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ISOMORPHISMS OF $\mathcal{B}_{2n}^{(n)}$

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ABSTRACT. In this paper, we will investigate certain two types of isomorphisms of $\mathcal{B}_{2n}^{(n)}$ which are closely related.

1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was only begun by W. B. Arveson [1] in 1974. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the sequence $\{\mathcal{B}_2, \mathcal{B}_4, \mathcal{B}_6, \cdots, \mathcal{B}_{2n}, \cdots\}$ which is defined in [5]. These algebras possess many surprising properties related to isomorphisms and cohomology. Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{f_1, f_2, \cdots, f_{2n}\}$. Then a member of $\mathcal{B}_{2n}$ on $\mathcal{H}$ has the form

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \ddots & & \\ & & & & & \ddots & * \\ & & & & & & * \end{pmatrix}$$

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with respect to the basis $\{f_1, f_2, \cdots, f_{2n}\}$, where all non-starred entries are zero. If we write the given basis in the order $\{f_1, f_3, f_5, \cdots, f_{2n-1}, f_2, f_4, \cdots, f_{2n}\}$, then the above matrix looks like this

$$\begin{pmatrix} * & & & & & & * & & & & & \\ & * & & & & & * & * & & & & \\ & & \cdot & & & & & * & & & & \\ & & & \cdot & & & & & \cdot & & & \\ & & & & \cdot & & & & & \cdot & & \\ & & & & & * & & & & & * & * \\ & & & & & & * & & & & & \\ & & & & & & & \cdot & & & & \\ & & & & & & & & \cdot & & & \\ & & & & & & & & & \cdot & & \\ & & & & & & & & & & \cdot & \\ & & & & & & & & & & & * \end{pmatrix},$$

where all non-starred entries are zero.

Let $\mathcal{B}_{2n}^{(n)} = \{ \begin{pmatrix} D_1 & S \\ \mathbf{0} & D_2 \end{pmatrix} : D_1$ and D_2 are (n,n)-diagonal matrices and S is an (n,n)-matrix whose $(1,n)$-entry is $\mathbf{0}$. $\}$ $(n \geq 2)$.

First we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$. If $\mathcal{A}$ is a vector space over C and if $\mathcal{A}$ is closed under the composition of maps, then $\mathcal{A}$ is called an algebra. $\mathcal{A}$ is called a self-adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$. Otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, $Alg\mathcal{L}$ denotes the algebra of all operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections which leave invariant each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is called a CSL-algebra. If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\varphi : ALg\mathcal{L}_1 \rightarrow Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\varphi : ALg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$.

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let $\mathcal{L}$ be the lattice generated by $\{[e_1], [e_2], \cdots, [e_n], [e_1, e_2, \cdots, e_n, e_i], [e_2, \cdots, e_n, e_{2n}] : i = n+1, \cdots, 2n-1\}$. Then $\mathcal{B}_{2n}^{(n)} = Alg\mathcal{L}$. Since $\mathcal{L}$ is commutative, $\mathcal{B}_{2n}^{(n)}$ and $\mathcal{L}$ are reflexive.

It is well-known that an automorphism need not be spatially implemented by the following theorem.

THEOREM 1. ([10]). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and let $\mathcal{L}_1$ be completely distributive. Let $\rho : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ be an algebraic isomorphism. The following are equivalent :*

i) *ρ is quasi-spatial, implemented by a closed, injective linear transformation $T : \mathcal{H}_1 \to \mathcal{H}_2$ whose range and domain are dense.*
ii) *ρ preserves the rank of every finite-rank operator ; that is, rank $(\rho(R)) =$ rank R for all finite-rank R.*

Throughout this work, we denote E_{ij} by the matrix whose (i,j)-component is 1 and all other components are 0.

2. Examples of Isomorphisms

EXAMPLE 1. Let $T = (t_{ii})$ be a $(2n, 2n)$-diagonal invertible matrix. Define $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ by $\rho(A) = TAT^{-1}$ for all A in $\mathcal{B}_{2n}^{(n)}$. Then ρ is an isomorphism,

$$\rho(E_{1,n+j}) = t_{11}t_{n+j,n+j}^{-1}E_{1,n+j} \; (1 \le j < n) \text{ and}$$
$$\rho(E_{i,n+j}) = t_{ii}t_{n+j,n+j}^{-1}E_{i,n+j} \; (2 \le i \le n, 1 \le j \le n).$$

Put $t_{11}t_{n+j,n+j}^{-1} = \gamma_{1,n+j}$ $(1 \le j < n)$ and $t_{ii}t_{n+j,n+j}^{-1} = \gamma_{i,n+j}$ $(2 \le i \le n, 1 \le j \le n)$. Let $A = (\gamma_{i,n+j})$ be an (n,n)-matrix $(1 \le i, j \le n)$ and $A_1 = (\gamma_{1,n+1}, \gamma_{1,n+2}, \cdots, \gamma_{1,2n-1})$ and $A_p = (\gamma_{p,n+1}, \cdots, \gamma_{p,2n})$ $(2 \le p \le n)$. Then A_p and A_q are linearly dependent $(2 \le p, q \le n)$.

EXAMPLE 2. Let $\gamma_{1,3+l}$, $\gamma_{i,3+j}$ be non-zero complex numbers and let $\alpha_{1,n+l}$, $\alpha_{i,n+j}$ be complex numbers ($i=2,3$; $l=1,2$; $j=1,2,3$). If $\varphi : \mathcal{B}_6^{(3)} \to \mathcal{B}_6^{(3)}$ be a linear map defined by $\varphi(E_{11}) = E_{11}+\alpha_{14}E_{14}+\alpha_{15}E_{15}$, $\varphi(E_{22}) = E_{33}+\alpha_{34}E_{34}+\alpha_{35}E_{35}+\alpha_{36}E_{36}$, $\varphi(E_{33}) = E_{22}+\alpha_{24}E_{24}+\alpha_{25}E_{25}+\alpha_{26}E_{26}$, $\varphi(E_{44}) = E_{55}-\alpha_{15}E_{15}-\alpha_{25}E_{25}-\alpha_{35}E_{35}$, $\varphi(E_{55}) = E_{44}-\alpha_{14}E_{14}-\alpha_{24}E_{24}-\alpha_{34}E_{34}$, $\varphi(E_{66}) = E_{66}-\alpha_{26}E_{26}-\alpha_{36}E_{36}$, $\varphi(E_{14}) = \gamma_{15}E_{15}$, $\varphi(E_{15}) = \gamma_{14}E_{14}$, $\varphi(E_{24}) = \gamma_{35}E_{35}$, $\varphi(E_{25}) = \gamma_{34}E_{34}$, $\varphi(E_{26}) = \gamma_{36}E_{36}$, $\varphi(E_{34}) = \gamma_{25}E_{25}$, $\varphi(E_{35}) = \gamma_{24}E_{24}$, and $\varphi(E_{36}) = \gamma_{26}E_{26}$, then φ is an isomorphism. Let $\rho : \mathcal{B}_6^{(3)} \to \mathcal{B}_6^{(3)}$ be an isomorphism defined by $\rho(E_{11}) = E_{11}+\alpha_{15}E_{14}+\alpha_{14}E_{15}$, $\rho(E_{22}) = E_{22}+\alpha_{35}E_{24}+\alpha_{34}E_{25}+\alpha_{36}E_{26}$, $\rho(E_{33}) = E_{33}+\alpha_{25}E_{34}+\alpha_{24}E_{35}+\alpha_{26}E_{36}$, $\rho(E_{44}) = E_{44}-\alpha_{15}E_{14}-\alpha_{35}E_{24}-\alpha_{25}E_{34}$, $\rho(E_{55}) = E_{55}-\alpha_{14}E_{15}-\alpha_{34}E_{25}-\alpha_{25}E_{35}$, $\rho(E_{66}) = E_{66}-\alpha_{36}E_{26}-\alpha_{26}E_{36}$, $\rho(E_{14}) = \gamma_{15}E_{14}$, $\rho(E_{15}) = \gamma_{14}E_{15}$, $\rho(E_{24}) = \gamma_{35}E_{24}$, $\rho(E_{25}) = \gamma_{34}E_{25}$, $\rho(E_{26}) = \gamma_{36}E_{26}$, $\rho(E_{34}) = \gamma_{25}E_{34}$, $\rho(E_{35}) = \gamma_{24}E_{35}$, $\rho(E_{36}) = \gamma_{26}E_{36}$.

Let $V = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$. Then $\varphi = V^*\rho V$, i.e., $\varphi(A) = V^*\rho(A)V$ for all A in $\mathcal{B}_6^{(3)}$. Let $\sigma_\varphi : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$ be the permutation induced by φ, i.e., $\sigma_\varphi(i) = k$ if $\varphi(E_{ii}) = E_{kk} + *$. Note that

$$\rho(E_{11}) = E_{11} + \sum_{j=1}^{2} \alpha_{1,\sigma_\varphi(3+j)} E_{1,3+j},$$

$$\rho(E_{ii}) = E_{ii} + \sum_{j=1}^{3} \alpha_{\sigma_\varphi(i),\sigma_\varphi(3+j)} E_{i,3+j} \qquad (i=2,3)$$

$$\rho(E_{3+k,3+k}) = E_{3+k,3+k} - \sum_{t=1}^{3} \alpha_{\sigma_\varphi(t),\sigma_{\varphi(3+k)}} E_{t,3+k} \quad (k=1,2)$$

$$\rho(E_{66}) = E_{66} - \sum_{t=2}^{3} \alpha_{\sigma_\varphi(t),6} E_{t6},$$
$$\rho(E_{1,3+p}) = \gamma_{1,\sigma_\varphi(3+p)} E_{1,3+p} \qquad (p = 1,2)$$
$$\rho(E_{i,3+q}) = \gamma_{\sigma_\varphi(i),\sigma_\varphi(n+q)} E_{i,3+q} \qquad (i = 1,2,\ ;\ q = 1,2,3)$$

and V is a $(2n, 2n)$-matrix whose $(i, \sigma_\varphi(i))$-component is 1 for all $i = 1, 2, \cdots, 6$ and all other components are zero.

3. Isomorphisms of $\mathcal{B}_{2n}^{(n)}$

First we will consider the special isomorphism $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ such that $\rho(E_{ii}) = E_{ii}\ (i = 1, 2, \cdots, 2n)$. Since $E_{i,n+j} = E_{ii}E_{i,n+j}E_{n+j,n+j}$, $\rho(E_{i,n+j}) = \rho(E_{ii}E_{i,n+j}E_{n+j,n+j}) = E_{ii}\rho(E_{i,n+j})E_{n+j,n+j}$ for each $E_{i,n+j}$ in $\mathcal{B}_{2n}^{(n)}$. So $\rho(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ for some non-zero complex number $\gamma_{i,n+j}$ for each $E_{i,n+j}$ in $\mathcal{B}_{2n}^{(n)}$.

We can get the following theorem by the above calculation.

THEOREM 2. *Let $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism such that $\rho(E_{ii}) = E_{ii}$ $(i = 1, 2, \cdots, 2n)$. Then there exist $n^2 - 1$ non-zero complex numbers $\gamma_{i,n+j}$ such that $\rho(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ for each $E_{i,n+j}$ in $\mathcal{B}_{2n}^{(n)}$.*

Let $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism such that $\rho(E_{ll}) = E_{ll}$ $(l = 1, 2, \cdots, 2n)$, $\rho(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}$ and $\rho(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ where $\gamma_{1,n+k}$ and $\gamma_{i,n+j}$ are non-zero complex numbers $(1 \le k \le n-1$, $2 \le i \le n$, $1 \le j \le n)$. Let $A = (\gamma_{i,n+j})$ be an (n,n)-matrix. We put $A_i = (\gamma_{i,n+1}, \gamma_{i,n+2}, \cdots, \gamma_{i,2n-1})$ $(i = 1, 2, \cdots, n)$.

THEOREM 3. *Let $\gamma_{1,n+k}$ and $\gamma_{i,n+j}$ be non-zero complex numbers $(1 \le k \le n-1,\ 2 \le i \le n,\ 1 \le j \le n)$. Let $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be defined by $\rho(E_{kk}) = E_{kk}$ $(k = 1, 2, \cdots, 2n)$, $\rho(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}$ and $\rho(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ $(1 \le k \le n-1,\ 2 \le i \le n,\ 1 \le j \le n)$. Then ρ is an isomorphism.*

And ρ is spatially implemented by some invertible operator T if and only if

i) *A_i and A_j are linearly dependent,*

ii) *if* $A_p = \alpha_{qp} A_q$, *then* $\gamma_{p,2n} = \alpha_{qp}\gamma_{q,2n}$ $(p, q \geq 2)$ *and*
iii) $\gamma_{1,n+k} = \alpha_{12}^{-1}\gamma_{2,n+k}$ *and* $\gamma_{i,n+j} = \alpha_{12}^{-1}\alpha_{1,i}\gamma_{2,n+j}$ $(1 \leq k \leq n-1,$ $2 \leq i \leq n,\ 1 \leq j \leq n)$.

PROOF. It is clear to show that ρ is an isomorphism.

If ρ is spatially implemented by a $(2n, 2n)$-invertible matrix $T = (t_{ij})$, then $\rho(E_{ii})T = TE_{ii}$. Since $\rho(E_{ii}) = E_{ii}$, $E_{ii}T = TE_{ii}$ $(i = 1, 2, \cdots, 2n)$ and hence T is a diagonal operator. Since T is invertible, $t_{ii} \neq 0$ $(i = 1, 2, \cdots, 2n)$. Since $\rho(E_{1,n+k})T = \gamma_{1,n+k}E_{1,n+k}T = TE_{1,n+k}$ and $\rho(E_{i,n+j})T = \gamma_{i,n+j}E_{i,n+j}T = TE_{i,n+j}$, $\gamma_{1,n+k}t_{n+k,n+k} = t_{11}$ and $\gamma_{i,n+j}t_{n+j,n+j} = t_{ii}$ $(1 \leq k \leq n-1,\ 2 \leq i \leq n,\ 1 \leq j \leq n)$. Hence $\gamma_{1,n+k} = t_{11}t_{n+k,n+k}^{-1}$ and $\gamma_{i,n+j} = t_{ii}t_{n+j,n+j}^{-1}$ $(1 \leq k \leq n-1,\ 2 \leq i \leq n,$ $1 \leq j \leq n)$.

From the above, we can prove easily *i*), *ii*) and *iii*).

Suppose that *i*), *ii*) and *iii*) are satisfied. Let $A_j = \alpha_{ij}A_i$ $(i, j = 1, 2, \cdots, n)$. Put $t_{11} = 1$, $t_{22} = \alpha_{12}$, $t_{33} = \alpha_{13}$, $\cdots$, $t_{nn} = \alpha_{1n}$, $t_{n+1,n+1} = \alpha_{12}\gamma_{2,n+1}^{-1}$, $t_{n+2,n+2} = \alpha_{12}\gamma_{2,n+2}^{-1}$, $\cdots$, $t_{2n,2n} = \alpha_{12}\gamma_{2,2n}^{-1}$. Let $T = (t_{ii})$ be a $(2n, 2n)$-diagonal operator. Then $TE_{ll}T^{-1} = E_{ll}$ $(l = 1, 2, \cdots, 2n)$, $TE_{1,n+k}T^{-1} = \gamma_{1,n+k}E_{1,n+k}$ and $TE_{i,n+j}T^{-1} = \gamma_{i,n+j}E_{i,n+j}$ because $\gamma_{1,n+k} = \alpha_{12}^{-1}\gamma_{2,n+k}$ and $\gamma_{i,n+j} = \alpha_{12}^{-1}\alpha_{1i}\gamma_{2,n+j}$ $(1 \leq k \leq n-1,\ 2 \leq i \leq n,\ 1 \leq j \leq n)$. Thus $\rho(B) = TBT^{-1}$ for all B in $\mathcal{B}_{2n}^{(n)}$. □

THEOREM 4. *Let* $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ *be an isomorphism. Then for all* $i = 1, 2, \cdots, n$,

$$\varphi(E_{ii}) = E_{kk} + \sum_{j=1}^{n \text{ or } n-1} \alpha_{k,n+j}E_{k,n+j}$$

for some k(*if* $k = 1$, *then* $j = 1, \cdots n-1$) *or*

$$\varphi(E_{ii}) = E_{n+k,n+k} + \sum_{j=1 \text{ or } 2}^{n} \alpha_{j,n+k}E_{j,n+k}$$

for some $k(1 \leq k \leq n)$ (*if* $k = n$, *then* $j = 2, \cdots n$), *where* $\alpha_{k,n+j}$ *and* $\alpha_{j,n+k}$ *are complex numbers.*

PROOF. Let $\varphi(E_{ii}) = (\alpha_{pq})$ be in $\mathcal{B}_{2n}^{(n)}$. Then $(\alpha_{pq})^2 = \varphi(E_{ii})^2 = \varphi(E_{ii}^2) = \varphi(E_{ii}) = (\alpha_{pq})$. So $\alpha_{pp} = 1$ or 0 for all $p = 1, 2, \cdots, 2n$. If $\alpha_{pp} = 0$ for all $p = 1, 2, \cdots, 2n$, then $(\alpha_{pq}) = (\alpha_{pq})^2 = (0)$. Thus $\alpha_{pp} = 1$ for some $p = 1, 2, \cdots, 2n$. Suppose that the (p,p)-component and the (q,q)-component of $\varphi(E_{ii})$ are 1 $(p \neq q)$. Then there exists j $(1 \leq j \leq 2n)$ such that the (p,p)-component or the (q,q)-component of $\varphi(E_{jj}) = 1$. If the (p,p)-component of $\varphi(E_{jj}) = 1$, then since $\varphi(E_{ii})\varphi(E_{jj}) = \varphi(E_{ii}E_{jj}) = (0)$ and the (p,p)-component of $\varphi(E_{ii})\varphi(E_{jj})$ is 1, we have a contradiction. So only one diagonal entry of $\varphi(E_{ii})$ is 1. Since $\varphi(E_{ii}) = \varphi(E_{ii})^2$, $\varphi(E_{ii}) = E_{kk} + \sum_{j=1}^{n \text{ or } n-1} \alpha_{k,n+j}E_{k,n+j}$ or $\varphi(E_{ii}) = E_{n+k.n+k} + \sum_{j=1 \text{ or } 2}^{n} \alpha_{j,n+k}E_{j,n+k}$ for complex numbers $\alpha_{i,n+j}$. □

THEOREM 5. *Let $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism. If*

$$\varphi(E_{11}) = E_{11} + \sum_{j=1}^{n-1} \alpha_{1,n+j}E_{1,n+j},$$

$$\varphi(E_{ii}) = E_{ii} + \sum_{j=1}^{n} \alpha_{i,n+j}E_{i,n+j},$$

$$\varphi(E_{n+k,n+k}) = E_{n+k,n+k} + \sum_{l=1}^{n} \beta_{l,n+k}E_{l,n+k} \text{ and}$$

$$\varphi(E_{2n,2n}) = E_{2n,2n} + \sum_{l=2}^{n} \beta_{l,2n}E_{l,2n},$$

then $\beta_{p,n+q} = -\alpha_{p,n+q}$ and $\varphi(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}$ and $\varphi(E_{i,n+l}) = \gamma_{i,n+l}E_{i,n+l}$ for some non-zero complex numbers $\gamma_{1,n+k}$ and $\gamma_{i,n+l}$ $(1 \leq k \leq n-1,\ 2 \leq i \leq n,\ 1 \leq l \leq n)$.

PROOF. Let $\varphi(E_{1,n+k}) = (\gamma_{uv})$ in $\mathcal{B}_{2n}^{(n)}$ $(1 \leq k \leq n-1)$. Since $E_{1,n+k} = E_{11}E_{1,n+k}\,E_{n+k,n+k}$, $\varphi(E_{1,n+k}) = \varphi(E_{11})\varphi(E_{1,n+k})\varphi(E_{n+k,n+k})$ and hence $\varphi(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}$ for some non-zero complex number $\gamma_{1,n+k}$. Let $\varphi(E_{i,n+j}) = (\gamma_{uv})$ in $\mathcal{B}_{2n}^{(n)}$ $(2 \leq i \leq n,\ 1 \leq j \leq n)$. Since

$E_{i,n+j} = E_{ii}E_{i,n+j}E_{n+j,n+j}$, $\varphi(E_{i,n+j}) = \varphi(E_{ii})\varphi(E_{i,n+j})\varphi(E_{n+j,n+j})$ and hence $\varphi(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ for some non-zero complex number $\gamma_{i,n+j}$. Let $A = E_{11} + E_{1,n+k} + E_{n+k,n+k}$ $(1 \leq k \leq n-1)$. Comparing the $(1, n+k)$-component of $\varphi(A)^2$ with the $(1, n+k)$-component of $\varphi(A^2)$, $\beta_{1,n+k} = -\alpha_{1,n+k}$. Let $A = E_{ii} + E_{i,n+j} + E_{n+j,n+j}$ $(2 \leq i \leq n,$ $1 \leq j \leq n)$. Since $\varphi(A^2) = \varphi(A)^2$, the $(i, n+j)$-component of $\varphi(A)^2$ is equal to the $(i, n+j)$-component of $\varphi(A)^2$. Hence $\beta_{i,n+j} = -\alpha_{i,n+j}$. □

Let $\varphi : \mathcal{B}_{2n}^{(n)} \rightarrow \mathcal{B}_{2n}^{(n)}$ be an isomorphism defined as in Theorem 5. Let $A = (\gamma_{i,n+j})$ be an (n,n)-matrix and let $A_i = (\gamma_{i,n+1}, \gamma_{i,n+2}, \cdots, \gamma_{i,2n-1})$ ($i = 1, 2, \cdots, n$).

THEOREM 6. *Let $\gamma_{1,n+k}$, $\gamma_{i,n+j}$ be non-zero complex numbers and let $\alpha_{1,n+k}$ and $\alpha_{i,n+j}$ be complex numbers $(1 \leq k \leq n-1,\ 2 \leq i \leq n,$ $1 \leq j \leq n)$. Let $\varphi : \mathcal{B}_{2n}^{(n)} \rightarrow \mathcal{B}_{2n}^{(n)}$ be a linear map defined by*

$$\varphi(E_{11}) = E_{11} + \sum_{j=1}^{n-1} \alpha_{1,n+j}E_{1,n+j}$$

$$\varphi(E_{ii}) = E_{ii} + \sum_{j=1}^{n} \alpha_{i,n+j}E_{i,n+j}\ (2 \leq i \leq n)$$

$$\varphi(E_{n+i,n+i}) = E_{n+i,n+i} - \sum_{j=1}^{n} \alpha_{j,n+i}E_{j,n+i}\ (1 \leq i \leq n-1)$$

$$\varphi(E_{2n,2n}) = E_{2n,2n} - \sum_{j=2}^{n} \alpha_{j,2n}E_{j,2n},$$

$$\varphi(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}\ (1 \leq k \leq n-1)\ \text{and}$$

$$\varphi(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}\ (2 \leq i \leq n,\ 1 \leq j \leq n).$$

Then φ is an isomorphism.

And φ is spatially implemented by $T = (t_{pq})$ if and only if

i) *A_p and A_q are linearly dependent,*

ii) *if $A_p = \beta_{qp}A_q$, then $\gamma_{p,2n} = \beta_{qp}\gamma_{q,2n}$ and*

iii) *$\gamma_{1,n+k} = \beta_{12}^{-1}\gamma_{2,n+k}$ and $\gamma_{i,n+j} = \beta_{12}^{-1}\beta_{1i}\gamma_{2,n+j}$ $(2 \leq p, q, i \leq n,$ $1 \leq k \leq n-1,\ 1 \leq j \leq n)$.*

In particular, T is in $\mathcal{B}_{2n}^{(n)}$.

PROOF. It is easy to prove that φ is an isomorphism.

Suppose that φ is spartially implemented by $T = (t_{pq})$. Since $\varphi(E_{1,n+k})\, T = TE_{1,n+k}$ and $\gamma_{1,n+k} \neq 0$,

$$t_{n+k,q} = 0 \text{ and } \gamma_{1,n+k} t_{n+k,n+k} = t_{11} \tag{1}$$
$$(1 \leq k \leq n-1, q = 1, 2, \cdots, n+k-1, n+k+1, \cdots, 2n)$$

Since $\varphi(E_{i,n+j})T = TE_{i,n+j}$ and $\gamma_{i,n+j} \neq 0$,

$$t_{mi} = 0 \text{ and } \gamma_{i,n+j} t_{n+j,n+j} = t_{ii} \tag{2}$$
$$(2 \leq i \leq n, 1 \leq j \leq n, m = 1, 2, \cdots, i-1, i+1, \cdots, n)$$

Since $\varphi(E_{11})T = TE_{11}$,

$$t_{21} = 0, \cdots, t_{n1} = 0, t_{1,2n} = 0, \text{ and } \alpha_{1,n+k} t_{n+k,n+k} + t_{1,n+k} = 0 \tag{3}$$
$$(k = 1, 2, \cdots, n-1)$$

Hence T is in $\mathcal{B}_{2n}^{(n)}$.

If $t_{11} = 0$, then $t_{n+1,n+1} = 0$, $t_{n+2,n+2} = 0$, $\cdots$, $t_{2n-1,2n-1} = 0$ by (1). So by (3), $t_{1,n+1} = 0$, $t_{1,n+2} = 0$, $\cdots$, $t_{1,2n-1} = 0$. Hence T is not invertible. Thus $t_{11} \neq 0$.

If $t_{ii} = 0$ for some i $(i \geq 2)$, then $t_{n+1,n+1} = 0$, $t_{n+2,n+2} = 0$, $\cdots$, $t_{2n,2n} = 0$ by (1). So by (3) $t_{1,n+1} = 0$, $\cdots$, $t_{1,2n-1} = 0$ and $t_{11} = 0$ by (1). Therefore T is not invertible. Hence $t_{ll} \neq 0$ $(l = 1, 2, \cdots, 2n)$, $\gamma_{1,n+k} = t_{11} t_{n+k,n+k}^{-1}$, $\gamma_{i,n+j} = t_{ii} t_{n+j,n+j}^{-1}$, and $\alpha_{1,n+k} = -t_{1,n+k} t_{n+k,n+k}^{-1}$ $(1 \leq k \leq n-1)$. Since $\varphi(E_{ii})T = TE_{ii}$, $\alpha_{i,n+j} = -t_{i,n+j} t_{n+j,n+j}^{-1}$ $(2 \leq i \leq n,\ 1 \leq j \leq n)$.

From the above, we can prove easily $i)$, $ii)$ and $iii)$.

Suppose that $i)$, $ii)$ and $iii)$ are satisfied. Let $A_p = \beta_{qp} A_q$ $(p,\ q \geq 2)$. Put $t_{11} = 1$, $t_{22} = \beta_{12}$, $t_{33} = \beta_{13}$, $\cdots$, $t_{nn} = \beta_{1n}$, $t_{n+1,n+1} = \beta_{12}\gamma_{2,n+1}^{-1}$, $t_{n+2,n+2} = \beta_{12}\gamma_{2,n+2}^{-1}$, $\cdots$, $t_{2n,2n} = \beta_{12}\gamma_{2,2n}^{-1}$, $t_{1,n+k} = -\alpha_{1,n+k}\beta_{12}\gamma_{2,n+k}^{-1}$

$(1 \le k \le n-1)$ and $t_{i,n+j} = -\alpha_{i,n+j}\beta_{12}\gamma_{2,n+j}^{-1}$ $(2 \le i \le n,\ 1 \le j \le n)$. Let $T = (t_{ij})$ be a $(2n,2n)$-matrix. Then T is in $\mathcal{B}_{2n}^{(n)}$. Since the determinant of T is non-zero, T is invertible. By a simple calculation, $\varphi(E_{ii})T = TE_{ii}$ $(i = 1,2,\cdots,2n)$. Since $\gamma_{1,n+k} = \beta_{12}^{-1}\gamma_{2,n+k}$, $\varphi(E_{1,n+k})T = TE_{1,n+k}$ $(1 \le k \le n-1)$. Since $\gamma_{i,n+j} = \beta_{12}^{-1}\beta_{1i}\gamma_{2,n+j}$, $\varphi(E_{i,n+j})T = TE_{i,n+j}$ $(2 \le i \le n,\ 1 \le j \le n)$. Hence φ is spatially implemented by T. □

Comparing components of $\varphi(A)T$ with those of $T\rho(A)$ for all A in $\mathcal{B}_{2n}^{(n)}$, we can get the following theorem.

THEOREM 7. *Let $\varphi : \mathcal{B}_{2n}^{(n)} \longrightarrow \mathcal{B}_{2n}^{(n)}$ be the isomorphism defined in Theorem 6. Let $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism defined by $\rho(E_{pp}) = E_{pp}$ $(p = 1,2,\cdots,2n)$, $\rho(E_{1,n+k}) = \gamma_{1,n+k}E_{1,n+k}$ and $\rho(E_{i,n+j}) = \gamma_{i,n+j}E_{i,n+j}$ $(1 \le k \le n-1,\ 2 \le i \le n,\ 1 \le j \le n)$. Then $\varphi(A) = T\rho(A)T^{-1}$ for all A in $\mathcal{B}_{2n}^{(n)}$, where T is a $(2n,2n)$-matrix in $\mathcal{B}_{2n}^{(n)}$ whose (p,p)-component is 1, $(1,n+k)$-component is $-\alpha_{1,n+k}$ and $(i,n+j)$-component is $-\alpha_{i,n+j}$ $(p = 1,2,\cdots,2n,\ 1 \le k \le n-1,\ 2 \le i \le n,\ 1 \le j \le n)$.*

THEROEM 8. *Let $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism.*

1) *If $\varphi(E_{ii}) = E_{pp} + *_1$ $(2 \le i \le n)$ and $\varphi(E_{n+j,n+j}) = E_{qq} + *_2$ $(1 \le j \le n-1)$, then $1 \le p \le n-1$ and $n+1 \le q \le 2n-1$.*
2) *$\varphi(E_{11}) = E_{11} + *$ and $\varphi(E_{2n,2n}) = E_{2n,2n} + *$.*

PROOF. 1). If $p = n+k$ for some k $(1 \le k \le n)$, then $\varphi(E_{i,n+j}) = \varphi(E_{ii})\varphi(E_{i,n+j})\ \varphi(E_{n+j,n+j}) = (E_{n+k,n+k} + *_3)\varphi(E_{i,n+j})\varphi(E_{n+j,n+j})$. Since the $(n+k,n+k)$-component of $\varphi(E_{n+j,n+j})$ is 0, $\varphi(E_{i,n+j}) = (0)$. We have a contradiction.

If $1 \le q \le n$, there is l such that $1 \le l \le n$ and $\varphi(E_{ll}) = E_{n+t,n+t} + *_4$ for some t $(1 \le t \le n)$. Since $E_{l,n+j} = E_{ll}E_{l,n+j}E_{n+j,n+j}$, $\varphi(E_{l,n+j}) = \varphi(E_{ll})\varphi(E_{l,n+j})\varphi(E_{n+j,n+j}) = (E_{n+t,n+t} + *_4)\varphi(E_{l,n+j})\varphi(E_{n+j,n+j})$. Since the $(n+t,n+t)$-component of $\varphi(E_{n+j,n+j})$ is 0, $\varphi(E_{l,n+j}) = (0)$. It is a contradiction because φ is injective. Hence $n+1 \le q \le 2n$.

Let $\varphi(E_{ii}) = E_{11} + *_5$ for some i $(2 \le i \le n)$. If $\varphi(E_{11}) = E_{2n,2n}$, then there is $n+p$ $(1 \le p \le n)$, l $(2 \le l \le n)$ such that $\varphi(E_{n+p,n+p})$

$= E_{ll} + *_6$. Then since $E_{i,n+p} = E_{ii}E_{i,n+p}E_{n+p,n+p}$, $\varphi(E_{i,n+p}) = \varphi(E_{ii})\varphi(E_{i,n+p})\varphi(E_{n+p,n+p}) = (E_{11} + *_5)\varphi(E_{i,n+p})(E_{ll} + *_6)$. Since the (l,l)-component of $(E_{11} + *_5)\varphi(E_{i,n+p})$ is 0, $\varphi(E_{i,n+p}) = (0)$. We have a contradiction.

If $\varphi(E_{n+u,n+u}) = E_{2n,2n} + *_6$ for some u $(1 \le u \le n)$, then since $E_{i,n+u} = E_{ii}E_{i,n+u}E_{n+u,n+u}$, $\varphi(E_{i,n+u}) = \varphi(E_{ii})\varphi(E_{i,n+u})\varphi(E_{n+u,n+u}) = (E_{11} + *_5)\varphi(E_{i,n+u})(E_{2n,2n} + *_6)$. Since $(E_{11} + *_5)\varphi(E_{i,n+u})(E_{2n,2n} + *_6) = (0)$, we have a contradiction. Thus $1 \le p \le n-1$ and $\varphi(E_{11}) = E_{11} + *$ or $\varphi(E_{11}) = E_{2n,2n} + *$.

Let $\varphi(E_{n+j,n+j}) = E_{2n,2n} + *$ for some j $(1 \le j \le n-1)$. Then $\varphi(E_{11}) = E_{11} + *$. Since $E_{1,n+j} = E_{11}E_{1,n+j}E_{n+j,n+j}$, $\varphi(E_{1,n+j}) = \varphi(E_{11})\varphi(E_{1,n+j})\varphi(E_{n+j,n+j}) = (E_{11} + *)\varphi(E_{1,n+j})(E_{2n,2n} + *)$. Since $(E_{11} + *)\varphi(E_{1,n+j})(E_{2n,2n} + *) = (0)$, it is a contradiction because φ is injective. Thus $n+1 \le q \le 2n-1$ and $\varphi(E_{2n,2n}) = E_{2n,2n} + *$ or $\varphi(E_{2n,2n}) = E_{11} + *$. □

PROOF. **2**). Suppose $\varphi(E_{11}) = E_{2n,2n} + *$ and $\varphi(E_{2n,2n}) = E_{11} + *$. Let $\varphi(E_{n+k,n+k}) = E_{n+j,n+j} + *$ for some $1 \le k, j \le n-1$. Since $E_{1,n+k} = E_{11}E_{1,n+k}E_{n+k,n+k}$, $\varphi(E_{1,n+k}) = \varphi(E_{11})\varphi(E_{1,n+k})\varphi(E_{n+k,n+k}) = (E_{2n,2n} + *)\varphi(E_{1,n+k})(E_{11} + *)$. Since $(E_{2n,2n} + *)\varphi(E_{1,n+k})(E_{11} + *) = (0)$, $\varphi(E_{1,n+k}) = (0)$. We have a contradiction because φ is injective. Therefore $\varphi(E_{11}) = E_{11} + *$ and $\varphi(E_{2n,2n}) = E_{2n,2n} + *$. □

THEOREM 9. *Let $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism. If*

$$\varphi(E_{ii}) = E_{kk} + \sum_{l=1}^{n} \alpha_{k,n+l}E_{k,n+l} \text{ and}$$

$$\varphi(E_{n+j,n+j}) = E_{n+p,n+p} + \sum_{t=1}^{n} \beta_{t,n+p}E_{t,n+p}, \text{ then}$$

$$\begin{aligned}\varphi(E_{1,n+j}) &= \gamma_{1,n+p}E_{1,n+p},\\ \varphi(E_{i,n+j}) &= \gamma_{k,n+p}E_{k,n+p}, \text{ and}\\ \varphi(E_{i,2n}) &= \gamma_{k,2n}E_{k,2n}\ (2 \le i,k,j,p \le n-1).\end{aligned}$$

Let $\alpha_{1,n+k}$ and $\alpha_{i,n+j}$ be complex numbers and let $\gamma_{1,n+k}$ and $\gamma_{i,n+j}$ be non-zero complex numbers $(1 \le k \le n-1,\ 2 \le i \le n,\ 1 \le j \le n)$.

Let $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism such that

$$\varphi(E_{11}) = E_{11} + \sum_{t=1}^{n-1} \alpha_{1,n+t} E_{1,n+t},$$

$$\varphi(E_{ii}) = E_{pp} + \sum_{t=1}^{n} \alpha_{p,n+t} E_{p,n+t},$$

$$\varphi(E_{n+l,n+l}) = E_{n+q,n+q} - \sum_{t=1}^{n} \alpha_{t,n+q} E_{t,n+q} \text{ and}$$

$$\varphi(E_{2n,2n}) = E_{2n,2n} - \sum_{t=2}^{n} \alpha_{t,2n} E_{t,2n}.$$

Let $\sigma_\varphi : \{1, 2, \cdots, n, n+1, \cdots, 2n\} \to \{1, 2, \cdots, n, n+1, \cdots, 2n\}$ be the permutation induced by φ, i.e., $\sigma_\varphi(i) = p$ if $\varphi(E_{ii}) = E_{pp} + *$. and $\sigma_\varphi(n+l) = n+q$ if $\varphi(E_{n+l,n+l}) = E_{n+q,n+q} + *$ $(2 \le i,\ p \le n,$ $1 \le l,\ q \le n-1)$.

From Example 2, we have some suggestion which we can derive the following theorem. We will omit its proof for it can be easily proven.

THEOREM 10. *Let $\varphi : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ be an isomorphism such that*

$$\varphi(E_{11}) = E_{11} + \sum_{t=1}^{n-1} \alpha_{1,n+t} E_{1,n+t},$$

$$\varphi(E_{ii}) = E_{kk} + \sum_{t=1}^{n} \alpha_{k,n+t} E_{k,n+t},$$

$$\varphi(E_{n+l,n+l}) = E_{n+q,n+q} - \sum_{t=1}^{n} \alpha_{t,n+q} E_{t,n+q},$$

$$\varphi(E_{2n,2n}) = E_{2n,2n} - \sum_{t=2}^{n} \alpha_{t,2n} E_{t,2n},$$

$$\varphi(E_{1,n+l}) = \gamma_{1,n+q} E_{1,n+q},$$

$$\varphi(E_{i,n+l}) = \gamma_{k,n+q} E_{k,n+q} \text{ and}$$

$$\varphi(E_{i,2n}) = \gamma_{k,2n} E_{k,2n} \ (2 \le i, k \le n; 1 \le l, q \le n-1).$$

Let $\rho : \mathcal{B}_{2n}^{(n)} \to \mathcal{B}_{2n}^{(n)}$ *be an isomorphism defined by*

$$\rho(E_{11}) = E_{11} + \sum_{j=1}^{n-1} \alpha_{1,\sigma_\varphi(n+j)} E_{1,n+j},$$

$$\rho(E_{ii}) = E_{ii} + \sum_{j=1}^{n} \alpha_{\sigma_\varphi(i),\sigma_\varphi(n+j)} E_{i,n+j},$$

$$\rho(E_{n+k,n+k}) = E_{n+k,n+k} - \sum_{t=1}^{n} \alpha_{\sigma_\varphi(t),\sigma_\varphi(n+k)} E_{t,n+k},$$

$$\rho(E_{2n,2n}) = E_{2n,2n} - \sum_{t=2}^{n} \alpha_{\sigma_\varphi(t),2n} E_{t,2n},$$

$$\rho(E_{1,n+l}) = \gamma_{1,\sigma_\varphi(n+l)} E_{1,n+l}, \quad \text{and}$$

$$\rho(E_{i,n+p}) = \gamma_{\sigma_\varphi(i),\sigma_\varphi(n+p)} E_{i,n+p}$$

$$(2 \le i \le n, 1 \le k, l \le n-1, 1 \le p \le n).$$

Let V *be a* $(2n, 2n)$*-matrix whose* $(i, \sigma_\varphi(i))$*-component is* 1 *for all* $i = 1, 2, \cdots, 2n$ *and all other components are zero. Then* $\varphi = V^*\rho V$, *i.e.,* $\varphi(A) = V^*\rho(A)V$ *for all* A *in* $\mathcal{B}_{2n}^{(n)}$.

COROLLARY 11. *Let* φ *and* ρ *be as in Theorem* 10. *Then* φ *is spatially implemented if and only if* ρ *is spatially implemented.*

Let φ be as in Theorem 10 and let $A = (\gamma_{i,n+j})$ be an (n,n)-matrix and let $A_i = (\gamma_{i,n+1}, \gamma_{i,n+2}, \cdots, \gamma_{i,2n-1})$ $(i = 1, 2, \cdots, n)$.

From Theorems 6 and 11, we can get the following.

THEOREM 12. *Let* φ *be as in Theorem* 10. *Then* φ *is spatially implemented if and only if*

i) A_p *and* A_q *are linearly dependent,*

ii) *if* $A_p = \beta_{qp} A_q$, *then* $\gamma_{p,2n} = \beta_{qp}\gamma_{q,2n}$ *and*

iii) $\gamma_{1,n+k} = \beta_{12}^{-1}\gamma_{2,n+k}$ *and* $\gamma_{i,n+j} = \beta_{12}^{-1}\beta_{1i}\gamma_{2,n+j}$. $(2 \le p, q, i \le n, 1 \le k \le n-1, 1 \le j \le n)$

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