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ON SELFSIMILAR AND SEMI-SELFSIMILAR PROCESSES WITH INDEPENDENT INCREMENTS

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ABSTRACT. After the review of known results on the connections between selfsimilar processes with independent increments (processes of class L) and selfdecomposable distributions and between semi-selfsimilar processes with independent increments and semi-selfdecomposable distributions, dichotomy of those processes into transient and recurrent is discussed. Due to the lack of stationarity of the increments, transience and recurrence are not expressed by finiteness and infiniteness of mean sojourn times on bounded sets. Comparison in transience-recurrence of the Lévy process and the process of class L associated with a common distribution of class L is made.

1. Introduction and definitions

A stochastic process $\{X_t\colon t \in [0,\infty)\}$ on the d-dimensional Euclidean space $\mathbb{R}^d$ is called *selfsimilar*, if, for every $a > 0$, there is $b > 0$ such that

$$\{X_{at}\colon t \in [0,\infty)\} \stackrel{\mathrm{d}}{=} \{bX_t\colon t \in [0,\infty)\}. \tag{1.1}$$

By (1.1) we mean that, for every choice of a finite number of times $0 \le t_1 < t_2 < \cdots < t_n$, $(X_{at_1},\ldots,X_{at_n})$ and $(bX_{t_1},\ldots,bX_{t_n})$ have a common distribution. We call a process $\{X_t\colon t \in [0,\infty)\}$ on $\mathbb{R}^d$ *semi-selfsimilar,* if, for some $a \in (0,1)\cup(1,\infty)$, there is $b > 0$ satisfying (1.1). The notion is introduced in [9]. Any $a > 1$ satisfying (1.1) with some $b > 0$ is called an *epoch* of the semi-selfsimilar process $\{X_t\}$. We denote by Γ the set of all $a > 0$ such that there is $b > 0$ satisfying (1.1).

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A stochastic process $\{X_t\colon t \in [0,\infty)\}$ on $\mathbb{R}^d$ is called a *Lévy process,* if it has stationary independent increments, it is stochastically continuous and starts at the origin, and its sample functions are, almost surely, right-continuous with left limits. If the stationarity of the increments is not assumed, we call it an *additive process.*

A stochastic process $\{X_t\colon t \in [0,\infty)\}$ with $P[X_t = 0] = 1$ for every $t \geq 0$ is called a zero process. Otherwise it is called non-zero.

Let $\{X_t\colon t \in [0,\infty)\}$ be an additive process on $\mathbb{R}^d$. We say that it is *transient* if

$$P[\lim_{t\to\infty} |X_t| = \infty] = 1, \tag{1.2}$$

where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$. For $s \geq 0$ we say that it is *s-recurrent* if

$$P[\liminf_{t\to\infty} |X_t - X_s| = 0] = 1. \tag{1.3}$$

The s-recurrence means that, starting at time s, the process returns to any neighborhood of the starting point after arbitrarily long time with probability 1. We say that the process is *recurrent* if it is s-recurrent for every $s \geq 0$. Since the stationarity of the increments is not assumed, the 0-recurrence does not always imply the recurrence. A trivial example of an additive process which is 0-recurrent but not recurrent is given by $\{X_t\}$ such that $P[X_t = f(t)] = 1$, using an appropriate nonrandom continuous function $f(t)$ with $f(0) = 0$ (private communication with Minoru Motoo).

The distribution (law) of a random variable X on $\mathbb{R}^d$ is denoted by $\mathcal{L}(X)$. The characteristic function of a probability measure μ on $\mathbb{R}^d$ is denoted by $\widehat{\mu}(z)$, that is, $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$, where $\langle z,x\rangle$ is the Euclidean inner product of $z, x \in \mathbb{R}^d$. The support of X is denoted by S_X or $S_{\mathcal{L}(X)}$, that is, S_X is the smallest closed set F with $P[X \in F] = 1$.

A probability measure μ on $\mathbb{R}^d$ is called *selfdecomposable*, or *of class L*, if there are sequences of independent $\mathbb{R}^d$-valued random variables Z_n, positive real numbers a_n, and vectors $c_n \in \mathbb{R}^d$, $n = 1, 2, \ldots$, such that $\max_{1\leq j\leq n} P[a_n|Z_j| > \varepsilon] \to 0$ for every $\varepsilon > 0$ and $\mathcal{L}(a_n \sum_{j=1}^n Z_j + c_n) \to \mu$ (weak convergence), as $n \to \infty$. It is well-known (see [5, 11]) that μ is selfdecomposable if and only if, for every $b \in (0,1)$, there is a probability measure ρ on $\mathbb{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z) \text{ for } z \in \mathbb{R}^d. \tag{1.4}$$

If μ is selfdecomposable, then the ρ in (1.4) is uniquely determined by μ and b, and both μ and ρ are infinitely divisible.

Maejima and Naito [8] extends the notion in the following way. Fix $b \in (0,1)$. A probability measure μ on $\mathbb{R}^d$ is said to be *of class* $L(b)$ if there exist sequences of independent $\mathbb{R}^d$-valued random variables Z_n, $a_n > 0$, $c_n \in \mathbb{R}^d$, and positive integers k_n, $n = 1, 2, \ldots$, such that $a_n \downarrow 0$, $k_n \uparrow \infty$, $a_{n-1}/a_n \to b$, $\max_{1\le j\le k_n} P[a_n|Z_j| > \varepsilon] \to 0$ for every $\varepsilon > 0$ and $\mathcal{L}(a_n \sum_{j=1}^{k_n} Z_j + c_n) \to \mu$, as $n \to \infty$. A probability measure μ on $\mathbb{R}^d$ is of class $L(b)$ if and only if there is an infinitely divisible distribution ρ on $\mathbb{R}^d$ that satisfies (1.4). If μ is of class $L(b)$, then μ is infinitely divisible and the ρ in (1.4) is uniquely determined by μ and b. A probability measure μ on $\mathbb{R}^d$ is said to be *semi-selfdecomposable* if, for some $b \in (0,1)$, μ is of class $L(b)$. Sometimes we say that μ is semi-selfdecomposable with *span* c in the same meaning as μ is of class $L(c^{-1})$.

In this paper we shall first formulate known results on the correspondence between selfsimilar additive processes and selfdecomposable distributions and between semi-selfsimilar additive processes and semi-selfdecomposable distributions. Then, in Section 3, we give dichotomy results for semi-selfsimilar additive processes into transient and recurrent. It is remarked in Section 4 that, unlike time-homogeneous Markov processes, transience and recurrence are not related to finiteness and infiniteness of mean sojourn times on bounded sets. In the final section we shall make comparison in transience and recurrence of the Lévy process and the selfsimilar additive process associated with a common selfdecomposable distribution.

2. Known results

The class of selfsimilar additive processes and the class of selfdecomposable distributions are in one-to-one correspondence, as shown in [13, 14]. The correspondence is preserved in a weaker form in the relation between the classes of semi-selfsimilar processes and semi-selfdecomposable distributions, which is given in [9]. We begin with the existence of exponents.

THEOREM 2.1. *Suppose that $\{X_t : t \in [0,\infty)\}$ is a non-zero semi-selfsimilar process on $\mathbb{R}^d$, stochastically continuous at $t = 0$. Then, b in*

(1.1) *is uniquely determined by* $a \in \Gamma$, *and there is a unique* $H \in \mathbb{R}$ *such that* $b = a^H$ *for all* $a \in \Gamma$.

The H is necessarily positive and called the *exponent* of $\{X_t\}$. As selfsimilarity is a stronger property than semi-selfsimilarity, any non-zero selfsimilar process stochastically continuous at $t = 0$ has its exponent H. Theorem 2.1 is proved in [9, 10] under a slightly different assumption that $\{X_t\}$ is a non-trivial, wide-sense semi-selfsimilar process stochastically continuous at $t = 0$. But the proof works as well.

The correspondence between selfsimilar additive processes and selfdecomposable distributions is as follows.

THEOREM 2.2. (i) *If* $\{X_t\}$ *is a selfsimilar additive process on* $\mathbb{R}^d$, *then, for every* $t \geq 0$, $\mathcal{L}(X_t)$ *is selfdecomposable.*

(ii) *Let* μ *be a selfdecomposable distribution on* $\mathbb{R}^d$ *and let* $H > 0$. *Suppose that* μ *is not a delta measure. Then there exists, uniquely in law, a non-zero selfsimilar additive process* $\{X_t\}$ *on* $\mathbb{R}^d$ *with exponent* H *and with* $\mathcal{L}(X_1) = \mu$.

Proof of (i) is simple. Namely, if $\{X_t\}$ is selfsimilar additive with exponent H, then, for any $b \in (0,1)$ and $t > 0$, choose $s < t$ so that $(s/t)^H = b$ and see that

$$\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\mu}_{s,t}(z) = \widehat{\mu}_t(bz)\widehat{\mu}_{s,t}(z),$$

where $\mu_t = \mathcal{L}(X_t)$ and $\mu_{s,t} = \mathcal{L}(X_t - X_s)$. Proof of (ii) is an application of Kolmogorov's extension theorem. See [13, 14]. The uniqueness in law comes from $\widehat{\mu}_t(z) = \widehat{\mu}_1(t^H z)$ and from the independent increments property.

PROPOSITION 2.3. *Let* $\{X_t\}$ *be a selfsimilar additive process on* $\mathbb{R}^d$ *with exponent* H. *Define* $Y_t = X_{t^\gamma}$, *where* $\gamma > 0$. *Then* $\{Y_t\}$ *is a selfsimilar additive process with exponent* γH.

Proof. Obviously $\{Y_t\}$ is an additive process. It is selfsimilar with exponent γH, because

$$\{Y_{at}\} = \{X_{a^\gamma t^\gamma}\} \stackrel{\mathrm{d}}{=} \{a^{\gamma H} X_{t^\gamma}\} = \{a^{\gamma H} Y_t\}.$$

This finishes the proof. □

We can consider $\{X_t\}$ and $\{Y_t\}$ in the proposition above as essentially identical processes. They have a common distribution at $t = 1$. Thus Theorem 2.2 shows that selfsimilar additive processes on $\mathbb{R}^d$ and selfdecomposable distributions on $\mathbb{R}^d$ are essentially in one-to-one correspondence. As selfdecomposable distributions are called of class L, we call selfsimilar additive processes *processes of class L*. Note that the structure of selfdecomposable distributions, namely the form of their Lévy measures, is known. See [5, 6, 11].

An extension of Theorem 2.2 to semi-selfsimilar additive processes is as follows.

THEOREM 2.4. (i) *Suppose that $\{X_t\}$ is a non-zero semi-selfsimilar additive process on $\mathbb{R}^d$. Let H be its exponent and a be an epoch of it. Then, for every $t \geq 0$, $\mathcal{L}(X_t)$ is semi-selfdecomposable with span a^H, that is, of class $L(a^{-H})$.*

(ii) *Let $0 < b < 1$ and $H > 0$. Let μ be semi-selfdecomposable on $\mathbb{R}^d$ with span b^{-1}, and suppose that it is not a delta measure. Then there exists a non-zero semi-selfsimilar additive process on $\mathbb{R}^d$ with exponent H and epoch $b^{-1/H}$ such that $\mathcal{L}(X_1) = \mu$.*

To see (i), note that $\mu_{s,t} = \mathcal{L}(X_t - X_s)$ is infinitely divisible for any $0 \leq s \leq t$ and that $\mu_t = \mathcal{L}(X_t)$ satisfies

$$\widehat{\mu}_t(z) = \widehat{\mu}_{at}(a^{-H}z) = \widehat{\mu}_t(a^{-H}z)\widehat{\mu}_{t,at}(a^{-H}z) \text{ for } z \in \mathbb{R}^d.$$

Proof of (ii) depends on the existence of an appropriate system $\{\mu_t : 1 \leq t \leq a\}$ of probability measures in Theorem 2.5. Unlike the case of selfsimilar additive processes, the semi-selfsimilar additive process $\{X_t\}$ in the assertion (ii) is not unique in law.

THEOREM 2.5. *Let $a > 1$ and $H > 0$. Let $\{\mu_t : 1 \leq t \leq a\}$ be a system of probability measures on $\mathbb{R}^d$ such that*

1. *μ_1 is not a delta measure,*
2. *$\widehat{\mu}_t(z) \neq 0$ for $1 \leq t \leq a$ and $z \in \mathbb{R}^d$,*
3. *for any s, t with $1 \leq s < t \leq a$, there is a probability measure $\mu_{s,t}$ satisfying $\widehat{\mu}_t(z) = \widehat{\mu}_s(z)\widehat{\mu}_{s,t}(z)$ for $z \in \mathbb{R}^d$,*
4. *μ_t is weakly continuous in $t \in [1, a]$,*
5. *$\widehat{\mu}_a(z) = \widehat{\mu}_1(a^H z)$ for $z \in \mathbb{R}^d$.*

Then μ_t, $1 \leq t \leq a$, are semi-selfdecomposable with span a^H and there exists, uniquely in law, a semi-selfsimilar additive process $\{X_t\}$ on $\mathbb{R}^d$ with exponent H and span a such that $\mathcal{L}(X_t) = \mu_t$ for $1 \leq t \leq a$.

This is a special case of a more general theorem in [9], where wide-sense semi-selfsimilar additive processes are treated.

REMARK 2.6. As is well-known, the class of Lévy processes $\{X_t\}$ on $\mathbb{R}^d$ is in one-to-one correspondence with the class of infinitely divisible distributions μ on $\mathbb{R}^d$ through the relation that $\mathcal{L}(X_1) = \mu$. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is selfsimilar if and only if μ is strictly stable. The index α ($0 < \alpha \leq 2$) of stability and the exponent H satisfy $\alpha H = 1$. Thus, if μ is selfdecomposable but not strictly stable, then it induces two different processes — one is a Lévy process and another is a process of class L (more rigorously the latter is an equivalence class of processes of class L with the equivalence relation defined by time change described in Proposition 2.3). The problem of comparison of these two processes is proposed in [13, 14]. Some results are given there. Path behaviors of increasing processes of class L on $\mathbb{R}$ are deeply investigated by Watanabe [17], showing clear differences with those of subordinators (i.e. increasing Lévy processes) studied by Fristedt [2] and Fristedt and Pruitt [3]. (Increase and decrease in this paper are in the wide sense allowing flatness.)

REMARK 2.7. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is semi-selfsimilar if and only if the corresponding infinitely divisible distribution μ is strictly semi-stable. Any non-trivial semi-stable distribution has its index $\alpha \in (0, 2]$. Again this α and the exponent H of semi-selfsimilarity are in the relation $\alpha H = 1$. See [15] for review on semi-stable processes.

3. Dichotomy of semi-selfsimilar additive processes

We prove the following three results.

THEOREM 3.1. *Let $\{X_t\}$ be a semi-selfsimilar additive process on $\mathbb{R}^d$. Suppose that it is not transient. Then it is 0-recurrent and, moreover, s-recurrent for any s satisfying $0 \in S_{X_s}$.*

THEOREM 3.2. *If $\{X_t\}$ is a selfsimilar additive process on $\mathbb{R}^d$, then it is either transient or recurrent.*

THEOREM 3.3. *If $\{X_t\}$ is a semi-selfsimilar additive process on the line $\mathbb{R}$, then it is either transient or recurrent.*

We do not have the proof of the dichotomy into transient and recurrent for semi-selfsimilar additive processes on $\mathbb{R}^d$ with $d \geq 2$.

REMARK 3.4. Lévy processes on $\mathbb{R}^d$ are transient or recurrent. However, there are additive processes which are neither transient nor 0-recurrent. For example, let $\{X_t\}$ be a non-zero Lévy process and let $a(t)$ be a strictly increasing continuous function such that $a(0) = 0$ and $a(\infty) < \infty$. Define $Y_t = X_{a(t)}$. Then Y_t tends to $X_{a(\infty)-}$ as $t \to \infty$, and $\{Y_t\}$ is neither transient nor 0-recurrent.

REMARK 3.5. A criterion of Spitzer type tells us that, for any fixed $\varepsilon > 0$, a Lévy process $\{X_t\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1) = \mu$ is transient or recurrent according as

$$\int_{|z|<\varepsilon} \mathrm{Re}\left(\frac{1}{-\psi(z)}\right) dz < \infty \text{ or } = \infty,$$

respectively, where $\psi(z)$ is the continuous function satisfying $e^{\psi(z)} = \widehat{\mu}(z)$ and $\psi(0) = 0$. See [15] for references. It is an interesting problem to find a criterion of transience and recurrence for selfsimilar additive processes in terms of the properties of the corresponding selfdecomposable distributions. Notice that transience and recurrence are invariant under the time change described in Proposition 2.3. One of the authors tackles this problem and obtains several sufficient conditions for transience in [18]. One of the results is that, if $d \geq 3$, then any non-degenerate selfsimilar additive process on $\mathbb{R}^d$ is transient.

Let us prove Theorems 3.1—3.3.

Proof of Theorem 3.1. The zero process is obviously recurrent. So we assume that $\{X_t\}$ is non-zero. Then $\{X_t\}$ has an exponent H by Theorem 2.1. Let us prove the 0-recurrence. First we claim that

$$(3.1) \qquad P[\lim_{t\to\infty} |X_t| = \infty] = P[\inf_{t>c} |X_t| > 0] \text{ for any } c > 0.$$

In fact, the semi-selfsimilarity shows that, for any $\varepsilon > 0$ and $a \in \Gamma$,

$$\begin{aligned} P[\,|X_t| > \varepsilon \text{ for every } t > ac] &= P[\,|X_{at}| > \varepsilon \text{ for every } t > c] \\ &= P[\,|X_t| > a^{-H}\varepsilon \text{ for every } t > c], \end{aligned}$$

which tends to $P[\inf_{t>c} |X_t| > 0]$ as a goes to ∞ in Γ. Note that Γ is an unbounded set, since $a \in \Gamma$ implies $a^n \in \Gamma$ for all $n \in \mathbb{Z}$. Therefore

$$P[\,|X_t| > \varepsilon \text{ for all large } t] = P[\inf_{t>c} |X_t| > 0].$$

Since ε is arbitrary, this shows (3.1). By the assumption that the process is not transient, the probability that $\lim_{t\to\infty} |X_t| = \infty$ is less than one. Notice that the event $\lim_{t\to\infty}|X_t| = \infty$ is, for every s, written as $\lim_{t\to\infty}|X_t - X_s| = \infty$, which is measurable with respect to $\{X_t - X_s : t \in [s,\infty)\}$. Hence, by Kolmogorov's 0-1 law, its probability is zero. Namely, by (3.1),

$$P[\inf_{t>c}|X_t| = 0] = 1.$$

Since c is arbitrary, this shows that $P[\liminf_{t\to\infty}|X_t| = 0] = 1$. That is, the process is 0-recurrent.

Next, let us prove the s-recurrence for any $s > 0$ such that 0 is in S_{X_s}, the support of X_s. Suppose, on the contrary, $\{X_t\}$ is not s-recurrent. Then,

$$P[\liminf_{t\to\infty}|X_t - X_s| > 0] > 0$$

and, hence, there are $c > s$, $\varepsilon > 0$, and $k > 0$ such that

$$P[\inf_{t>c}|X_t - X_s| > 2\varepsilon] > k.$$

Let $a \in \Gamma \cap (1,\infty)$. Using the independence of the increments and the semi-selfsimilarity, we have

$$\begin{aligned}
&P[\inf_{t>ac}|X_t - X_s| > a^H\varepsilon] \\
&= \int_{\mathbb{R}^d} P[\inf_{t>ac}|X_t - X_{as} + x| > a^H\varepsilon]P[X_{as} - X_s \in dx] \\
&\geq \int_{\mathbb{R}^d} P[\inf_{t>ac}|X_t - X_{as}| > |x| + a^H\varepsilon]P[X_{as} - X_s \in dx] \\
&= \int_{\mathbb{R}^d} P[\inf_{t>c}|X_t - X_s| > a^{-H}|x| + \varepsilon]P[X_{as} - X_s \in dx] \\
&\geq P[\inf_{t>c}|X_t - X_s| > 2\varepsilon]P[|X_{as} - X_s| < a^H\varepsilon] \\
&\geq kP[|X_{as} - X_s| < a^H\varepsilon] \\
&= kP[|X_s - X_{s/a}| < \varepsilon]
\end{aligned}$$

Let

$$A = \limsup_{\Gamma\ni a\to\infty} P[\inf_{t>ac}|X_t - X_s| > a^H\varepsilon].$$

Then, the estimate above shows that

$$A \geq kP[|X_s| < \varepsilon] > 0,$$

by the assumption that $0 \in S_{X_s}$. On the other hand,

$$\begin{aligned} A &\le P[\limsup_{\Gamma \ni a \to \infty}\{\omega\colon \inf_{t > ac}|X_t - X_s| > a^H\varepsilon\}] \\ &\le P[\lim_{\Gamma \ni a \to \infty} \inf_{t > ac} |X_t - X_s| = \infty] \\ &= P[\lim_{\Gamma \ni a \to \infty} \inf_{t > ac} |X_t| = \infty] \\ &= P[\lim_{t \to \infty} |X_t| = \infty]. \end{aligned}$$

By Kolmogorov's 0-1 law it follows that $P[\lim_{t\to\infty}|X_t| = \infty] = 1$, that is, $\{X_t\}$ is transient, in contradiction to the assumption. This proves s-recurrence. □

Proof of Theorem 3.2. Let $\{X_t\}$ be a selfsimilar additive process on $\mathbb{R}^d$. Let $\mu = \mathcal{L}(X_1)$. The zero process is recurrent. So we assume that $\{X_t\}$ is non-zero with exponent H. If $0 \in S_\mu$, then, $0 \in S_{X_s}$ for any $s > 0$, since

$$P[|X_s| < \varepsilon] = P[|X_1| < s^{-H}\varepsilon] > 0 \text{ for } \varepsilon > 0$$

by virtue of $\Gamma = (0, \infty)$. Therefore, in case $0 \in S_\mu$, the process is either transient or recurrent by Theorem 3.1. If $0 \notin S_\mu$, then $\{X_t\}$ is transient, because there is $\varepsilon > 0$ such that $P[|X_1| \ge \varepsilon] = 1$, and hence

$$P[|X_t| \ge t^H\varepsilon] = P[|X_1| \ge \varepsilon] = 1$$

again by virtue of $\Gamma = (0, \infty)$. □

Proof of Theorem 3.3. Here we consider a semi-selfsimilar additive process $\{X_t\}$ on the line. We assume that $\{X_t\}$ is non-zero with exponent H. Let $\mu_t = \mathcal{L}(X_t)$. Then

$$\widehat{\mu}_t(z) = \exp\left[-\tfrac{1}{2}A_t z^2 + i\gamma_t z + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_{[-1,1]}(x))\nu_t(dx)\right]$$

with $A_t \ge 0$, $\gamma_t \in \mathbb{R}$, $\nu_t\{0\} = 0$, and $\int(1 \wedge |x|^2)\nu_t(dx) < \infty$. We have $A_s \le A_t$ and $\nu_s \le \nu_t$ for $s \le t$. Moreover, if $a \in \Gamma$, then $A_{at} = a^{2H}A_t$ and $\nu_{at} = T_{a^H}\nu_t$, where, in general, we define $(T_c\nu_t)(B) = \nu_t(c^{-1}B)$ for any Borel set B. We use the results of Tucker [16] (see [12] for exposition) on supports of infinitely divisible distributions on the line. We use also the fact that $a \in \Gamma$ implies $a^n \in \Gamma$ for all $n \in \mathbb{Z}$. We say that ν_t is one-sided if $\nu_t(0,\infty) = 0$ or $\nu_t(-\infty, 0) = 0$. We say that ν_t is two-sided if $\nu_t(0,\infty) > 0$ and $\nu_t(-\infty,0) > 0$.

If $A_1 > 0$, then $A_t > 0$ for any $t > 0$. If $\int_{|x|\le 1}|x|\nu_1(dx) = \infty$, then $\int_{|x|\le 1}|x|\nu_t(dx) = \infty$ for any $t > 0$. If ν_1 is two-sided, then ν_t is two-sided for any $t > 0$. Notice that $0 \in S_{\nu_t}$ if $\nu_t \neq 0$. Thus, in these three cases, $S_{X_t} = \mathbb{R}$ for any $t > 0$ by Tucker's result, which implies, by Theorem 3.1, that $\{X_t\}$ is either transient or recurrent.

Consider the remaining case, that is, $A_1 = 0$, $\int_{|x|\leq 1} |x|\nu_1(dx) < \infty$, $\nu_1 \neq 0$, and ν_1 is one-sided. Note that these are true also for A_t and ν_t with $t > 0$. Assume that $\nu_1(0,\infty) > 0$ and $\nu_1(-\infty,0) = 0$. The other case is treated similarly. Now

$$\widehat{\mu}_t(z) = \exp\left[\int_0^\infty (e^{izx} - 1)\nu_t(dx) + i\gamma_t^0 z\right]$$

with some $\gamma_t^0 \in \mathbb{R}$. We have $\gamma_{at}^0 = a^H \gamma_t^0$ for $a \in \Gamma$. Suppose that $\{X_t\}$ is not transient. Then it is 0-recurrent by Theorem 3.1. Given $s > 0$, let us show that $\{X_t\}$ is s-recurrent. We consider two cases: $\gamma_s^0 < 0$ and $\gamma_s^0 \geq 0$.

Case 1. $\gamma_s^0 < 0$. Let $a \in \Gamma \cap (1,\infty)$. Let $\mathbb{N}$ be the set of positive integers. We have

$$\begin{aligned}
1 &\geq \sum_{n=1}^{\infty} P[X_{a^n s} < 0,\ X_{a^{n-k} s} > 0 \text{ for } \forall k \in \mathbb{N}] \\
&\geq \sum_{n=1}^{\infty} P[X_{a^n s} < 0 < X_{a^{n-1} s},\ X_{a^{n-k+1} s} - X_{a^{n-1} s} > 0 \text{ for } \forall k \in \mathbb{N}] \\
&\geq \sum_{n=1}^{\infty} P[X_{a^n s} < 0 < X_{a^{n-1} s}]\ P[X_{a^{n+k-1} s} - X_{a^{n+1} s} > 0 \text{ for } \forall k \in \mathbb{N}].
\end{aligned}$$

Hence, using the semi-selfsimilarity, we have

$$1 \geq \sum_{n=1}^{\infty} P[X_s < 0 < X_{as}]\ P[X_{a^k s} - X_s > 0 \text{ for } \forall k \in \mathbb{N}]. \tag{3.2}$$

Notice that

$$\begin{aligned}
P[X_s < 0 < X_{as}] &\geq P[\gamma_s^0 \leq X_s < 0,\ X_{as} > X_s - \gamma_s^0] \\
&= P[\gamma_s^0 \leq X_s < 0]\ P[X_{as} - X_s > -\gamma_s^0],
\end{aligned} \tag{3.3}$$

and that $P[\gamma_s^0 \leq X_s < 0] > 0$, since Tucker's result says that $S_{X_s} = [\gamma_s^0,\infty)$ by $0 \in S_{\nu_s}$. We have $T_{a^H}\nu_s = \nu_{as} \geq \nu_s$, that is,

$$\nu_s(a^{-H}c_1, a^{-H}c_2] \geq \nu_s(c_1,c_2] \text{ for any interval } (c_1,c_2]. \tag{3.4}$$

Moreover $T_{a^H}\nu_s \neq \nu_s$. In fact, if $T_{a^H}\nu_s = \nu_s$, then the equality holds in (3.4), and hence $\nu_s(a^{nH}, a^{(n+1)H}] = \nu_s(1, a^H]$ for every $n \in \mathbb{Z}$, which, combined with $\nu_s(1,\infty) < \infty$, implies $\nu_s = 0$, contrary to our assumption. Since $X_{as} - X_s$ is infinitely divisible with Lévy measure $T_{a^H}\nu_s - \nu_s$, $S_{X_{as}-X_s}$ is unbounded above. Hence $P[X_{as} - X_s > -\gamma_s^0] > 0$. Thus it follows from (3.3) that $P[X_s < 0 < X_{as}] > 0$. Therefore, by (3.2),

$$(3.5) \qquad P[X_{a^k s} - X_s > 0 \text{ for } \forall k \in \mathbb{N}] = 0.$$

Similarly to (3.2), we have

$$(3.6) \qquad 1 \geq \sum_{n=1}^{\infty} P[X_s > 0 > X_{as}]\ P[X_{a^k s} - X_s < 0 \text{ for } \forall k \in \mathbb{N}].$$

Let $\eta > 0$. Then

$$\begin{aligned} P[X_s > 0 > X_{as}] &\geq P[\eta \geq X_s > 0,\ X_{as} < X_s - \eta] \\ &= P[\eta \geq X_s > 0]\ P[X_{as} - X_s < -\eta]. \end{aligned}$$

Since the infimum of $S_{X_{as}-X_s}$ is $(a^H - 1)\gamma_s^0$, we have

$$P[X_{as} - X_s < -\eta] > 0$$

for any sufficiently small η. On the other hand, $P[\eta \geq X_s > 0] > 0$ for any $\eta > 0$, since $S_{X_s} = [\gamma_s^0, \infty)$. Therefore $P[X_s > 0 > X_{as}] > 0$. Consequently,

$$(3.7) \qquad P[X_{a^k s} - X_s < 0 \text{ for } \forall k \in \mathbb{N}] = 0.$$

It follows from (3.5) and (3.7) that

$$\begin{aligned} P[X_{a^k s} \leq X_s \text{ for some } k \in \mathbb{N}] = 1, \\ P[X_{a^k s} \geq X_s \text{ for some } k \in \mathbb{N}] = 1. \end{aligned}$$

Since $a \in \Gamma$ can be chosen arbitrarily large, we see that

$$(3.8) \qquad P[\ \exists\, t_n \uparrow \infty \text{ in } (s,\infty) \text{ such that } X_{t_{2n}} \geq X_s \geq X_{t_{2n-1}}] = 1.$$

Since the sample functions of $\{X_t\}$ do not have downward jumps, it follows that

$$(3.9) \qquad P[\ \exists\, t'_n \uparrow \infty \text{ in } (s,\infty) \text{ such that } X_{t'_n} = X_s] = 1.$$

Hence $\{X_t\}$ is s-recurrent.

Case 2. $\gamma_s^0 \geq 0$. We claim that, for any $a \in \Gamma \cap (1,\infty)$,

$$(3.10) \qquad P[\lim_{n\to\infty} X_{a^n s} = \infty] = 1.$$

Note that $P[X_s > 0] = 1$, since $S_{X_s} = [\gamma_s^0, \infty)$ and ν_s has infinite total mass. If (3.10) is proved, then, combining it with the 0-recurrence, we obtain (3.8) again, from which (3.9) follows. The proof of (3.10) is as follows. We have

$$X_{a^n s} = X_s + \sum_{k=1}^{n} (X_{a^k s} - X_{a^{k-1} s})$$

and $X_{a^n s} - X_{a^{n-1} s}$ has Lévy measure $T_{a^{nH}} \nu_s - T_{a^{(n-1)H}} \nu_s$ and drift $(a^{nH} - a^{(n-1)H})\gamma_s^0 \geq 0$. Thus, $X_{a^n s}$ is increasing in n. Suppose that (3.10) is not true. Then, by Kolmogorov's 0-1 law,

$$P[\lim_{n\to\infty} X_{a^n s} < \infty] = 1.$$

Hence

$$P[\lim_{n\to\infty} (X_{a^n s} - X_{a^{n-1} s}) = 0] = 1.$$

Thus $\mathcal{L}(X_{a^n s} - X_{a^{n-1} s}) \to \delta_0$, the unit mass at 0. It follows that

(3.11) $$(T_{a^{nH}} \nu_s - T_{a^{(n-1)H}} \nu_s)(\varepsilon, \infty) \to 0 \text{ as } n \to \infty \text{ for } \forall \varepsilon > 0$$

by Section 19 of [5]. But

$$(T_{a^{nH}} \nu_s - T_{a^{(n-1)H}} \nu_s)(1, \infty) = \nu_s(a^{-nH}, a^{-(n-1)H}]$$

and, by (3.4),

$$\nu_s(a^{-(n+1)H}, a^{-nH}] \geq \nu_s(a^{-nH}, a^{-(n-1)H}] \text{ for } \forall n \in \mathbb{Z}.$$

Since $\nu_s \neq 0$, there is some $n \in \mathbb{Z}$ such that $\nu_s(a^{-nH}, a^{-(n-1)H}] > 0$. Hence (3.11) is not true for $\varepsilon = 1$. This is absurd. Hence (3.10) must be true. □

4. Transience and recurrence and mean sojourn times on bounded sets

A Lévy process $\{X_t : t \in [0, \infty)\}$ on $\mathbb{R}^d$ is transient if and only if

$$E\left[\int_0^\infty 1_K(X_t)dt\right] < \infty$$

for every compact set K. On the other hand, a Lévy process $\{X_t\}$ is recurrent if and only if

$$E\left[\int_0^\infty 1_D(X_t)dt\right] = \infty$$

for every open set D containing 0. Similar expression of transience and recurrence by sojourn times persists in a wide class of time-homogeneous Markov processes, although not in all irreducible time-homogeneous Markov processes, as Uchiyama's example in [7] shows. See Getoor [4] and Chung [1] for other equivalent conditions.

It should be kept in mind that the connection of transience and recurrence with finiteness and infiniteness of mean sojourn times does not exist in time-inhomogeneous Markov processes. It does not exist even in selfsimilar additive processes, as Proposition 4.1 below indicates.

It is well-known that a strictly stable process $\{X_t^{(\alpha)}\}$ on $\mathbb{R}^d$ of index α $(0 < \alpha \le 2)$ is transient or recurrent according as $\alpha < d \vee 1$ or $\alpha \ge d \vee 1$, respectively. Strictly stable processes are selfsimilar Lévy processes (see Remark 2.6). Thus any selfsimilar additive process $\{Y_t\}$ obtained from $\{X_t^{(\alpha)}\}$ through time change described in Proposition 2.3 is transient or recurrent according as $\alpha < d \vee 1$ or $\alpha \ge d \vee 1$. But finiteness and infiniteness of mean sojourn times on bounded sets for $\{Y_t\}$ depend on the time change function. The following proposition shows it in the symmetric one-dimensional case.

Proposition 4.1. *Let $\{X_t^{(\alpha)}\}$ be a non-zero symmetric stable process on $\mathbb{R}$ of index α and let $\{Y_t\}$ be a selfsimilar additive process obtained from $\{X_t^{(\alpha)}\}$ by time change, $Y_t = X_{t^\gamma}^{(\alpha)}$ with $\gamma > 0$. If $\gamma > \alpha$, then*

$$(4.1) \qquad E\left[\int_s^\infty 1_K(Y_t - Y_s)dt\right] < \infty \text{ for every compact set } K.$$

If $\gamma \le \alpha$, then

$$(4.2) \quad E\left[\int_s^\infty 1_D(Y_t - Y_s)dt\right] = \infty \text{ for every open set } D \text{ containing } 0.$$

Proof. Let

$$f_b(x) = \frac{1}{2b}\left(1 - \frac{|x|}{2b}\right) 1_{[-2b,2b]}(x)$$

with $b > 0$. The assertion (4.1) is equivalent to

$$E\left[\int_s^\infty f_b(Y_t - Y_s)dt\right] < \infty \text{ for every } b > 0,$$

while (4.2) is equivalent to

$$E\left[\int_s^\infty f_b(Y_t - Y_s)dt\right] = \infty \text{ for every } b > 0.$$

The Fourier transform $\widehat{f_b}(z)$ of $f_b(x)$ is

$$\widehat{f_b}(z) = \int_{-\infty}^\infty e^{izx} f_b(z)dx = \left(\frac{\sin bz}{bz}\right)^2.$$

We write $f_b = f$, suppressing the subscript. We have

$$E\left[\int_s^\infty f(Y_t - Y_s)dt\right] = \int_s^\infty E[f(X_{t^\gamma}^{(\alpha)} - X_{s^\gamma}^{(\alpha)})]dt.$$

Since

$$E[\exp(izX_t^{(\alpha)})] = \exp(-tc|z|^\alpha)$$

with some $c > 0$, we get

$$\begin{aligned} E[f(X_{t^\gamma}^{(\alpha)} - X_{s^\gamma}^{(\alpha)})] &= E[f(X_{t^\gamma - s^\gamma}^{(\alpha)})] \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)dx \int_{-\infty}^\infty e^{-izx} e^{-c(t^\gamma - s^\gamma)|z|^\alpha} dz \\ &= \frac{1}{\pi}\int_0^\infty \widehat{f}(z) e^{-c(t^\gamma - s^\gamma)z^\alpha} dz. \end{aligned}$$

We denote by $C_1, C_2, \dots$ positive constants. By change of variables,

$$\begin{aligned} E\left[\int_s^\infty f(Y_t - Y_s)dt\right] &= C_1 \int_0^\infty \widehat{f}(z)dz \int_s^\infty e^{-c(t^\gamma - s^\gamma)z^\alpha} dt \\ &= C_2 \int_0^\infty \widehat{f}(z) z^{-\alpha/\gamma} dz \int_0^\infty e^{-t}(t + cs^\gamma z^\alpha)^{\frac{1}{\gamma}-1} dt \\ &= I \quad \text{(say)} \end{aligned}$$

Consider two cases: $\gamma \geq 1$ and $\gamma < 1$.

Case 1. $\gamma \geq 1$. If $\alpha/\gamma < 1$, then

$$I \leq C_2 \Gamma(1/\gamma) \int_0^\infty \widehat{f}(z) z^{-\alpha/\gamma} dz < \infty.$$

Suppose $\alpha/\gamma \geq 1$. Then

$$\begin{aligned} I &\geq C_3 \int_0^\infty \widehat{f}(z) z^{-\alpha/\gamma} dz \int_0^1 (t + cs^\gamma z^\alpha)^{\frac{1}{\gamma}-1} dt \\ &\geq C_3 \int_0^\infty \widehat{f}(z) z^{-\alpha/\gamma} (1 + cs^\gamma z^\alpha)^{\frac{1}{\gamma}-1} dz = \infty. \end{aligned}$$

Case 2. $\gamma < 1$. If $\alpha/\gamma \geq 1$, then

$$I \geq C_2\Gamma(1/\gamma)\int_0^\infty \widehat{f}(z)z^{-\alpha/\gamma}dz = \infty.$$

Suppose $\alpha/\gamma < 1$. If $s = 0$, then it is easy to see that $I < \infty$. Suppose that $s > 0$. Partition the domain of integration in I with respect to z into $(0,1]$ and $(1,\infty)$. We have

$$\begin{aligned}\int_0^1 \widehat{f}(z)z^{-\alpha/\gamma}dz \int_0^\infty e^{-t}(t + cs^\gamma z^\alpha)^{\frac{1}{\gamma}-1}dt \\ \leq \int_0^1 \widehat{f}(z)z^{-\alpha/\gamma}dz \int_0^\infty e^{-t}(t + cs^\gamma)^{\frac{1}{\gamma}-1}dt \\ < \infty\end{aligned}$$

and

$$\begin{aligned}&\int_1^\infty \widehat{f}(z)z^{-\alpha/\gamma}dz \int_0^\infty e^{-t}(t + cs^\gamma z^\alpha)^{\frac{1}{\gamma}-1}dt \\ &= C_4\int_1^\infty \widehat{f}(z)z^{-\alpha}dz \int_0^\infty e^{-t}(t(cs^\gamma z^\alpha)^{-1} + 1)^{\frac{1}{\gamma}-1}dt \\ &\leq C_4\int_1^\infty \widehat{f}(z)z^{-\alpha}dz \int_0^\infty e^{-t}(t(cs^\gamma)^{-1} + 1)^{\frac{1}{\gamma}-1}dt \\ &< \infty.\end{aligned}$$

Thus, in both cases, we have proved that I is finite or infinite according as $\alpha/\gamma < 1$ or $\alpha/\gamma \geq 1$, respectively. □

5. Comparison of Lévy process and process of class L associated with a common selfdecomposable distribution

Let μ be a selfdecomposable distribution on $\mathbb{R}^d$. Let $\{X_t\}$ be the unique (in law) Lévy process such that $\mathcal{L}(X_1) = \mu$, and let $\{Y_t^{(H)}\}$ be the unique (in law) process of class L with exponent H such that $\mathcal{L}(Y_1^{(H)}) = \mu$. Unless μ is strictly stable, they are different processes (see Remark 2.6). Can we say anything about transience or recurrence of $\{Y_t^{(H)}\}$ from that of $\{X_t\}$? The answer is negative, as we show below.

PROPOSITION 5.1. *Let μ be a Gaussian distribution on $\mathbb{R}$ with non-zero mean. Let $\{X_t\}$ and $\{Y_t^{(H)}\}$ be the processes associated with μ described above. Then $\{X_t\}$ is transient, but $\{Y_t^{(H)}\}$ is recurrent.*

Proof. The Lévy process $\{X_t\}$ is transient. This is a consequence of the strong law of large numbers for $\{X_t\}$, because it has non-zero mean. For different H and H', the processes $\{Y_t^{(H)}\}$ and $\{Y_t^{(H')}\}$ are transformed to each other by nonrandom time change described in Proposition 2.3. Hence they are transient or recurrent simultaneously. Let us show recurrence of $\{Y_t^{(1/2)}\}$. We have

$$\widehat{\mu}(z) = e^{-cz^2+i\gamma z}$$

with $c > 0$ and $\gamma \neq 0$. Let $\{B_t\}$ be the Brownian motion on $\mathbb{R}$. Define

$$Y_t = B_{2ct} + t^{1/2}\gamma.$$

Clearly $\{Y_t\}$ is an additive process. Since

$$E[\exp(izY_t)] = \exp(-tcz^2 + it^{1/2}\gamma z),$$

we have

$$E[\exp(izY_{at})] = E[\exp(iza^{1/2}Y_t)] \text{ for any } a > 0.$$

It follows that $\{Y_t\}$ is self-similar with exponent $1/2$. Since $\mathcal{L}(Y_1) = \mu$, we have $\{Y_t\} \stackrel{\mathrm{d}}{=} \{Y_t^{(1/2)}\}$. The law of the iterated logarithm says that, almost surely,

$$\limsup_{t\to\infty} \frac{B_t}{(2t\log\log t)^{1/2}} = 1 \text{ and } \liminf_{t\to\infty} \frac{B_t}{(2t\log\log t)^{1/2}} = -1.$$

Hence $Y_t/(4ct\log\log t)^{1/2}$ has the upper limit and the lower limit equal to 1 and -1, respectively, as $t \to \infty$. Therefore, almost surely, there is a sequence $t_n = t_n(\omega) \uparrow \infty$ such that $Y_{t_n}(\omega) = 0$, since $Y_t(\omega)$ is continuous in t. Hence $\{Y_t^{(1/2)}\}$ is 0-recurrent. This implies the recurrence by Theorem 3.2. □

PROPOSITION 5.2. *Let μ be a selfdecomposable distribution on $\mathbb{R}$ such that*

$$\widehat{\mu}(z) = \exp\left[\int_{-\infty}^{\infty} (e^{izx} - 1)\frac{k(x)}{|x|}dx\right], \tag{5.1}$$

where $k(x)$ is nonnegative, increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$. Suppose that $0 < k(0-) < \infty$, $0 < k(0+) < \infty$, and $\int_{-\infty}^{0} k(x)dx = \int_0^{\infty} k(x)dx < \infty$. Then, the associated Lévy process $\{X_t\}$ is recurrent and the associated process $\{Y_t^{(H)}\}$ of class L is transient.

Proof is based on the following fact, which is proved in [18].

PROPOSITION 5.3. *Let μ be a selfdecomposable distribution with $\widehat{\mu}(z)$ of the form* (5.1) *with $k(x)$ being nonnegative, increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$ and satisfying $k(0-) < \infty$, $k(0+) < \infty$, and $k(0-) + k(0+) > 0$. Then the associated process $\{Y_t^{(H)}\}$ of class L is transient.*

Proof of Proposition 5.2. The process $\{Y_t^{(H)}\}$ is transient by Proposition 5.3. On the other hand, the Lévy process $\{X_t\}$ has finite mean since $\int_{|x|>1} k(x)dx < \infty$, and the mean is zero since the derivative of $\widehat{\mu}(z)$ of (5.1) vanishes at $z = 0$. Hence $\{X_t\}$ is recurrent (see [12]). □

General conditions for occurrence of such phenomena as in Propositions 5.1 and 5.2 are unknown to us.

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