J. Korean Math. Soc. **35** (1998), No. 1, pp. 191–205

ITERATIVE PROCESS WITH ERRORS FOR m-ACCRETIVE OPERATORS

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ABSTRACT. In this paper, we prove that the Mann and Ishikawa iteration sequences with errors converge strongly to the unique solution of the equation $x + Tx = f$, where T is an m-accretive operator in uniformly smooth Banach spaces. Our results extend and improve those of Chidume, Ding, Zhu and others.

1. Introduction

Let X be a real Banach space, X^* be the dual space of X and $\langle \cdot, \cdot \rangle$ be the generalized duality pairing between X and X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2, \quad \|f\| = \|x\|\}, \quad x \in X.$$

A real Banach space X is said to be uniformly smooth if its modulus of smoothness $\rho_X(\tau)$ defined by

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\}$$

for $\tau > 0$ satisfies $\frac{\rho_X(\tau)}{\tau} \to 0$ as $\tau \to 0$. It is known ([21], [31]) that X is uniformly smooth (convex) if and only if X^* is uniformly convex (smooth). If X^* is uniformly convex, then the normalized duality mapping J is single-valued and uniformly continuous on any bounded

Received October 8, 1997.

1991 Mathematics Subject Classification: 47H17, 47H06, 47H10.

Key words and phrases: m-accretive operator, the Mann and Ishikawa iteration sequences with errors.

Supported by the academic research fund of Ministry of Education, Korea, 1997, Project No. BSRI-97-1405, and the National Natural Science Foundation of China, 1988.

subset of X. Let $D(T)$ and $R(T)$ denote the domain and range of an operator T, respectively.

An operator $T : D(T) \subset X \to X$ is said to be accretive if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0.$$

An accretive operator T is said to be m-accretive if $R(T + \lambda I) = X$ for all $\lambda > 0$ (or, equivalently, for some $\lambda > 0$), where I is the identity operator.

The notion of accretive operators was introduced and studied independently by Browder ([3]) and Kato ([20]). In [3], Browder proved that if X is a Banach space and T is a locally Lipschitzian and accretive operator with $D(T) = X$, then T is m-accretive, and so, for any given $f \in X$, the equation $x + Tx = f$ has a solution. The result was generalized by Martin ([24]) to the continuous accretive operators.

Recently, many authors proved that the Mann and Ishikawa iteration sequences ([19], [23]) converge strongly to a solution of the equation $x + Tx = f$, where T is a Lipschitzian accretive operator on a Hilbert space or L_p space ([8], [9]) or T is a continuous accretive operator on uniformly smooth Banach spaces ([10], [15]) or T is a Lipschitzian accretive operator on p-uniformly smooth Banach spaces ([16], [18], [29]), which generalizes the corresponding results of Chidume ([8], [9]) or T is strongly accretive and strongly pseudo-contractive operators on uniformly smooth Banach spaces or L_p spaces ([7], [11]-[13]). In [34], Zhu proved that the Mann iteration sequence converges strongly to the unique solution of the equation $x + Tx = f$ under slightly different conditions, where $T : D(T) \subset X \to X$ is a Lipschitzian m-accretive operator and $D(T)$ is an open subset of a uniformly smooth Bnanch space. Recently, Chidume and Osilike ([14]) extended the above results to the Ishikawa iteration sequence, where T is a Lipschitzian m-accretive and $D(T)$ is a closed subset of a real Banach space which is both uniformly convex and q-uniformly smooth. Further, Liu ([22]) and others([4], [5], [6], [10], [17], [33]) proved also some convergence theorems on the Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings and monotone type mapping in Banach spaces. Very recently, Chang and Tan ([6]) extended these in uniformly smooth Banach spaces by using the new inequality which is more convenient and simple than Reich's inequality ([27]).

On the other hand, a class of operators closely related to the class of accretive operators is the class of dissipative operators. An operator $T : D(T) \subset X \to X$ is said to be dissipative ([1]) if $-T$ is accretive. The dissipative operator T is said to be m-dissipative if $R(I - \lambda T) = X$ for all $\lambda > 0$. In [2], Browder proved that T is a locally Lipschitzian dissipative operator on $D(T) = X$, then T is m-dissipative. Some related results of the equation $x - \lambda Tx = f$ are given in [8]-[10], [14] and [18], where $\lambda > 0$ and T is a Lipschitzian dissipative operator on Banach spaces.

In this paper, we prove that if X is a uniformly smooth Banach space and $T : D(T) \subset X \to X$ is m-accretive with the closed domain $D(T)$ and the bounded range $R(T)$, then the Mann and Ishikawa iteration sequences with errors converge strongly to the unique solution of the equation $x + Tx = f$. Our results extend, generalize and improve Chidume and Osilike ([14]), Ding ([17]), Zhu ([34]) and the known results mentioned above.

2. Preliminaries

In this section, we give some lemmas for our main results. In [27], Reich proved that if X^* is a uniformly convex Banach space, then there exists a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ and $j(x) \in J(x)$ such that $b(0) = 0$, $b(ct) \leq cb(t)$ for all $c \geq 1$ and

$$(2.1) \qquad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|)$$

for all $x, y \in X$. Further, Nevanlinna and Reich ([25]) proved also that, for any given continuous nondecreasing function $b(t)$ with $b(0) = 0$, there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers such that

(i) $0 < \lambda_n < 1$ for all $n = 0, 1, 2, \cdots$,
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(iii) $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$.

For $X = L_p$, $1 < p < \infty$, we can choose any sequence $\{\lambda_n\}_{n=0}^{\infty}$ in $l^s \setminus l^1$ with $s = p$ if $1 < p \leq 2$ and $s = 2$ if $p \geq 2$.

By using the results of Reich and Nevanlinna-Reich, Chidume ([8]-[10], [13]), Ding ([17]), Zeng ([32], [33]), Weng ([30]) and many authors proved some convergence theorems of the Mann and Ishikawa iteration sequences in Banach spaces.

Recently, Chang et al. ([5]-[7]) introduced an inequality which is more simple and convenient than Reich's inequality (2.1) as follows:

LEMMA 2.1. ([7]) *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any given $x, y \in X$*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

LEMMA 2.2. ([22]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1-t_n)a_n + b_n + c_n,$$

where $0 \le t_n \le 1$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

In Lemma 2.2, if we put $c_n = 0$ for all $n = 0, 1, 2, \cdots$, then we have the lemma proved by Weng ([30]).

LEMMA 2.3. ([34]) *Let X be a Banach space and $T : D(T) \subset X \to X$ be an m-accretive operator. Then, for any given $f \in X$, the equation $x + Tx = f$ has a unique solution in $D(T)$.*

3. Main Results

Now, we give our main results in this paper.

THEOREM 3.1. *Let X be a uniformly smooth Banach space and $T : D(T) \subset X \to X$ be an m-accretive operator with the closed domain $D(T)$ and the bounded range $R(T)$. Let $\{u_n\}$, $\{v_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$ such that*

(i) $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\|v_n\| \to 0$ as $n \to \infty$,

(ii) $\beta_n \to 0$ as $n \to \infty$,

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ as $n \to \infty$.

For any given $f \in X$, define $Sx = f - Tx$ for all $x \in D(T)$. If there exists $x_0 \in D(T)$ such that the sequences $\{x_n\}$, $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n + v_n \end{cases} \tag{3.1}$$

for all $n = 0, 1, 2, \cdots$ are contained in $D(T)$, then the Ishikawa iteration sequence $\{x_n\}$ with errors defined by (3.1) converges strongly to the unique solution x^* of the equation $x + Tx = f$.

Proof. Since $T : D(T) \subset X \to X$ is m-accretive, by Lemma 2.3, the equation $x + Tx = f$ has a unique solution $x^* \in D(T)$. Since $Sx^* = f - Tx^* = x^*$, the point x^* is a fixed point of S. Note that $R(S)$ is also bounded. Further, for all $x, y \in D(T)$,

$$\begin{aligned} \langle Sx - Sy, j(x-y)\rangle &= \langle f - Tx - (f - Ty), j(x-y)\rangle \\ &= -\langle Tx - Ty, j(x-y)\rangle \\ &\leq 0. \end{aligned} \tag{3.2}$$

Since $R(S)$ is bounded, let

$$d = \sup\{\|Sx - x^*\| : x \in D(T)\} + \|x_1 - x^*\|, \tag{3.3}$$

$$M = d + \sum_{n=0}^{\infty} \|u_n\| + 1. \tag{3.4}$$

By induction, we can prove

$$\|x_{n+1} - x^*\| \leq d + \sum_{j=1}^{n} \|u_j\| \leq M \tag{3.5}$$

for all $n = 0, 1, 2, \cdots$. In fact, if $n = 0$, then, from (3.3) and (3.4), it follows that (3.5) is true. If $n = 1$, then we have

$$\begin{aligned} \|x_2 - x^*\| &= \|(1-\alpha_1)(x_1 - x^*) + \alpha_1(Sy_1 - x^*) + u_1\| \\ &\leq (1-\alpha_1)\|x_1 - x^*\| + \alpha_1\|Sy_1 - x^*\| + \|u_1\| \\ &\leq d + \|u_1\| \leq M \end{aligned}$$

and so (3.5) is also true. Assume that (3.5) is true for $n = k - 1$. Then we have

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|(1-\alpha_k)(x_k - x^*) + \alpha_k(Sy_k - x^*) + u_k\| \\ &\leq (1-\alpha_k)\|x_k - x^*\| + \alpha_k\|Sy_k - x^*\| + \|u_k\| \\ &\leq (1-\alpha_k)\{d + \sum_{j=1}^{k-1}\|u_j\|\} + \alpha_k d + \|u_k\| \\ &= d + \sum_{j=1}^{k}\|u_j\| \leq M. \end{aligned}$$

Therefore, (3.5) is true for all $n = 0, 1, 2, \cdots$. By (3.1) and Lemma 2.1, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*) + u_n\|^2 \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle Sy_n - x^*, j(x_{n+1} - x^* - u_n)\rangle \\
&\quad + 2\langle u_n, j(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{3.6}
$$

Now, consider the third term of (3.6):

$$
2\langle u_n, j(x_{n+1} - x^*)\rangle \le 2\|u_n\|\|x_{n+1} - x^*\| \le 2\|u_n\|M. \tag{3.7}
$$

Next, consider the second term of (3.6):

$$
\begin{aligned}
&\langle Sy_n - x^*, j(x_{n+1} - x^* - u_n)\rangle \\
&\quad = \langle Sy_n - x^*, j(x_n - x^*)\rangle \\
&\qquad + \langle Sy_n - x^*, j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\rangle \\
&\quad \le \langle Sy_n - x^*, j(x_n - x^*)\rangle \\
&\qquad + |\langle Sy_n - x^*, j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\rangle| \\
&\quad = d_n + e_n,
\end{aligned} \tag{3.8}
$$

where

$$
d_n = \langle Sy_n - x^*, j(x_n - x^*)\rangle,
$$

$$
e_n = |\langle Sy_n - x^*, j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\rangle|.
$$

From (3.2)-(3.4), it follows that

$$
\begin{aligned}
d_n &= \langle Sy_n - x^*, j(x_n - x^*)\rangle \\
&= \langle Sy_n - x^*, j(y_n - x^*)\rangle \\
&\quad - \langle Sy_n - x^*, j(y_n - x^*) - j(x_n - x^*)\rangle \\
&\le |\langle Sy_n - x^*, j(y_n - x^*) - j(x_n - x^*)\rangle| \\
&\le \|Sy_n - x^*\|\|j(y_n - x^*) - j(x_n - x^*)\| \\
&\le d \cdot \|j(y_n - x^*) - j(x_n - x^*)\|
\end{aligned} \tag{3.9}
$$

and, as $n \to \infty$,

$$\begin{aligned}\|y_n - x^* - (x_n - x^*)\| &= \|y_n - x_n\| \\ &= \|\beta_n(Sx_n - x_n) + v_n\| \\ &\le \beta_n\{\|Sx_n - x^*\| + \|x_n - x^*\|\} + \|v_n\| \\ &\le 2\beta_n M + \|v_n\| \to 0.\end{aligned}$$

Since X is uniformly smooth, the normalized duality mapping j is uniformly continuous on any bounded subset of X and so, as $n \to \infty$,

$$\|j(y_n - x^*) - j(x_n - x^*)\| \to 0.$$

Now, we prove $e_n \to 0$ as $n \to \infty$. In fact, we have

$$\begin{aligned}e_n &= |\langle Sy_n - x^*, j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\rangle| \\ &\le \|Sy_n - x^*\|\|j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\| \\ &\le d\|j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\|\end{aligned}$$

and, as $n \to \infty$,

$$\begin{aligned}\|x_{n+1} - x^* - u_n - (x_n - x^*)\| &= \|x_{n+1} - x_n - u_n\| \\ &= \alpha_n\|Sy_n - x_n\| \\ &\le \alpha_n\{\|Sy_n - x^*\| + \|x_n - x^*\|\} \\ &\le 2\alpha_n M \to 0.\end{aligned}$$

Thus, by the uniform continuity of j,

$$\|j(x_{n+1} - x^* - u_n) - j(x_n - x^*)\| \to 0$$

and so $e_n \to 0$ as $n \to \infty$. From (3.6)-(3.8), it follows that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(k_n + e_n) + 2\|u_n\|M \qquad (3.10)\\ &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n(k_n + e_n) + 2\|u_n\|M,\end{aligned}$$

where k_n is defined by

$$k_n = d \cdot \|j(y_n - x^*) - j(x_n - x^*)\|$$

and so $k_n \to 0$ as $n \to \infty$. Taking $\|x_n - x^*\|^2 = a_n$, $\alpha_n = t_n$, $2\alpha_n(k_n + e_n) = b_n$ and $2\|u_n\|M = c_n$, the inequality (3.10) reduces to

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

By the conditions (i)-(iii), it is easy to see that

$$\sum_{n=0}^{\infty} t_n = \infty, \quad b_n = o(t_n), \quad \sum_{n=0}^{\infty} c_n < \infty$$

and so, by Lemma 2.2, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - x^*\|^2 = 0$, i.e., the sequence $\{x_n\}$ defined by (3.1) converges strongly to the unique solution x^* of the equation $x + Tx = f$. This completes the proof. □

REMARK 1. Theorem 3.1 improves Theorem 3.1 in Ding [17] in several ways, i.e., in our theorem, we need not the conditions of the uniform convexity of X and $\lim_{n\to\infty} \alpha_n b(\alpha_n) = 0$. In [17], also Ding used Lemma 2.4 ([28]) to prove that the Ishikawa iteration sequence defined by him newly converges strongly to the unique solution of the equation $x + Tx = f$. By using Lemma 2.1, our proof of Theorem 3.1 is more simple and easier than the proof in Theorem 3.1 [17].

COROLLARY 3.2. *Let $X, T, D(T)$ be the same as in* Theorem 3.1. *Let $\{u_n\}$ be a sequence in X and $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that*

(i) $\sum_{n=0}^{\infty} \|u_n\| < \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ *as* $n \to \infty$.

For any given $f \in X$, define $Sx = f - Tx$ for all $x \in D(T)$. If there exists $x_0 \in D(T)$ such that the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S x_n + u_n \tag{3.11}$$

for all $n = 0, 1, 2, \cdots$ is contained in $D(T)$, then the Mann iteration sequence $\{x_n\}$ with errors defined by (3.11) converges strongly to the unique solution of the equation $x + Tx = f$.

Proof. Taking $v_n = 0$ and $\beta_n = 0$ for all $n = 0, 1, 2, \cdots$ in Theorem 3.1, we have the conclusion. □

REMARK 2. Corollary 3.2 improves and generalizes Chidume and Osilike [14, Corollary 1, Theorem 5], Zhu [34, Theorem 3], Zeng [33, Theorems 1 and 2] and many others.

THEOREM 3.3. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a continuous accretive operator with the bounded range $R(T)$. Let $\{u_n\}$, $\{v_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ satisfying the conditions* (i)-(iii) *in* Theorem 3.1. *For any given $f \in X$, define $Sx = f - Tx$ for all $x \in X$. Then for any $x_0 \in X$, the Ishikawa iteration sequence $\{x_n\}$ with errors defined by* (3.1) *converges strongly to the unique solution of the equation $x + Tx = f$.*

Proof. By the result of Martin [24], since T is continuous accretive, T is m-accretive and so the equation $x + Tx = f$ has a unique solution $x^* \in X$. Therefore, the conclusion follows as in the proof of Theorem 3.1 □

COROLLARY 3.4. *Let X and T be the same as in* Theorem 3.3. *Let $\{u_n\}$ be a sequence in X and $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that*

(i) $\sum_{n=0}^{\infty} \|u_n\| < \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$ *as* $n \to \infty$.

For any given $f \in X$, define $Sx = f - Tx$ for all $x \in X$. Then for any $x_0 \in X$, the Mann iteration sequence $\{x_n\}$ with errors defined by (3.11) *converges strongly to the unique solution of the equation $x + Tx = f$.*

Proof. Taking $v_n = 0$ and $\beta_n = 0$ for all $n = 0, 1, 2, \cdots$ in Theorem 3.3, we have the conclusion. □

REMARK 3. Theorem 3.3 and Corollary 3.4 improve and generalize Theorem 4, Theorem 6 and Corollary 2 in Chidume and Osilike ([14]) and many others.

Next, we prove some convergence theorems for dissipative operators, i.e., some theorems on the approximation of a unique solution of the equation $x - Tx = f$, where $T : D(T) \subset X \to X$ is an m-dissipative operator.

THEOREM 3.5. *Let X be a uniformly smooth Banach space and $T : D(T) \subset X \to X$ be an m-dissipative operator with the bounded range $R(T)$. Let $\{u_n\}$, $\{v_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$ satisfying the condition* (i)-(iii) *in* Theorem 3.1. *For any given $f \in X$, define $Sx = f + Tx$ for all $x \in D(T)$. If there exists $x_0 \in D(T)$ such that the sequences $\{x_n\}$, $\{y_n\}$ defined by (3.1) are contained in $D(T)$, then the Ishikawa iteration sequence $\{x_n\}$ with errors defined by (3.1) converges strongly to the unique solution of the equation $x - Tx = f$.*

Proof. Since T is an m-dissipative operator with the bounded range $R(T)$, $-T$ is an m-accretive operator with the bounded range $R(-T)$ and so the result follows from Theorem 3.1. □

COROLLARY 3.6. *Let X and T be the same as in* Theorem 3.5. *Let $\{u_n\}$ be a sequence in X and $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the conditions* (i) *and* (ii) *in* Corollary 3.2. *For any given $f \in X$, define $Sx = f + Tx$ for all $x \in D(T)$. If there exists $x_0 \in D(T)$ such that the sequence $\{x_n\}$ defined by (3.11) is contained in $D(T)$, then the Mann iteration sequence $\{x_n\}$ with errors defined by (3.11) converges strongly to the unique solution of the equation $x - Tx = f$.*

Proof. The conclusion follows from Theorem 3.5 with $v_n = 0$ and $\beta_n = 0$ for $n = 0, 1, 2, \cdots$. □

Finally, we give also some convergence theorems of the Mann iteration sequence for m-accretive and locally Lipschitzian operators in real Banach spaces.

THEOREM 3.7. *Let X be a real Banach space, $T : D(T) \subset X \to X$ be an m-accretive and locally Lipschitzian operator with the local Lipschitz constant $L \geq 1$ of T. Suppose that $D(T)$ is open and x^* is the unique solution of the equation $x + Tx = f$ for all $f \in X$ and $x \in D(T)$ and that $\{\alpha_n\}$ is a real sequence such that*

(i) $0 \leq \alpha_n \leq \frac{1}{2(1+L)^2}$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If there exists a closed convex neighborhood B of x^* contained in $D(T)$ and a point $x_0 \in B$ such that T is Lipschitzian on B and the *sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Tx_n) \tag{3.12}$$

for all $n = 0, 1, 2, \cdots$ *is contained in* B, *then the sequence* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the equation* $x + Tx = f$ *and we have the following error estimate:*

$$\|x_n - x^*\| \leq \exp(-\frac{1}{2}\sum_{j=0}^{n}\alpha_j)\|x_0 - x^*\|.$$

Proof. Define $S : D(T) \subset X \to X$ by $Sx = f - Tx$ for all $x \in D(T)$. It is obvious that x^* is a fixed point of S, S is also a locally Lipschitzian operator with the local Lipschitz constant $L \geq 1$ on B and $(-S)$ is accretive on $D(T)$, i.e., for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \leq 0.$$

From (3.12) and Lemma 2.1, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|(1-\alpha_n)(x_n - x^*) - \alpha_n(Sx_n - x^*)\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Sx_n - x^*, j(x_{n+1} - x^*)\rangle
\end{aligned} \tag{3.13}$$

for all $j(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$. Since $(-S)$ is accretive, there exists $\tilde{j}(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$ such that

$$\langle Sx_{n+1} - x^*, \tilde{j}(x_{n+1} - x^*)\rangle \leq 0. \tag{3.14}$$

Thus, from (3.13) and (3.14), it follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle Sx_n - Sx_{n+1} - Sx_{n+1} - x^*, \tilde{j}(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle Sx_n - Sx_{n+1}, \tilde{j}(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|Sx_n - Sx_{n+1}\|\|x_{n+1} - x^*\| \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n L\|x_n - x_{n+1}\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{3.15}$$

On the other hand, we have

$$\tag{3.16} \begin{aligned} \|x_n - x_{n+1}\| &= \alpha_n \|x_n - Sx_n\| \\ &\le \alpha_n\{\|x_n - x^*\| + \|x^* - Sx_n\|\} \\ &\le \alpha_n(1 + L)\|x_n - x^*\| \end{aligned}$$

and, from (3.16),

$$\tag{3.17} \begin{aligned} \|x_{n+1} - x^*\| &= \|x_{n+1} - x_n + x_n - x^*\| \\ &\le \|x_{n+1} - x_n\| + \|x_n - x^*\| \\ &\le \{\alpha_n(1 + L) + 1\}\|x_n - x^*\|. \end{aligned}$$

Substituting (3.16) and (3.17) into (3.15), we have, by the condition (i),

$$\tag{3.18} \begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\le \{(1-\alpha_n)^2 + \alpha_n L(2\alpha_n^2(1+L)^2 + 2\alpha_n(1+L))\}\|x_n - x^*\|^2 \\ &\le \{(1-\alpha_n)^2 + \alpha_n L[\alpha_n + 2\alpha_n(1+L)]\}\|x_n - x^*\|^2 \\ &= \{(1-\alpha_n) + \alpha_n[\alpha_n(1+3L+2L^2) - 1]\}\|x_n - x^*\|^2 \\ &\le (1-\alpha_n)\|x_n - x^*\|^2 \end{aligned}$$

for $n = 0, 1, 2, \cdots$. By induction and (3.18), we can prove

$$\tag{3.19} \|x_{n+1} - x^*\|^2 \le \exp(-\sum_{j=0}^{n} \alpha_j)\|x_0 - x^*\|^2$$

for $n = 0, 1, 2, \cdots$. Hence, by the condition (ii), we have

$$\|x_{n+1} - x^*\| \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., $x_n \to x^*$ as $n \to \infty$. From (3.19), we obtain the error estimate

$$\|x_{n+1} - x^*\| \le \exp(-\frac{1}{2}\sum_{j=0}^{n} \alpha_j)\|x_0 - x^*\|.$$

This completes the proof. □

REMARK 4. Theorem 3.7 improves and extends Osilike [26, Theorem 1], Chidume and Osilike [14], Zhu [34] and others.

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