$ lies on the curve follows from (b). □

For the convenience of notations, we parameterized the curve $v=V(u)$ by $(U(s),V(s))$ where s is the length of the arc of $v=V(u)$ joining (u_-,v_-) and the point $(U(s),V(s))$. Since the curve $v=V(u)$ does not intersect itself, the parameterization is bijective. In this kind of parameterization, s increases when ξ increases. We call the curve $(U(s), V(s))$ the base curve of the solution $(u(\xi), v(\xi))$.

Now we study the discontinuities of $(u(\xi), v(\xi))$. Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. Denote C_{ξ_0} by the portion of the base curve in the (u, v)-plane that connects points $(u(\xi_0-), v(\xi_0-))$ and $(u(\xi_0+), v(\xi_0+))$. We fix $(\bar{u}, \bar{v}) \in C_{\xi_0}$. For n large, we define $\xi_{\epsilon_n}(u; \bar{u}, \bar{v})$ to be the branch of the inverse function of $u = u_{\epsilon_n}(\xi)$ for which

$$v_{\epsilon_n}(\xi_{\epsilon_n}(\bar{u}; \bar{u}, \bar{v})) \to \bar{v} \tag{4.6}$$

as $n \to \infty$. For n large, we define ξ_{ϵ_n}, $\hat{u}_{\epsilon_n}$, $\hat{v}_{\epsilon_n}$ by the relations

$$\xi_{\epsilon_n} = \xi_{\epsilon_n}(\bar{u}) + \epsilon\zeta, \tag{4.7}$$

$$\hat{v}_{\epsilon_n}(\zeta) = v_{\epsilon_n}(\xi_n), \tag{4.8}$$

$$\hat{u}_{\epsilon_n}(\zeta) = u_{\epsilon_n}(\xi_n), \tag{4.9}$$

LEMMA 4.2. *Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. For $(\hat{u}_{\epsilon_n}(\zeta), \hat{v}_{\epsilon_n}(\zeta))$ defined above, there is a subsequence of $\{\epsilon_n\}$, also denoted by $\{\epsilon_n\}$, such that $(\hat{u}_{\epsilon_n}(\zeta), \hat{v}_{\epsilon_n}(\zeta)) \to (\hat{u}(\zeta), \hat{v}(\zeta)) \in C^1(\mathbb{R}; \mathbb{R}^2)$ as $n \to \infty$ uniformly for ζ in a compact subset of $\mathbb{R}$. $(\hat{u}(\zeta), \hat{v}(\zeta))$ satisfies the following initial value problem:*

$$\frac{d\hat{u}(\zeta)}{d\zeta} = -\xi_0(\hat{u}(\zeta) - u(\xi_0\mp)) - (f(\hat{v}(\zeta)) - f(v(\xi_0\mp))) \tag{4.11a}$$

$$\frac{d\hat{v}(\zeta)}{d\zeta} = -\xi_0(\hat{v}(\zeta) - v(\xi_0\mp)) - (g(\hat{u}(\zeta)) - g(u(\xi_0\mp))) \tag{4.11b}$$

$$\hat{v}(0) = \bar{v}, \hat{u}(0) = \bar{u}. \tag{4.11c}$$

Furthermore, $(\hat{u}(\xi), \hat{v}(\xi))$ lies on C_{ξ_0}.

PROOF. Clearly, the $(\hat{u}_{\epsilon_n}(\zeta), \hat{v}_{\epsilon_n}(\zeta))$ have uniformly bounded total variation since the $(u_\epsilon(\xi), v_\epsilon(\xi))$ do. Thus, there is a subsequence of $\{\epsilon_n\}$, again denoted by $\{\epsilon_n\}$, such that

$$(\hat{u}_{\epsilon_n}(\zeta), \hat{v}_\epsilon(\zeta)) \to (\hat{u}(\zeta), \hat{v}(\zeta)) \qquad \text{as } n \to \infty \tag{4.12}$$

for any $\zeta \in \mathbb{R}$. By Lemma 4.1(b), we can choose a small neighborhood V_{ξ_0} of $(u(\xi_0-), v(\xi_0-))$ in the (u,v)-plane such that

$$\{\frac{du_{\epsilon_n}}{dv_{\epsilon_n}}(\xi) | \xi \in \mathbb{R} \text{ such that } (u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)) \in V_{\epsilon_0}\} \tag{4.13a}$$

$$\{\frac{dv_{\epsilon_n}}{du_{\epsilon_n}}(\xi) | \xi \in \mathbb{R} \text{ such that } (u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)) \in V_{\epsilon_0}\} \tag{4.13b}$$

bounded uniformly in n. Since $U(s)$ is composed of as many monotone pieces as $u_\epsilon(\xi)$, we can further choose V_{ξ_0} small and $(u_\delta, v_\delta) \in C_{\xi_0} \cap V_{\xi_0}$ such that $U(s)$ is monotone when $(U(s), V(s)) \in V_{\xi_0}$ runs from $(u(\xi_0-), v(\xi_0-))$ to (u_δ, v_δ), along C_{ξ_0}. For definiteness, we can assume without loss of generality that (4.13b) holds. For n large, there is a

$$\theta_{\epsilon_n} \in (\xi_{\epsilon_n}(u_\delta) - \sqrt{\epsilon_n}, \xi_{\epsilon_n}(u_\delta)) \tag{4.14}$$

such that

$$|v'_{\epsilon_n}(\theta_n)| \leq \frac{3}{\sqrt{\epsilon_n}} \operatorname{TV}(v_{\epsilon_n}) \leq \frac{3M}{\sqrt{\epsilon_n}},$$
$$|u'_{\epsilon_n}(\theta_n)| \leq \frac{3}{\sqrt{\epsilon_n}} \operatorname{TV}(u_{\epsilon_n}) \leq \frac{3M}{\sqrt{\epsilon_n}}.$$

From (3.9) and

$$(u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)) \to (u(\xi), v(\xi)),$$

it is easily seen that

$$\xi_{\epsilon_n}(u_\delta) \to \xi_0$$

and hence $\theta_{\epsilon_n} \to \xi_0$. Since $U(s)$ is monotone, so is $u_{\epsilon_n}(\xi)$. Hence $\liminf_{n\to\infty} u_{\epsilon_n}(\theta_n)$ lies between $u(\xi_0-)$ and u_δ. Thus, extracting, if necessary, another subsequence, we deduce that

$$u_{\epsilon_n}(\theta_n) \to u_2 \text{ as } n \to \infty \tag{4.17}$$

for some u_2 between $u(\xi_0-)$ and u_δ. Then we have that

$$\lim_{n\to\infty} v_{\epsilon_n}(u_{\epsilon_n}(\theta_n)) = V(u_2) = v_2 \tag{4.18}$$

where $(u_2, v_2) \in V_{\xi_0}$. For simplicity, we shall write ϵ instead of ϵ_n in the rest of this paper. Integrating (1.4) from θ_ϵ to $\tau_\epsilon = \xi_\epsilon(\bar{u}) + \epsilon\zeta$, we get

$$\begin{aligned}
\frac{d\hat{u}_\epsilon(\zeta)}{d\zeta} &= -\, \xi_0(\hat{u}_\epsilon(\zeta) - u_\epsilon(\theta_\epsilon)) - f(\hat{v}_\epsilon(\zeta)) \\
&\quad + f(v_\epsilon(\theta_\epsilon)) + \epsilon u'_\epsilon(\theta_\epsilon) - \int_{\theta_\epsilon}^{\tau_\epsilon} (\xi - \xi_0) u'_\epsilon(\xi)\, d\xi \\
\frac{d\hat{v}_\epsilon(\zeta)}{d\zeta} &= -\, \xi_0(\hat{v}_\epsilon(\zeta) - v_\epsilon(\theta_\epsilon)) - g(\hat{u}_\epsilon(\zeta)) \\
&\quad + g(u_\epsilon(\theta_\epsilon)) + \epsilon v'_\epsilon(\theta_\epsilon) - \int_{\theta_\epsilon}^{\tau_\epsilon} (\xi - \xi_0) v'_\epsilon(\xi)\, d\xi
\end{aligned} \tag{4.19}$$

By (4.15) $\epsilon u'_\epsilon(\theta_\epsilon)$ and $\epsilon v'_\epsilon(\theta_\epsilon)$ approach 0 as $\epsilon \to 0$ uniformly in ζ. Recalling that $\theta_\epsilon \to \xi_0$, $\tau_\epsilon \to \xi_0$ as $n \to \infty$, uniformly in ζ for ζ in compact subsets of $\mathbb{R}$, we see that the last terms in (4.19) vanish as $n \to \infty$, uniformly in ζ in a compact set. A classical theorem of the ordinary differential equations implies that $(\hat{u}_\epsilon(\zeta), \hat{v}_\epsilon(\zeta)) \to (\hat{u}(\zeta), \hat{v}(\zeta))$ as $n \to \infty$, uniformly on compact subsets of $\mathbb{R}$, and that

$$\begin{aligned}
&\frac{d\hat{u}}{d\zeta}(\zeta) = -\xi_0(\hat{u}(\zeta) - u_2) - f(\hat{v}(\zeta)) + f(v_2), \\
&\frac{d\hat{v}}{d\zeta}(\zeta) = -\xi_0(\hat{v}(\zeta) - u_2) - g(\hat{u}(\zeta)) + g(u_2), \\
&\hat{u}(0) = u_2, \quad \hat{v}_2(0) = v_2.
\end{aligned} \tag{4.20}$$

By letting V_{ξ_0} shrink to $(u(\xi_0-), v(\xi_0-))$ so as to force $(u_2, v_2) \to (u(\xi_0-), v(\xi_0-))$, we obtain (3.6) and (3.7). The last assertion of the lemma is an immediate consequence of Lemma 4.1(b) and the uniqueness of (4.11).□

LEMMA 4.3. *Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. Then for any $(\bar{u}, \bar{v}) \in C_{\xi_0}$ it follows that*

(a) *if $U(s)$ is increasing (decreasing) at $(\bar{u},\bar{v})$,then*

$$-\xi_0(\bar{u}-u(\xi_0-))-(f(\bar{v})-f(v(\xi_0-)))\geq 0 \quad (\leq 0).$$

(b) *if $V(s)$ is increasing (decreasing) at $(\bar{u},\bar{v})$,then*

$$-\xi_0(\bar{v}-v(\xi_0-))-(g(\bar{u})-g(u(\xi_0-)))\geq 0 \quad (\leq 0).$$

Moreover we can change all ξ_0- to ξ_0+.

THEOREM 4.4. *$u(\xi)$ takes no value in (α,β) and may take at most one of α and β as a value.*

PROOF. From Theorem 1.1, $u(\xi)$ is increasing when $u(\xi)\in(\alpha,\beta)$. Assume that there is a $\xi_0\in\mathbb{R}$ such that $u(\xi_0-)\in(\alpha,\beta)$. We assert that $u(\xi)=u(\xi_0-)$ for $\xi\in(\xi_0-\delta,\xi_0)$ for some $\delta>0$. Indeed, if not, there are two possibilities:

Case 1: There is a sequence $\{\xi_n\}$ of points of discontinuity of $(u(\xi), v(\xi))$ such that $\xi_n\to\xi_0-$ as $n\to\infty$.

Case 2: $(u(\xi),v(\xi))$ is continuous in $(\xi_0-\delta,\xi_0)$ for some $\delta>0$ and there is a sequence $\{\xi_n\}\subset(\xi_0-\delta,\xi_0)$ such that $\xi_n\to\xi_0-$ as $n\to\infty$ and $u(\xi_n+)\neq u(\xi_0-)$.

Case 1 is impossible because $u(\xi_n\pm)\in(\alpha,\beta)$ for large n and the Rankine-Hugoniot conditions

$$-\xi_n(u(\xi_n+)-u(\xi_n-))-(f(v(x_n+)-f(v(x_n-))=0$$
$$-\xi_n(v(\xi_n+)-v(\xi_n-))-(g(u(x_n+)-g(u(x_n-))=0$$

cannot hold. We assert that Case 2 is also impossible. Indeed, we can integrate (1.5) from ξ_n to ξ_0, to get

$$\xi_0\frac{\Delta_n u}{\Delta_n v}=-f'(\theta)-\frac{1}{\Delta_n v}\int_{\xi_n}^{\xi_0}(\xi-\xi_0)u'(\xi)\,d\xi$$
$$\xi_0\frac{\Delta_n v}{\Delta_n u}=-g'(\tau)-\frac{1}{\Delta_n u}\int_{\xi_n}^{\xi_0}(\xi-\xi_0)v'(\xi)\,d\xi$$

where $\Delta_n u = u(\xi_0-) - u(\xi_n+) > 0$, $\Delta_n v = v(\xi_0-) - v(\xi_n+) > 0$, and $\tau \in (u(\xi_n+), u(\xi_0-))$ and θ lies in between $v(\xi_n+)$ and $v(\xi_0-)$. It follows from Lemma 4.1 that

$$\xi_0 \lim_{n\to\infty} \frac{\Delta_n u}{\Delta_n v} = -f'(v(\xi_0-))$$

$$\xi_0 \lim_{n\to\infty} \frac{\Delta_n v}{\Delta_n u} = -g'(u(\xi_0-)).$$

Then we have

$$\left(\lim_{n\to\infty} \frac{\Delta_n v}{\Delta_n u}\right)^2 = \frac{g'(u(\xi_0-))}{f'(v(\xi_0-))} < 0,$$

which is a contradiction. It follows that there exists a ξ_1 which is a point of discontinuity of $(u(\xi), v(\xi))$ such that $\xi_1 < \xi_0$ and $u(\xi_1+) = u(\xi_0-)$. Since $u'(\xi) > 0$ whenever $u(\xi) \in (\alpha, \beta)$, Lemma 4.3 and Rankine-Hugoniot condition shows that

$$-\xi_1(\bar{u} - u(\xi_1+)) - (f(\bar{v}) - f(v(\xi_1+)) \geq 0 \tag{4.21}$$

for any $(\bar{u}, \bar{v}) \in C_{\xi_1}$ with $\bar{u} \in (\alpha, \beta)$. We assume that (4.21) holds as a strictly inequality. If not, then from (4.20)

$$\left.\frac{d\hat{u}}{d\zeta}(\zeta)\right|_{\zeta=0} = 0.$$

Since $\hat{u}(\zeta)$ has no local extremum in (α, β), it follows that

$$\left.\frac{d^2\hat{u}}{d\zeta^2}(\zeta)\right|_{\zeta=0} = 0.$$

Thus, by differentiating (4.11a) with respect to ζ, we have

$$\left.\frac{d\hat{v}}{d\zeta}(\zeta)\right|_{\zeta=0} = 0.$$

Combining

$$\left.\frac{d\hat{u}}{d\zeta}(\zeta)\right|_{\zeta=0} = \left.\frac{d\hat{v}}{d\zeta}(\zeta)\right|_{\zeta=0} = 0 \tag{4.22}$$

and (4.11), we obtain

$$\bar{u} - u(\xi_1+) = f'(\theta)(g(\bar{u}) - g(u(\xi_1+)))$$

where θ lies in between $\bar{v}$ and $v(\xi_1+)$. This is impossible since $g'(u) < 0$ for $u \in (\alpha, \beta)$. Thus Lemma 4.2 leads to

$$\left.\frac{dV(u)}{du}\right|_{u=\bar{u}} = \frac{\xi_1(\bar{v} - v(\xi_1+)) + g(\bar{u}) - g(u(\xi_1+))}{\xi_1(\bar{u} - u(\xi_1+)) + f(\bar{v}) - f(v(\xi_1+))},$$

or equivalently,

$$\left(f'(\theta)\frac{\Delta v}{\Delta u} + \xi_1\right)\frac{dV}{du} = g'(\tau) + \xi_1\frac{\Delta v}{\Delta u}, \tag{4.23}$$

where $\Delta u = \bar{u} - u(\xi_1+)$, $\Delta v = \bar{v} - v(\xi_1+)$, θ lies in between $\bar{v}$ and $v(\xi_1+)$, and $\tau \in (\bar{u}, u(\xi_1+))$. Note that $V(u)$ is convex for $u \in (\alpha, \beta)$, we infer that, as $\bar{u} \to u(\xi_1+)-$,

$$\left.\frac{dV(u)}{du}\right|_{u=\bar{u}} \to \left.\frac{dV(u)}{du}\right|_{u=u(\xi_1+)-}, \quad \frac{\Delta v}{\Delta u} \to \left.\frac{dV(u)}{du}\right|_{u=u(\xi_1+)-}$$

Therefore, (4.23) implies that

$$f'(v(\xi_1+)-)\left(\frac{dV(u)}{du}|_{u=u(\xi_1+)-}\right)^2 = g'(u(\xi_1+)) < 0,$$

which is a contradiction. Thus $u(\xi_0-) \notin (\alpha, \beta)$ for any $\xi_0 \in \mathbb{R}$. Similarly we can show that $u(\xi+) \notin (\alpha, \beta)$ for any $\xi \in \mathbb{R}$. The last part of our assertion follows easily from the Rankine-Hugoniot condition. □

5. The Structure of the Solution

Since the base curve $(U(s), V(s))$ is oriented in the direction in which s increase, we can talk about the right and left sides of $(U(s_0), V(s_0))$

for the portions of the curve with $s < s_0$ and $s > s_0$ respectively. We define

(5.1a)

$$S(U(s),V(s)) = \begin{cases} 1 & \text{if both } U(s) \text{ and } V(s) \text{ are strictly increasing} \\ & \text{or strictly decreasing at } s, \\ -1 & \text{if both } U(s) \text{ and } -V(s) \text{ are strictly increasing} \\ & \text{or strictly decreasing at } s, \\ 0 & \text{otherwise} \end{cases}$$

(5.1b)

$$S(U(s_0),V(s_0);+) = \lim_{s\to s_0+} S(U(s),V(s))$$
$$S(U(s_0),V(s_0);-) = \lim_{s\to s_0-} S(U(s),V(s))$$

THEOREM 5.1. *Let ξ_0 be a point of discontinuity of $(u(\xi),v(\xi))$. Then*

(5.2)

$$\begin{aligned} &S(u(\xi_0-),v(\xi_0-);+)\sqrt{-f'(v(\xi_0-))g(u(\xi_0-))} \\ &\qquad \geq \xi_0 \\ &\qquad \geq S(u(\xi_0+),v(\xi_0+);-)\sqrt{-f'(v(\xi_0+))g(u(\xi_0+))}. \end{aligned}$$

THEOREM 5.2. (a) *If $u(\xi)$ or $v(\xi)$ is strictly monotone from the left at $\xi_0 \in \mathbb{R}$, then*

$$\xi_0 = S(u(\xi_0-),v(\xi_0-);-)\sqrt{-f'(v(\xi_0-))g(u(\xi_0-))}. \tag{5.3a}$$

(b) *If $u(\xi)$ or $v(\xi)$ is strictly monotone from the right at $\xi_0 \in \mathbb{R}$, then*

$$\xi_0 = S(u(\xi_0+),v(\xi_0+);+)\sqrt{-f'(v(\xi_0+))g(u(\xi_0+))}. \tag{5.3b}$$

LEMMA 5.3. *In the region $u \leq \alpha$ (or $u \geq \beta$) of the (u,v)-plane, the number of extrema for $U(s)$ and $V(s)$ is at most one.*

PROOF. We only consider the case when $U(s)$ has at least one local minimum in the region $u \leq \alpha$. The proof for other cases is similar. In this case, $V(s)$ has no local extremum since otherwise Lemma would be violated. Suppose that $U(v)$ has two local extrema at $v = v_1, v_2$ with $v_1 < v_2$. Then

$$U(v) = U(v_1) \text{ for } v \in [v_1, v_2] \tag{5.4}$$

because otherwise $u_{\epsilon_n}(\xi)$ would have at least two local extrema for n large. We assert that the curve $U(v), v \in (v_1, v_2)$ must have some common parts with C_{ξ_0} for some $\xi_0 \in \mathbb{R}$. Indeed, otherwise, there would be a point of continuity $\xi = \xi_1$ of $(u(\xi), v(\xi))$ such that

$$v(\xi_1) \in (v_1, v_2), \quad u(\xi_1) = U(v). \tag{5.5}$$

Thus $u(\xi) = U(v_1)$ in some neighborhood W of ξ_1 while $v(\xi)$ is not constant there. This is impossible by (1.5). Now we can choose

$$(\bar{u}, \bar{v}) \in C_{\xi_0} \cap \{(U(v_1), v_1) \mid v \in (v_1, v_2)\}.$$

Let $(\hat{u}(\zeta), \hat{v}(\zeta))$ be the solution of (4.11) with $(\bar{u}, \bar{v})$ as in (5.5). Clearly, $\frac{d\hat{u}}{d\zeta}(\zeta) = 0$ for $\zeta \in (-\delta, \delta)$ for some $\delta > 0$. The same argument as (4.22) implies

$$-\xi_0(\hat{u}(\zeta) - u(\xi_0-)) - (f(\hat{v}(\zeta)) - f(v(\xi_0-)) = 0,$$
$$-\xi_0(\hat{v}(\zeta) - v(\xi_0-)) - (g(\hat{u}(\zeta)) - g(u(\xi_0-)) = 0,$$

for $\zeta \in (-\delta, \delta)$. Thus

$$f'(\bar{v}) \left(\frac{dU}{dv}(v)|_{v=\bar{v}} \right)^2 = g'(\bar{u}),$$

which contradicts (5.4). □

If $U(s)$ or $V(s)$ attains a local extremum at $s = s_\alpha (or s = s_\beta)$ in the region $u < \alpha$ (or $u > \beta$), we set

$$(u_\alpha, v_\alpha) = (U(s_\alpha), V(s_\alpha))$$
$$(\text{ or } (u_\beta, v_\beta) = (U(s_\beta), V(s_\beta)).)$$

(u_1, v_1) is called *a constant state* of $(u(\xi), v(\xi))$ if $(u(\xi), v(\xi))$ is constant in some interval of $\mathbb{R}$.

COROLLARY 5.4. *The solution* $(u(\xi), v(\xi))$ *has no constant state other than* (u_-, v_-), (u_+, v_+) *and possibly* (u_α, v_α) *and* (u_β, v_β).

Combining Theorems 5.1, 5.2 and Corollary 5.4, we have

COROLLARY 5.5. *Let* ξ_0 *be a point of discontinuity of* $(u(\xi), v(\xi))$. *If* $(u(\xi_0-), v(\xi_0-))$ *(or* $(u(\xi_0+), v(\xi_0+))$*) is differ from* (u_-, v_-), (u_+, v_+), (u_α, v_α), (u_β, v_β), *then* ξ_0 *is a contact discontinuity from the left (or right).*

COROLLARY 5.6. (a) *At least one of* $(u(0-), v(0-))$ *and* $(u(0+), v(0+))$ *is a constant state of* $(u(\xi), v(\xi))$. *Furthermore,* $\xi = 0$ *is either a point of continuity of* $(u(\xi), v(\xi))$ *or the phase boundary (at which the shock jumps from one phase to another).*
(b) *Besides the constant states and the phase boundary,* $(u(\xi), v(\xi))$ *consists of shocks and simple waves of the first kind for* $\xi < 0$ *and of the second kind for* $\xi > 0$.

PROOF. In view of Theorem 4.5, at most one of α and β is in the range of $u(\xi)$. Thus, only the following two cases can occur: (i) $u(0-) = u(0+) = \alpha$ or β.
(ii) $u(0-) \neq u(0+)$ and hence $g'(u(0-)) > 0$ or $g'(u(0+)) > 0$.

In case (i), without loss of generality, we can assume that $u(0\pm) = \alpha$. Since $u(\xi)$ is nondecreasing when $u_- < u(\xi) < u_+$ and $u(\xi) \notin (\alpha, \beta)$, $u(\xi) = u(0+) = \alpha$ in $(0, \delta)$ for some $\delta > 0$.

For case (ii), we assume for definiteness that $g'(u(0-)) > 0$. The proof for the other cases is similar. We assert that $(u(0-), v(0-))$ is a constant state of $(u(\xi), v(\xi))$. Indeed, otherwise, $u(\xi)$ would be strictly monotone from the left of $\xi = 0$. By virtue of Theorem 5.2, we have

$$0 = S(u(0-), v(0-); -)\sqrt{-f'(v(0-))g'(u(0-))},$$

which implies that

$$S(u(0-), v(0-); -) = 0. \tag{5.6}$$

In view of Lemma 5.3, equation (5.6) cannot hold. Thus, $(u(0-), v(0-))$ is a constant state of $(u(\xi), v(\xi))$. Suppose that $\xi = 0$ is a point of discontinuity of $(u(\xi), v(\xi))$. We may assume that $u(0-) < \alpha$. If $u(0+) \leq \alpha$, then either $u(0+) = \alpha$ or $g'(u(0+)) > 0$. We assert that $u(0+) \neq \alpha$. Otherwise, the same arguments used in case (i) above would yield that $(u(\xi), v(\xi)) = (u(0+), v(0+)) = (\alpha, v(0+))$ in $(0, \delta)$ for some $\delta > 0$. Let δ_1 to be the maximum of such δ. Then $u(\delta_1 -) = u(0+) = \alpha$ and $u(\delta_1) > \alpha$. Applying Theorem 5.1, we have

$$\delta_1 \leq S(u(\delta_1), v(\delta_1); +)\sqrt{-f'(v(0+))g'(u(0+))} = 0,$$

which is impossible. On the other hand, the first part of this proof shows that if $g'(u(0+)) > 0$ and $g'(u(0-)) > 0$, then $(u(0+), v(0+))$ and $(u(0-), v(0-))$ are constant states of $(u(\xi), v(\xi))$. Therefore, $(u(0-), v(0-)) = (u_-, v_-)$ and $(u(0+), v(0+)) = (u_\alpha, v_\alpha)$. Applying the inequality (5.2) to the shock $\xi = 0$, we again obtain a contradiction. Thus $u(0+) > \alpha$ and hence $u(0+) \geq \beta$.

(b) At a point of continuity of $(u(\xi), v(\xi))$ which is not the phase boundary, i.e., both $u(\xi_0 -)$ and $u(\xi_0 +)$ are $\leq \alpha$ or $\geq \beta$. We assume that $u(\xi_0 \pm) \leq \alpha$. The proof of the other case is similar. We assume the contrary of our assertion about ξ_0, i.e., in the inequality (5.2),

$$\begin{aligned} &S(u(\xi_0 -), v(\xi_0 -); +) = -1, \\ &S(u(\xi_0 +), v(\xi_0 +); -) = 1. \end{aligned} \tag{5.7}$$

We can see from the definition (5.1) that (5.7) implies that while $U(s) < \alpha$, either $V(s)$ is increasing and $U(s)$ has a minimum or there are at least two extrema for $U(s)$ and $V(s)$. This is impossible by the property of $(U(s), V(s))$. □

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Department of Mathematics
College of Natural Sciences
Hoseo University
Asan, ChoongNam 336-795, Korea
E-mail: chlee@math.hoseo.ac.kr