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THE GENERALIZED NORMAL STATE SPACE AND UNITAL NORMAL COMPLETELY POSITIVE MAP

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ABSTRACT. By introducing the notion of *a generalized normal state space*, we give a necessary and sufficient condition for that there exists a unital normal completely positive map from a von Neumann algebra into another, in terms of their generalized normal state spaces.

1. Introduction

The generalized normal state space of a von Neumann algebra will be defined as the analog of the generalized state space of a unital C^*-algebra ([19] Definition 1.1). The main result is Theorem 3.2 that gives us a necessary and sufficient condition for the existence of a unital normal completely positive map from a von Neumann algebra into another, in terms of their generalized normal state spaces.

In this introductory section, we will explain the notations that will be frequently in use and define the generalized normal state space of a von Neumann algebra (Definition 1.1).

Section 2 provides us the matrix-valued functional representation of a von Neumann algebra, employing its generalized normal state space (Theorem 2.5). Our main theorem (Theorem 3.2) in Section 3 relies heavily on Theorem 2.5.

For any von Neumann algebra M, Hilbert space $\mathcal{H}$ and positive integer n, we shall use the following notation.

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$$
\begin{aligned}
1_M &= \text{the unit of } M.\\
M_h &= \text{the set of hermitian elements of } M.\\
M_+ &= \text{the set of positive elements of } M.\\
M^* &= \text{the normed dual space of } M, \text{ where we regard } M\\
&\quad\ \text{as a Banach space.}\\
(M^*)_h &= \text{the real normed space of bounded hermitian linear}\\
&\quad\ \text{functionals on } M.\\
(M^*)_+ &= \text{the positive cone of positive linear functionals on } M.\\
M_* &= \text{the normed predual space of } M.\\
(M_*)_h &= \text{the real normed space of hermition elements of } M_*.\\
(M_*)_+ &= \text{the positive cone of positive linear functionals in } (M_*)_h,\\
&\quad\ \text{i.e. the cone of normal positive linear functionals on } M.\\
\oplus^n\mathcal{H} &= \text{the direct sum Hilbert space of } n \text{ copies of } \mathcal{H}.\\
\mathcal{B}(\mathcal{H}) &= \text{the von Neumann algebra of (bounded linear) operators}\\
&\quad\ \text{on } \mathcal{H}.\\
M_n &= \text{the } C^* - \text{algebra of } n\times n \text{ complex matrices for the}\\
&\quad\ \text{elements of } \mathcal{B}(\oplus^n\mathbb{C}), \text{with respect to the standard}\\
&\quad\ \text{orthonormal basis of } \oplus^n\,\mathbb{C}.\\
&\quad\ \text{We identify } M_n \text{ with the von Neumann algebra } \mathcal{B}(\oplus^n\mathbb{C})).\\
1_n &= \text{the unit matrix of } M_n.\\
M_n(M) &= \text{the von Neumann algebra of } n\times n \text{ matrices with entries}\\
&\quad\ \text{taken from } M.\\
1_{n,M} &= \text{the unit of } M_n(M).\\
M^{**} &= \text{the von Neumann algebra of the second norm dual space}\\
&\quad\ \text{of } M.\\
M_n(M^{**}) &= \text{the von Neumann algebra of } n\times n \text{ matrices with entries}\\
&\quad\ \text{taken from } M^{**}.\\
M_n(M^*) &= \text{the normed space of } n\times n \text{ matries with entries taken}\\
&\quad\ \text{from } M^*, \text{ equipped with the norm with respect to the}\\
&\quad\ \text{natural pairing of } M_n(M^*) \text{ and } M_n(M).
\end{aligned}
$$

$M_n(M_*) =$ the normed space of $n \times n$ matrices with entries taken M_*, from equipped with the norm with respect to the natural pairing $M_n(M_*)$ and $M_n(M)$.

If $\Phi = [\varphi_{ij}]_{1\le i,j\le n} \in M_n(M^*)$, we may regard Φ as a bounded linear map from M into M_n by declaring

$$\Phi(x) = [\varphi_{ij}(x)]_{1\le i,j\le n}. \tag{1.1}$$

Conversely, any bounded linear map from M into M_n is given by an element $\Phi = [\varphi_{ij}]_{1\le i,j\le n} \in M_n(M^*)$. By the norm $\|\Phi\|$ of $\Phi = [\varphi_{ij}]_{1\le i,j\le n} \in M_n(M^*)$, we mean the supremum norm of Φ as a bounded linear map from M into M_n.

Similarly we may regard $M_n(M_*)$ as the Banach space of all (σ-weak)-(σ-weak) continuous linear maps Φ from M into M_n, equipped with norm $\|\Phi\| = \sup_{\|x\|\le 1} \|\Phi(x)\|$. It is well-known that $M_n(M_*)$ is a closed subspace of $M_n(M^*)$.

Furthermore $M_n(M^*)$ is a M_n-bimodule, by defining $\lambda\Phi$, $\Phi\lambda \in M_n(M^*)$, for $\lambda = [\lambda_{ij}]_{1\le i,j\le n} \in M_n$, $\Phi = [\varphi_{ij}] \in M_n(M^*)$, as $(\lambda\Phi)(x) = \lambda\Phi(x) \in M_n$, $(\Phi\lambda)(x) = \Phi(x)\lambda \in M_n$ where $x \in M$. In this case, it is clear that $M_n(M_*)$ is a submodule of the M_n-bimodule $M_n(M^*)$.

Finally, we put

> $\mathcal{M} =$ the topological direct sum of $\{M_n : n \in \mathbb{N}\}$: As a set this is the disjoint union $\bigcup_{n\in\mathbb{N}} M_n$, and a subset of $\mathcal{M}$ is open if and only if its intersection with M_n is open in M_n, for each $n \in \mathbb{N}$.

DEFINITION 1.1. (cf. [16] p.437). Let $n \in \mathbb{N}$ and $\Phi = [\varphi_{ij}] \in M_n(M_*)$. Then Φ is called a *normal n-positive linear functional* on M, if Φ is a completely positive map from M into M_n. Note that Φ is automatically (σ-weak)-(σ-weak) continuous, since $\varphi_{ij} \in M_*$ for all $i, j \in \{1, \dots, n\}$. If, in particular, Φ is unital, i.e., $\Phi(1_M) = 1_n$, then Φ is called a *normal n-state* of M. The set $\mathcal{N}_n(M)$ of the normal n-states of M, equipped with the metric induced from the norm on $M_n(M^*)$ is called the *normal n-state space* of M. The topological direct sum of all these $\mathcal{N}_n(M)$ ($n \in \mathbb{N}$) is called the *generalized normal state space* of M, which will be denoted by $\mathcal{N}(M)$. Thus a subset of $\mathcal{N}(M)$ is open if and only if its intersection with each $\mathcal{N}_n(M)$ ($n \in \mathbb{N}$) is an open subset of $\mathcal{N}_n(M)$.

Our definition of a normal n-state $\Phi = [\varphi_{ij}]_{1\le i,j\le n}$ differs from that of A. Kaplan ([16] p.437, p.444 Proposition 2 6) in the sense that we assume $\varphi_{ij}(1_M) = 0$ whenever $i \neq j$, not just because of σ-weak continuity.

It is clear that $\mathcal{N}_n(M)$ is a norm closed convex subset of $M_n(M_*)$ for every $n \in \mathbb{N}$.

A representation π of a von Neumann algebra M on a Hilbert space $\mathcal{H}$ will be called *normal* if it is (σ-weak)-(σ-weak) continuous as a map from M into $\mathcal{B}(\mathcal{H})$. The following analog of Kaplan's results ([16] p.439 Theorem 2.1, p.444 Proposition 2.6) together with the uniqueness up to unitary transformation ([15] p.43 Theorem 3.2) can be easily verified and hence the proofs are omitted.

PROPOSITION AND DEFINITION 1.2. *Let $n \in \mathbb{N}$ and M be a von Neumann algebra. If $\Phi = [\varphi_{ij}] \in M_n(M_*)$ is a normal n-positive linear functional on M, then there exists a triple $(\mathcal{H}, \pi, (\xi_i : 1 \le i \le n))$, where π is a unital normal representation of M on a Hilbert space $\mathcal{H}$ and $(\xi_i : 1 \le i \le n)$ is an n-tuple of vectors in $\mathcal{H}$ such that*

$$\varphi_{ij}(x) = \langle \pi(x)\xi_j, \xi_i\rangle \tag{1.2}$$

for all $x \in M$, $1 \le i, j \le n$.

Such a triple $(\mathcal{H}, \pi, (\xi_i : 1 \le i \le n))$, satisfying (1.2) is called a Kaplan's triple for Φ. One can always find a Kaplan's triple $(\mathcal{H}, \pi, (\xi_i : 1 \le i \le n))$ for Φ such that the closed linear span $[\pi(x)\xi_i : x \in M,\ 1 \le i \le n]$ of $\{\pi(x)\xi_i : x \in M,\ 1 \le i \le n\}$ is the whole $\mathcal{H}$, by considering the subrepresentation of π restricted to $[\pi(x)\xi_i : x \in M,\ 1 \le i \le n]$. When $\mathcal{H} = [\pi(x)\xi_i : x \in M,\ 1 \le i \le n]$, the Kaplan's triple $(\mathcal{H}, \pi, (\xi_i : 1 \le i \le n))$ for Φ will be called nondegenerate.

If $(\mathcal{H}, \pi, (\xi_i : 1 \le i \le n))$, $(\mathcal{K}, \sigma, (\eta_i : 1 \le i \le n))$ are nondegenerate Kaplan's triples for a normal n-positive linear functional Φ on M, then there exists a unique unitary transformation U from $\mathcal{H}$ onto $\mathcal{K}$ transforming π to σ and sending ξ_i to η_i ($1 \le i \le n$), i.e.,

$$U\pi(x)U^* = \sigma(x), \quad \text{for all} \quad x \in M, \tag{1.3}$$

$$U\xi_i = \eta_i, \quad \text{for all} \quad i = 1, \dots, n. \tag{1.4}$$

2. Functional Representation of a von Neumann Algebra

Let M be a von Neumann algebra. By the normal state space we mean the normal 1-state space $\mathcal{N}_1(M)$ that will be sometimes abbreviated by $\mathcal{N}_1$.

Recall that a complex valued function f on $\mathcal{N}_1$ is called *affine* if

$$f(\lambda\varphi + (1-\lambda)\psi) = \lambda f(\varphi) + (1-\lambda)f(\psi) \tag{2.1}$$

whenever $\lambda \in [0,1]$ and $\varphi, \psi \in \mathcal{N}_1$.

By $\mathcal{A}(\mathcal{N}_1)$ we denote the linear space of complex valued bounded continuous affine functions on $\mathcal{N}_1$. The norm $\|f\|$ of $f \in \mathcal{A}(\mathcal{N}_1)$ is defined by

$$\|f\| = \sup\{|f(\varphi)| : \varphi \in \mathcal{N}_1\}. \tag{2.2}$$

Then, it is not hard to see that $\mathcal{A}(\mathcal{N}_1)$ is a Banach space with respect to this norm.

Define a map $\Lambda : M \to \mathcal{A}(\mathcal{N}_1)$ by sending $x \in M$ to $\Lambda(x) \in \mathcal{A}(\mathcal{N}_1)$, where

$$\Lambda(x)(\varphi) = \varphi(x) \tag{2.3}$$

for each $x \in M$ and all $\varphi \in \mathcal{N}_1$.

Let $\mathcal{A}_{\mathbb{R}}(\mathcal{N}_1)$ denote that $\mathbb{R}$-linear space of real valued functions f belonging to $\mathcal{A}(\mathcal{N}_1)$. Clearly $\mathcal{A}_{\mathbb{R}}(\mathcal{N}_1)$ is a real Banach space with respect to the norm defined in (2.2). Furthermore, it is also a real ordered vector space ([23] p.2). The following is an analog of the Kadison's function representation ([22] p.70 Theorem 3.10.3), of which proof can be adopted with a slight modification to verify our version. We shall omit the proof.

LEMMA 2.1. (i) *The map $\Lambda : M \to \mathcal{A}(\mathcal{N}_1)$ defined in (2.3) is surjecive and*

$$\frac{1}{2}\|x\| \le \|\Lambda(x)\| \le \|x\|, \tag{2.4}$$

for all $x \in M$.

(ii) *$\Lambda|M_h : M_h \to \mathcal{A}_{\mathbb{R}}(\mathcal{N}_1)$ is an $\mathbb{R}$-linear order isomorphism from M_h onto $\mathcal{A}_{\mathbb{R}}(\mathcal{N}_1)$.*

DEFINITION 2.2. Let $n \in \mathbb{N}$. The normal n-state space $\mathcal{N}_n(M)$ and the generalized normal state space $\mathcal{N}(M)$ of a von Neumann algebra M will be abbreviated by $\mathcal{N}_n$ and $\mathcal{N}$ respectively.

A map $F : \mathcal{N}_n \to M_n$ is called *operator affine*, if for any $r \in \mathbb{N}$ and any r-tuple $(A_i : 1 \leq i \leq r)$ of matrices $A_i \in M_n$ such that

$$(2.5) \qquad \sum_{i=1}^{r} A_i^* A_i = 1_n \in M_n$$

one has that

$$(2.6) \qquad F(\sum_{i=1}^{r} A_i^* \Phi A_i) = \sum_{i=1}^{r} A_i^* F(\Phi) A_i \in M_n$$

for all $\Phi \in \mathcal{N}_n$.

A map $F : \mathcal{N} \to \mathcal{M}$ is called *operator affine*, if $F(\mathcal{N}_n) \subset M_n$ and $F|\mathcal{N}_n : \mathcal{N}_n \to M_n$ is operator affine for every $n \in \mathbb{N}$.

If $n \leq m$ $(n, m \in \mathbb{N})$, $\Phi = [\varphi_{ij}]_{1 \leq i,j \leq m} \in M_m(M_*)$, we call $\Phi_n = [\varphi_{ij}]_{1 \leq i,j \leq n} \in M_n(M_*)$ the nth *compression* of Φ. Similarly, if $\lambda = [\lambda_{ij}]_{1 \leq i,j \leq n} \in M_m$, we call $\lambda_n = [\lambda_{ij}]_{1 \leq i,j \leq n} \in M_n$ the nth *compression* of λ. A map $F : \mathcal{N} \to \mathcal{M}$ is called *hereditary* if $F(\mathcal{N}_n) \subset M_n$ for every $n \in \mathbb{N}$ and $F(\Phi_n) = F(\Phi)_n$, for each $\Phi \in \mathcal{N}_m$ and every $n \in \mathbb{N}$ with $n \leq m$.

Now let N be another von Neumann algebra and $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ be a mapping. Then Θ is called *operator affine*, if $\Theta(\mathcal{N}_n(N)) \subset \mathcal{N}_n(M)$ for every $n \in \mathbb{N}$, and if for every $r \in \mathbb{N}$ and any r-tuple $(A_i : 1 \leq i \leq r)$ of matrices $A_i \in M_n$ such that (2.5) is satisfied, one has that

$$(2.7) \qquad \Theta(\sum_{i=1}^{r} A_i^* \Phi A_i) = \sum_{i=1}^{r} A_i^* \Theta(\Phi) A_i$$

for every $\Phi \in \mathcal{N}_n(N)$ and every $n \in \mathbb{N}$.

A map $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ is called *hereditary* if for any pair n, $m \in \mathbb{N}$ with $n \leq m$, one has that

$$(2.8) \qquad \Theta(\mathcal{N}_m(N)) \subset \mathcal{N}_m(M)$$

and

$$\Theta(\Phi|\mathcal{N}_n(N)) = \Theta(\Phi)_n$$

for every $\Phi \in \mathcal{N}_m(N)$.

Recall that $\mathcal{N}$ denotes $\mathcal{N}(M)$. Let $F : \mathcal{N} \to \mathcal{M}$ be a map such that $F(\mathcal{N}_n) \subset M_n$ for every $n \in \mathbb{N}$. It is called *bounded* if the quantity $\|F\|$ defined (2.9) below is finite, and in that case $\|F\|$ is called the *norm* of F:

$$\|F\| = \sup\{\|F(\Phi)\| : \Phi \in \mathcal{N}\}. \tag{2.9}$$

Let $\mathcal{C}_M$ denote the set of all bounded, operator affine, hereditary continuous maps $F : \mathcal{N}(M) \to \mathcal{M}$. It is not hard to see that $\mathcal{C}_M$ is a Banach space over $\mathbb{C}$. We can define an involution $*$ on $\mathcal{C}_M$ by

$$F^*(\Phi) = F(\Phi)^*, \tag{2.10}$$

for every $\Phi \in \mathcal{N}(M)$.

As the natural extension of Λ in (2.3), we now define the map $\Lambda_M : M \to \mathcal{C}_M$ by

$$\Lambda_M(x)(\Phi) = [\varphi_{ij}(x)] \in M_n \ (\subset \mathcal{M}) \tag{2.11}$$

whenever $\Phi \in \mathcal{N}$ and $\Phi \in M_n$ for some $n \in \mathbb{N}$, for all $x \in M$.

It is immediate to see that $\Lambda_M(x) \in \mathcal{C}_M$ for each $x \in M$. The main result of this section, Theorem 2.5, in fact, asserts that $\Lambda_M : M \to \mathcal{C}_M$ is surjective. To prove Theorem 2.5, we need to define an ordered vector space over $\mathbb{C}$, mimicking the usual definition of an ordered vector space over $\mathbb{R}$ ([23] p.2).

DEFINITION 2.3. Let $\mathcal{C}$ be a complex vector space and $\leq$ be a partial order on $\mathcal{C}$. We say that $\mathcal{C}$ is an *ordered vector space*, with respect to the partial order $\leq$, if the following conditions (2.12) and (2.13) are satisfied, for any pair $F, G \in \mathcal{C}$ such that $F \leq G$.

$$F + H \leq G + H, \quad \text{for all} \quad H \in \mathcal{C}, \tag{2.12}$$

and

$$tF \leq tG, \quad \text{for all} \quad t \in [0, \infty) \tag{2.13}$$

In this case, the set $\mathcal{C}^+$ defined by

$$\mathcal{C}^+ = \{F \in \mathcal{C} : F \geq 0\}, \tag{2.14}$$

where 0 is the zero element of $\mathcal{C}$, is called the *positive cone* of $\mathcal{C}$.

Let $\{F_\alpha : \alpha \in A\}$ be an indexed subset of $\mathcal{C}^+$. An element $F \in \mathcal{C}^+$ is called the *least upperbound* of $\{F_\alpha : \alpha \in A\}$ and denoted by $\mathrm{lub}\{F_\alpha : \alpha \in A\}$ if the following conditions (i) and (ii) are satisfied.

(i) $F_\alpha \leq F$ for all $\alpha \in A$.

(ii) Whenever $G \in \mathcal{C}^+$ satisfies that $F_\alpha \leq G$ for all $\alpha \in A$, then $F \leq G$.

Note that $\mathrm{lub}\{F_\alpha : \alpha \in A\}$ is uniquely determined, once it exists. An ordered vector space $(\mathcal{C}, \leq)$ over $\mathbb{C}$ is called a *complete ordered vector space* if for every increasing net $\{F_\alpha : \alpha \in A\}$ in $\mathcal{C}^+$, $\mathrm{lub}\{F_\alpha : \alpha \in A\}$ exists as an element of $\mathcal{C}^+$.

Let $(\mathcal{C}, \leq)$ and $(\mathcal{D}, \leq)$ be two ordered vector spaces over $\mathbb{C}$. Then a linear mapping γ from $\mathcal{C}$ into $\mathcal{D}$ is called an *order preserving* if the following holds: For $F, G \in \mathcal{C}$, if $F \leq G$, then $\gamma(F) \leq \gamma(G)$. A linear isomorphism γ from $\mathcal{C}$ onto $\mathcal{D}$ is called an *order isomorphism* if both γ and γ^{-1} are order preserving.

Finally, let γ be a linear mapping from a complete ordered vector space $(\mathcal{C}, \leq)$ into another $(\mathcal{D}, \leq)$. Then γ is called *normal*, if it is order preserving and

$$\gamma(\mathrm{lub}\{F_\alpha : \alpha \in A\}) = \mathrm{lub}\{\gamma(F_\alpha) : \alpha \in A\}, \tag{2.15}$$

whenever $\{F_\alpha : \alpha \in A\}$ is an increasing net in $\mathcal{C}^+$.

DEFINITION 2.4. Let M be a von Neumann algebra. We equip M with the order $\leq$ by declaring $x \leq y$ if $y - x \in M_+$ $(x, y \in M)$. Also we equip $\mathcal{C}_M$ with the order $\leq$ by declaring $F \leq G$ if $G(\Phi) - F(\Phi) \in (M_n)_+$ whenever $\Phi \in \mathcal{N}(M)$, $\Phi \in \mathcal{N}_n(M)$, for some $n \in \mathbb{N}$.

By the following theorem and Lemma 2.1, one can immediately verify that $\mathcal{C}_M$ as well as M is a complete ordered vector space over $\mathbb{C}$. The identity element $1_{\mathcal{C}}$ of $\mathcal{C}_M$ is defined as the element of $\mathcal{C}_M$ such that $1_{\mathcal{C}}(\Phi) = 1_n$ for every $\Phi \in \mathcal{N}(M)$ whenever $\Phi \in \mathcal{N}_n(M)$ for some $n \in \mathbb{N}$.

THEOREM 2.5. *Let M be a von Neumann algebra. Then the map $\Lambda_M : M \to \mathcal{C}_M$ defined by (2.11) is a $*$-preserving linear isometry from M onto $\mathcal{C}_M$, sending 1_M to $1_{\mathcal{C}}$ such that both Λ_M and Λ_M^{-1} are normal.*

Proof. We have already noted that $\Lambda_M(M) \subset \mathcal{C}_M$. To get the reverse inclusion let $F \in \mathcal{C}_M$. For each $n \in \mathbb{N}$, define

$$F_n = F|\mathcal{N}_n : \mathcal{N}_n(M) \to M_n. \tag{2.16}$$

where $\mathcal{N}_n$ denotes $\mathcal{N}_n(M)$.

Thus, when $n = 1$, we have

$$F_1 \in \mathcal{N}_1. \tag{2.17}$$

By Lemma 2.1, there exists a unique element $x \in M$ such that

$$F_1(\varphi) = \varphi(x) \quad \text{for all} \quad \varphi \in \mathcal{N}_1. \tag{2.18}$$

We will show that

$$F = \Lambda_M(x). \tag{2.19}$$

For this, it suffices to verify that for every $n \in \mathbb{N}$ and every $\Phi = [\varphi_{ij}]_{1 \le i,j \le n}$, one has

$$\Lambda_M(x)(\Phi) = F(\Phi) \in M_n \tag{2.20}$$

i.e.,

$$[\varphi_{ij}(x)]_{1 \le i,j \le n} = F(\Phi) \in M_n \tag{2.21}$$

Fixing $n \in \mathbb{N}$ and $\Phi \in \mathcal{N}_n$, let us write $F(\Phi)$ in the matrix form:

$$F(\Phi) = [\lambda_{ij}]_{1 \le i,j \le n} \in M_n. \tag{2.22}$$

It suffices to show that

$$\lambda_{ij} = \varphi_{ij}(x), \tag{2.23}$$

for every $i, j \in \{1, \dots, n\}$. We will first show that

$$\lambda_{11} = \varphi_{11}(x). \tag{2.24}$$

The 1st compression Φ_1 of Φ is φ_{11} and the 1st compression $F(\Phi)_1$ of $F(\Phi)$ is λ_{11}. Since F is hereditary,

$$\lambda_{11} = F(\Phi)_1 = F_1(\Phi_1) = F_1(\varphi_{11}) = \varphi_{11}(x)$$

by (2.18). Hence (2.24) holds.

Next, we will show that

$$\lambda_{ii} = \varphi_{ii}(x) \quad \text{for every} \quad i \in \{1, \dots, n\}. \tag{2.25}$$

To see this let u_{ij} denote the permutation matrix belonging to M_n, interchanging the ith row and jth row of $1_n (\in M_n)$ with each other. By (2.7) with $r = 1$, we get

$$\begin{aligned}
\lambda_{ii} &= (u_{1i}^* F(\Phi) u_{1i})_1 \\
&= (F(u_{1i}^* \Phi u_{1i}))_1 \\
&= F_1((u_{1i}^* \Phi u_{1i})_1) \\
&= F_1(\varphi_{ii}) \\
&= \varphi_{ii}(x), \quad \text{by (2.24)},
\end{aligned}$$

for each $i \in \{1, \dots, n\}$. This proves (2.25).

Now for every pair $k, l \in \{1, \dots, n\}$, let e_{kl} denote the element of M_n with 1 at (k, l)-entry and zeros elsewhere. When $k \neq l$, $(k, l \in \{1, \dots, n\})$, let us fix these k and l temporarily and define $v, w \in M_n$ by

$$\begin{cases} v = \frac{1}{\sqrt{2}}(e_{ll} + e_{kl}), \\ w = \frac{1}{\sqrt{2}}(e_{ll} - ie_{kl}). \end{cases} \tag{2.26}$$

On can easily verify that

$$v^* v = w^* w = e_{ll} \quad \text{and} \tag{2.27}$$

$$v^*v + \sum_{i\neq l} e_{ii}^* e_{ii} = w^*w + \sum_{i\neq l} e_{ii}^* e_{ii} \tag{2.28}$$
$$= 1_n \in M_n.$$

Also, it is not hard to see that

$$\begin{aligned} & v^*\Phi v + \sum_{i\neq l} e_{ii}^*\Phi e_{ii} \\ &= \mathrm{diag}(\varphi_{11}, \dots, \varphi_{l-1,l-1}, \\ &\qquad \frac{1}{2}(\varphi_{ll} + \varphi_{kk} + \varphi_{kl} + \varphi_{lk}), \varphi_{l+1,l+1}, \dots, \varphi_{n,n}) \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} & w^*\Phi w + \sum_{i\neq l} e_{ii}^*\Phi e_{ii} \\ &= \mathrm{diag}(\varphi_{11}, \dots, \varphi_{l-1,l-1}, \\ &\qquad \frac{1}{2}(\varphi_{ll} + \varphi_{kk} + i(\varphi_{kl} - \varphi_{lk})), \varphi_{l+1,l+1}, \dots, \varphi_{nn}), \end{aligned} \tag{2.30}$$

where $\mathrm{diag}(\varphi_1, \varphi_2, \dots, \varphi_n)$ denotes the element

$$\begin{pmatrix} \varphi_1 & & O & \\ & \varphi_2 & & \\ & & \ddots & \\ & O & & \varphi_n \end{pmatrix} \quad (\varphi_1, \varphi_2, \dots, \varphi_n \in \mathcal{N}_1(M)).$$

We evaluate both sides of (2.29) and (2.30) by F. From the fact that F is operator affine and (2.25), we get the following.

(2.31)

$$\begin{aligned} & v^*F(\Phi)v + \sum_{i\neq l} e_{ii}^*F(\Phi)e_{ii} \\ &= \mathrm{diag}(\varphi_{11}(x), \dots, \varphi_{l-1,l-1}(x), \frac{1}{2}(\varphi_{ll}(x) + \varphi_{kk}(x) + \varphi_{lk}(x) + \varphi_{kl}(x)), \\ &\qquad \varphi_{l+1,l+1}(x), \dots, \varphi_{nn}(x)), \end{aligned}$$

and
(2.32)

$$w^*F(\Phi)w + \sum_{i\neq l} e_{ii}^* F(\Phi)e_{ii}$$

$$= \operatorname{diag}(\varphi_{11}(x), \ldots, \varphi_{l-1,l-1}(x), \frac{1}{2}(\varphi_{ll}(x) + \varphi_{kk}(x) + i(\varphi_{kl}(x) - \varphi_{lk}(x)), \varphi_{k+1,k+1}(x), \ldots, \varphi_{nn}(x)),$$

where $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ denotes the diagonal matrix in M_n with the diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n (\in \mathbb{C})$.

By (2.22), we can rewrite the left hand sides of (2.31) and (2.32) as follows just as the way how we obtained the right hand sides of (2.29) and (2.30) from the left hand sides of them respectively, i.e.,

$$\operatorname{diag}(\lambda_{11}, \ldots, \lambda_{l-1,l-1}, \frac{1}{2}(\lambda_{ll} + \lambda_{kk} + \lambda_{kl} + \lambda_{lk}), \lambda_{l+1,l+1}, \ldots, \lambda_{nn}), \quad \text{and}$$

$$\operatorname{diag}(\lambda_{11}, \ldots, \lambda_{l-1,l-1}, \frac{1}{2}(\lambda_{ll} + \lambda_{kk} + i(\lambda_{kl} - \lambda_{lk})), \lambda_{l+1,l+1}, \ldots, \lambda_{nn}).$$

Because these two expressions are equal to the right hand sides of (2.31) and (2.32) respectively, we can deduce that

$$\lambda_{ll} + \lambda_{kk} + \lambda_{kl} + \lambda_{lk} = \varphi_{ll}(x) + \varphi_{kk}(x) + \varphi_{kl}(x) + \varphi_{lk}(x) \tag{2.33}$$

and

$$\lambda_{ll} + \lambda_{kk} + i(\lambda_{kl} - \lambda_{lk}) = \varphi_{ll}(x) + \varphi_{kk}(x) + i(\varphi_{kl}(x) - \varphi_{lk}(x)). \tag{2.34}$$

By aid of (2.25), we can solve the above simultaneous equations (2.33) and (2.34) to obtain that $\lambda_{kl} = \varphi_{kl}(x)$ for all $k, l \in \{1, \ldots, n\}$, proving (2.23) i.e., that (2.21) holds.

Since the element $x \in M$ depends only on $F \in \mathcal{C}_M$, but not on the way how we choose $\Phi \in \mathcal{N}(M)$, we see that Λ_M is indeed surjective. It is routine, and in fact well known that M is an ordered vector space over $\mathbb{C}$. Also it is easy to verify that Λ_M is order preserving. Then the surjectivity of Λ_M can be used to prove that $\mathcal{C}_M$ is also a complete ordered vector space over $\mathbb{C}$ and that both Λ_M and Λ_M^{-1} are normal.

The fact that Λ_M is isometric can be proven by a modification of Theorem 4.7 ([25] p.1054). We shall omit the easy verification for the fact that $\Lambda_M(1_M) = 1_C$ and $\Lambda_M(x^*) = (\Lambda_M(x))^*$ for every $x \in M$. □

From the above Theorem 2.5 every element $F \in \mathcal{C}_M$ turns out to be necessarily bounded.

3. Unital Normal Completely Positive Map

To prove the main result, Theorem 3.2, we have to know about the functorial property of the generalized normal state spaces of von Neumann algebras.

Let $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ be a continuous, operator affine, hereditary map, where M and N are von Neumann algebras. Then it is clear that we can define a map $\Theta : \mathcal{C}_M \to \mathcal{C}_N$ by

$$\tilde{\Theta}(F) = F \circ \Theta \quad \text{(composition)} \tag{3.1}$$

for all $F \in \mathcal{C}_M$. Furthermore, $\tilde{\Theta}$ is linear and $*$-preserving, and sends the unit of $\mathcal{C}_M$ to the unit of $\mathcal{C}_N$, i.e. unital, whose easy proofs will be omitted. But among the properties of $\tilde{\Theta}$, we now prove the normality (Definition 2.3) of $\tilde{\Theta}$.

LEMMA 3.1. *If $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ is a continuous, operator affine, hereditary map, then $\tilde{\Theta}$ is normal.*

Proof. Let $\{F_\alpha : \alpha \in A\}$ be an increasing net in $\mathcal{C}_M^+$ having $\operatorname{lub}\{F_\alpha : \alpha \in A\} = F \in \mathcal{C}_M^+$. Then, obviously $\{F_\alpha \circ \Theta : \alpha \in A\}$ is an increasing net in $\mathcal{C}_N^+$ such that

$$F_\alpha \circ \Theta \leq F \circ \Theta \tag{3.2}$$

for all $\alpha \in A$, by Definition 2.4. Since $\mathcal{C}_N$ is a complete ordered vector space over $\mathbb{C}$ as we have observed just before Theorem 2.5, there exists the least upper bound $G \in \mathcal{C}_N^+$ of $\{F_\alpha \circ \Theta : \alpha \in A\}$. Thus, by the definition of a least upper bound, we have

$$G \leq F \circ \Theta. \tag{3.3}$$

To get the reverse inequality, let $y \in N_+$, $x_\alpha \in M_+$ be such that

$$G = \Lambda_N(y), \quad F_\alpha = \Lambda_M(x_\alpha), \tag{3.4}$$

which is possible by Theorem 2.5.

Since G is an upper bound of $\{F_\alpha \circ \Theta : \alpha \in A\}$, we get

$$\Lambda_M(x_\alpha) \circ \Theta \leq \Lambda_N(y) \tag{3.5}$$

for all $\alpha \in A$. Consequently, for every $\Psi \in \mathcal{N}(N)$, we have

$$\Lambda_M(x_\alpha)(\Theta(\Psi)) \leq \Lambda_N(y)(\Psi)$$

i.e.

$$\Theta(\Psi)(x_\alpha) \leq \Psi(y) \tag{3.6}$$

for every $\alpha \in A$.

Put $x = \mathrm{lub}\{x_\alpha : \alpha \in A\} \in M_+$. As we know that $\Lambda_M : M \to \mathcal{C}_M$ is normal, by Theorem 2.5, we have

$$F = \Lambda_M(x), \tag{3.7}$$

since $F = \mathrm{lub}\{F_\alpha : \alpha \in A\}$ in $\mathcal{C}_M$.

Thus, for our desired inequality $F \circ \Theta \leq G$, it suffices to show that

$$(\Lambda_M(x) \circ \Theta)(\Psi) \leq \Lambda_N(y)(\Psi) \tag{3.8}$$

for every $\Psi \in \mathcal{N}(N)$.

i.e.,

$$\Theta(\Psi)(x) \leq \Psi(y) \tag{3.9}$$

for every $\Psi \in \mathcal{N}(N)$.

From (3.6), (3.9) will be fulfilled if we can show

$$\Theta(\Psi)(x) = \mathrm{lub}\{\Theta(\Psi)(x_\alpha) : \alpha \in A\} \tag{3.10}$$

in M_n, whenever $\Psi \in \mathcal{N}(N)$ and $\Psi \in \mathcal{N}_n(N)$ for some $n \in \mathbb{N}$.

By assuming $\Psi \in \mathcal{N}(N)$ and $\Psi \in \mathcal{N}_n(N)$ for some $n \in \mathbb{N}$, (3.10) will be fulfilled if we can show

$$(3.11) \qquad \Phi(x) = \mathrm{lub}\{\Phi(x_\alpha) : \alpha \in A\}$$

in M_n, for every $\Phi \in \mathcal{N}_n(M)$. But this last (3.11) is obvious as we have assumed that each $\Phi \in \mathcal{N}_n(M)$ is (σ-weak)-(σ-weak) continuous (Definition 1.1) and $\{x_\alpha : \alpha \in A\}$ is well-known to converge to x σ-weakly ([28] p.39). Hence we can conclude that $F \circ \Theta \leq G$. This together with (3.3) now says that $G = F \circ \Theta$, i.e. $G = \tilde{\Theta}(F)$. We thus have shown that $\mathrm{lub}\{\tilde{\Theta}(F_\alpha) : \alpha \in A\} = \tilde{\Theta}(\mathrm{lub}\{F_\alpha : \alpha \in A\})$, proving $\tilde{\Theta}$ is normal. □

THEOREM 3.2. *Let M, N be von Neumann algebras. Assume that*

$$(3.10) \qquad \Theta : \mathcal{N}(N) \to \mathcal{N}(M)$$

is a continuous, operator affine, hereditary map. If we define a linear map $\pi : M \to N$ by

$$(3.11) \qquad \pi = \Lambda_N^{-1} \circ \tilde{\Theta} \circ \Lambda_M \text{ (compositions)},$$

then π is a unital normal completely positive map from M into N.

π has the following properties (i) *and* (ii).

(i) *For each $\Psi = [\psi_{ij}]_{1 \leq i,j \leq n} \in \mathcal{N}(N)$ if we put*

$$(3.12) \qquad \Phi = [\varphi_{ij}]_{1 \leq i,j \leq n} = \Theta(\Psi) \in \mathcal{N}(M),$$

then we have

$$(3.13) \qquad \varphi_{ij}(x) = \psi_{ij}(\pi(x)),$$

for every $i, j \in \{1, \ldots, n\}$ and all $x \in M$.

(ii) *Let $\theta : N_* \to M_*$ be the bounded linear map such that*

$$(3.14) \qquad \pi = \theta^* \quad \text{(the adjoint of } \theta).$$

Then, we have

$$(3.15) \qquad \varphi_{ij} = \theta(\psi_{ij})$$

for every pair $i, j \in \{1, \ldots, n\}$.

Conversely, let π be a unital normal completely positive (linear) map from M into N. Let $\theta : N_* \to M_*$ be defined as in (3.14) above. Then the map $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ sending every $\Psi = [\psi_{ij}]_{1 \le i,j \le n} \in \mathcal{N}(N)$ to $\Theta(\Psi) = \Phi = [\varphi_{ij}]_{1 \le i,j \le n}$, where φ_{ij} is defined by (3.15) above is a continuous, operator affine, hereditary map from $\mathcal{N}(N)$ into $\mathcal{N}(M)$.

Finally, when Θ and π are related as above, we have

(iii) $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ is injective if and only if π is surjective.

(iv) $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ is surjective if and only if π is injective.

Proof. The fact that $\pi : M \to N$ is a unital normal completely positive map from M into N follows from (3.11), Theorem 2.5 and Lemma 3.1.

We will verify (3.13) in (i). Then, (3.15) in (ii) follows from (3.13). To verify (3.13) in (i), let $x \in M$. From (3.11), we have

$$\tilde{\Theta}(\Lambda_M(x)) = \Lambda_N(\pi(x)), \tag{3.16}$$

and from (3.1) with $F = \Lambda_M(x)$, we get

$$\tilde{\Theta}(\Lambda_M(x)) = \Lambda_M(x) \circ \Theta \tag{3.17}$$

By comparing (3.16) and (3.17) we get

$$\Lambda_M(\pi(x)) = \Lambda_M(x) \circ \Theta. \tag{3.18}$$

For each $\Psi = [\psi_{ij}]_{1 \le i,j \le n} \in \mathcal{N}_n(N)$, we put $\Phi = [\varphi_{ij}]_{1 \le i,j \le n} = \Theta(\Psi) \in \mathcal{N}_n(M)$. Then, by evaluating both sides of (3.18) at Ψ, we have

$$\Psi(\pi(x)) = \Phi(x). \tag{3.19}$$

This is exactly (3.13) in (i) that we wanted to verify.

Now, let us show that $\pi : M \to N$ in (3.11) is completely positive. So, let $n \in \mathbb{N}$ and $[x_{ij}]_{1 \le i,j \le n} \in M_n(M)_+$ be taken arbitrarily. We have to show

$$[\pi(x_{ij})]_{1 \le i,j \le n} \in M_n(N)_+. \tag{3.20}$$

This will be fulfilled, if we can show that, for any $f \in (M_n(N)_*)_+$,

$$f([\pi(x_{ij})]_{1 \le i,j \le n}) \ge 0. \tag{3.21}$$

From the obvious analog of Theorem 5.1 ([21] p.64. Also see Proposition 1.1 [16] p.438), it suffices to show that, for every normal completely positive map $\Psi' = [\psi'_{ij}]_{1\leq i,j\leq n} : N \to M_n$,

$$\sum_{1\leq i,j\leq n} \psi'_{ij}(\pi(x_{ij})) \geq 0. \tag{3.22}$$

On the other hand, by Lemma 9.5 ([27] p.120), there exists a unital completely positive map $\Psi : N \to M_n$, i.e., $\Psi(1_N) = 1_n$, such that

$$\Psi'(y) = \Psi'(1)^{1/2}\Psi(y)\Psi'(1)^{1/2} \tag{3.23}$$

for every $y \in N$.

If we regard M_n as $\mathcal{B}(\oplus^n\mathbb{C})$, and put E as the range projection of the operator $\Psi'(1)^{1/2} \in \mathcal{B}(\oplus^n\mathbb{C})$, the above (3.23) says that Ψ' is also considered as a normal completely positive map from N into $\mathcal{B}(E(\oplus^n\mathbb{C}))$. Because the range $E(\oplus^n\mathbb{C})$ of $\Psi'(1)^{1/2}$ reduces $\Psi'(1)^{1/2}$, and $\Psi'(1)^{1/2}|E(\oplus^n\mathbb{C})$ is exactly $(\Psi'(1)|E(\oplus^n\mathbb{C}))^{1/2}$ where $\Psi'(1)|E(\oplus^n\mathbb{C})$ is the restriction of $\Psi'(1)$ to $E(\oplus^n\mathbb{C})$, we may assume that, without loss of generality, $\Psi'(1)$ and hence $(\Psi'(1))^{1/2}$ is an invertible operator on $\oplus^n\mathbb{C}$. Then from (3.23), we get

$$\Psi(y) = \Psi'(1)^{-(1/2)}\Psi'(y)\Psi'(1)^{-(1/2)} \tag{3.24}$$

for every $y \in N$.

This (3.24) says that Ψ is normal, just as Ψ' is (Definition 1.1). Thus, in (3.23), we can take a unital completely positive map $\Psi : N \to M_n$ that is also normal, i.e., $\Psi \in \mathcal{N}_n(N)$.

We put

$$A = [a_{ij}]_{1\leq i,j\leq n} = \Psi'(1)^{1/2} \tag{3.25}$$

$$\Psi = [\psi_{ij}]_{1\leq i,j\leq n} \tag{3.26}$$

in (3.23). Then

$$\psi'_{ij}(y) = \sum_{1\leq p,q\leq n} a_{ip}\psi_{pq}(y)a_{qj} \tag{3.27}$$

for every $y \in N$.

To verify (3.22), let us now compute

$$
\begin{aligned}
\sum_{1\le i,j\le n} \psi'_{ij}(\pi(x_{ij})) &= \sum_{1\le i,j\le n} \Big(\sum_{1\le p,q\le n} a_{ip}\psi_{pq}(\pi(x_{ij}))a_{qj} \\
&= \sum_{1\le p,q\le n} \psi_{pq}(\pi(\sum_{1\le i,j\le n} a_{ip}x_{ij}a_{qj})) \qquad (3.28)\\
&= \sum_{1\le p,q\le n} \psi_{pq}(\pi(\sum_{1\le i,j\le n} \overline{a}_{pi}x_{ij}\overline{a}_{jq}))
\end{aligned}
$$

since $A^* = A$. If we put

$$B = [b_{ij}]_{1\le i,j\le n}, \quad \text{where} \tag{3.29}$$

$$b_{ij} = \overline{a}_{ij} \quad \text{(the complex conjugate of } a_{ij}) \tag{3.30}$$

for $i, j \in \{1, \dots, n\}$, then it is easy to see that $B \in (M_n)_+$, since $A \in (M_n)_+$. Now from (3.28) we get

$$\sum_{1\le i,j\le n} \psi'_{ij}(\pi(x_{ij})) = \sum_{1\le p,q\le n} \psi_{pq}(\pi(\sum_{1\le i,j\le n} b_{pi}x_{ij}b_{qj})). \tag{3.31}$$

By putting

$$z_{pq} = \sum_{1\le i,j\le n} b_{pi}x_{ij}b_{qj} \tag{3.32}$$

where $p, q \in \{1, \dots, n\}$, we see that $[z_{pq}]_{1\le p,q\le n} \in M_n(M)_+$, since $B \in (M_n)_+$ and $[x_{ij}]_{1\le i,j\le n} \in M_n(M)_+$. Then from (3.31) and (3.13) we have

$$
\begin{aligned}
\sum_{1\le i,j\le n} \psi'_{ij}(\pi(x_{ij})) &= \sum_{1\le p,q\le n} \psi_{pq}(\pi(z_{pq})) \qquad (3.33)\\
&= \sum_{1\le p,q\le n} \varphi_{pq}(z_{pq})\\
&\ge 0,
\end{aligned}
$$

by virtue of the obvious analog of Theorem 5.1 ([21] p.64) mentioned above. This verifies (3.22) as desired, showing that π is completely positive.

We omit the converse part of the assertion, which can be verified easily.

To prove (iii) and (iv), let π and θ be related as above. Then one can immediately verify that Θ is injective (respectively, surjective) if and only if θ is injective (respectively, surjective). Now (iii) and (iv) follow from the dualities in the Banach spaces ([24] pp.92–97). $\square$

COROLLARY 3.3. *If $\Theta : \mathcal{N}(N) \to \mathcal{N}(M)$ is a one to one continuous operator affine, hereditary map such that $\Theta^{-1} : \mathcal{N}(M) \to \mathcal{N}(N)$ is also a continuous operator affine, hereditary map, then π in Theorem is an isomorphism from M onto N as a morphism between $*$-algebras. The converse is trivially ture.*

Proof. This is clear from Theorem 3.2 and a result of M. D. Choi ([3] p.570 Corollary 3.2). $\square$

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