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WEIGHTED BLOCH SPACES IN $\mathbb{C}^n$

KYONG TAIK HAHN AND KI SEONG CHOI

ABSTRACT. In this paper, weighted Bloch spaces $\mathcal{B}_q$ $(q > 0)$ are considered on the open unit ball in $\mathbb{C}^n$. These spaces extend the notion of Bloch spaces to wider classes of holomorphic funcitons. It is proved that the functions in a weighted Bloch space admit certain integral representation. This representation formula is then used to determine the degree of growth of the functions in the space $\mathcal{B}_q$. It is also proved that weighted Bloch space is a Banach space for each weight $q > 0$, and the little Bloch space $\mathcal{B}_{q,0}$ associated with $\mathcal{B}_q$ is a separable subspace of $\mathcal{B}_q$ which is the closure of the polynomials for each $q \geq 1$.

1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$. The Bloch space of D consists of analytic functions f on D such that

$$\sup\{(1 - |z|^2)|f'(z)| \mid z \in D\} < +\infty.$$

The functions in the Bloch space on the unit disk in the complex plane are well known and have been studied by many authors [1, 2].

In this paper, we will consider Bloch type functions on the open unit ball B in the complex n-space $\mathbb{C}^n$. The Bergman metric (on B) $b_B : B \times \mathbb{C}^n \longrightarrow R$ is given by

$$b_B^2(z,\xi) = \frac{n+1}{(1 - \| z \|^2)^2}[(1 - \| z \|^2)\| \xi \|^2 + | < z, \xi > |^2].$$

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Let $f \in C^1(B)$ and $\xi \in \mathbb{C}^n$. The maximal derivative of f with respect to the Bergman metric b_B is defined by

$$\hat{Q}f(z) = \sup_{|\xi|=1} \frac{|df(z)\cdot\xi|}{b_B(z,\xi)},\ z \in B$$

where

$$df(z)\cdot\xi = \sum_{i=1}^{n} [\frac{\partial f(z)}{\partial z_i}\xi_i + \frac{\partial f(z)}{\partial \overline{z}_i}\overline{\xi}_i].$$

The quantity $\hat{Q}f$ is invariant under the group $Aut(B)$ of holomorphic automorphisms of B. Namely, $\hat{Q}(f\circ\varphi) = (\hat{Q}f)\circ\varphi$ for all $\varphi \in Aut(B)$. If $f \in H(B)$ where $H(B)$ is the set of holomorphic functions on B, then the quantity $\hat{Q}f$ is reduced to

$$Qf(z) = \sup_{|\xi|=1} \frac{|\nabla f(z)\cdot\xi|}{b_B(z,\xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \quad , \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f.

A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function if

$$\sup_{z\in B} Qf(z) < \infty.$$

Bloch functions on bounded homogeneous domains were first studied by K. T. Hahn[5]. In [8], Timoney showed that the linear space of all holomorphic functions $f : B \to C$ which satisfy

$$\sup_{z\in B}(1 - \| z \|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space $\mathcal{B}$ of Bloch functions on B.

The little Bloch space $\mathcal{B}_0$ is the subspace of $\mathcal{B}$ consisting of those functions $f : B \to \mathbb{C}$ which satisfy:

$$\lim_{\|z\|\to 1}(1 - \| z \|^2) \| \nabla f(z) \| = 0.$$

In this paper, we introduce the Weighted Bloch Space $\mathcal{B}_q (q > 0)$ on the open unit ball B in $\mathbb{C}^n$ which extend the notion of Bloch space $\mathcal{B}$ to larger classes of holomorphic functions on B.

For each $q > 0$, let $\mathcal{B}_q$ denote the space of holomorphic functions $f : B \to \mathbb{C}$ which satisfy:

$$\sup_{z \in B}(1 - \| z \|^2)^q \| \nabla f(z) \| < \infty.$$

The corresponding little Bloch space $\mathcal{B}_{q,0}$ is defined by the functions f in $\mathcal{B}_q$ such that

$$\lim_{\|z\| \to 1}(1 - \| z \|^2)^q \| \nabla f(z) \| = 0.$$

Clearly, both $\mathcal{B}_q$ and $\mathcal{B}_{q,0}$ are increasing function spaces of $q > 0$. In particular, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_{1,0} = \mathcal{B}_0$.

In §2, we prove certain integral representation theorems (See Theorem 2 and Theorem 3) for the functions in $\mathcal{B}_q$ for $q > 0$. The space $\mathcal{B}_q$ is a Banach space with respect to the norm, as defined in §3, for each $q > 0$ (See Theorem 4), and for $q > 1$, $\mathcal{B}_q$ can be identified with the space of holomorphic functions f with the conditions:

$$\sup\{(1 - \| z \|^2)^{q-1} |f(z)| \mid z \in B\} < \infty.$$

These results are given in §3. In §4, it is shown that the weighted little Bloch space $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_q$ for each $q \geq 1$.

2. Integral representations on the space $\mathcal{B}_q$

Let $a \in B$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{< z, a >}{< a, a >} a, \quad if \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - < z, a >}.$$

It is easily shown that the mapping φ_a belongs to $Aut(B)$ where $Aut(B)$ is the group of all biholomorphic mappings of B onto itself, and satisfies,

$$\varphi_a(0) = a, \varphi_a(a) = 0 \quad and \quad \varphi_a(\varphi_a(z)) = z.$$

Furthermore, for all $z, w \in \overline{B}$, we have

$$1 - < \varphi_a(z), \varphi_a(w) > = \frac{(1 - \| a \|^2)(1 - < z, w >)}{(1 - < z, a >)(1 - < a, w >)}.$$

In particular, for $a \in B, z \in \overline{B}$,

$$1 - \| \varphi_a(z) \|^2 = \frac{(1 - \| a \|^2)(1 - \| z \|^2)}{|1 - < z, a > |^2}$$

[See [7] Theorem 2.2.2].

THEOREM 1. *Let ψ be a biholomorphic mapping of B onto itself and $a = \psi^{-1}(0)$. The determinant $J_R\psi$ of the real Jacobian matrix of ψ satisfies the following identity:*

$$J_R\psi(z) = |J\psi(z)|^2 = \left(\frac{1 - \| a \|^2}{|1 - < z, a > |^2} \right)^{n+1} = \left(\frac{1 - \| \psi(z) \|^2}{1 - |z|^2} \right)^{n+1}.$$

Proof. See [7] Theorem 2.2.6. □

Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. Let σ be the rotation invariant surface measure on S normalized by $\sigma(S) = 1$. The measure μ_q is the weighted Lebesgue measure:

$$d\mu_q = c_q(1 - \| z \|^2)^q d\nu(z)$$

where $q > -1$ is fixed, and c_q is a normalization constant such that $\mu_q(B) = 1$.

THEOREM 2. *If* $f \in L^1_{\mu_q}(B) \cap H(B)$, $q > 0$, *then*

$$f(z) = c_q \int_B \frac{(1 - \| w \|^2)^q}{(1 - < z, w >)^{n+q+1}} f(w) d\nu(w).$$

Proof. Since $f \in H(B)$, by the mean value theorem,

$$f(0) = \int_S f(r\zeta) d\sigma(\zeta), \quad 0 < r < 1. \tag{1}$$

By integrating both side of (1) with respect to the measure $2n(1 - r^2)^q r^{2n-1} dr$ over [0, 1], we have

$$2n \int_0^1 \int_S f(r\zeta)(1 - r^2)^q r^{2n-1} d\sigma(\zeta) dr = f(0) c_q^{-1}.$$

Namely,

$$f(0) = c_q \int_B f(w)(1 - \| w \|^2)^q d\nu(w).$$

Replace f by $f \circ \varphi_z$ and apply Theorem 1, then

$$\begin{aligned}
& f(z) \\
&= c_q \int_B f(w)(1 - \| \varphi_z(w) \|^2)^q \left(\frac{(1 - \| z \|^2)}{|1 - < w, z > |^2} \right)^{n+1} d\nu(w) \\
&= c_q \times \\
& \int_B f(w) \left(\frac{(1 - \| z \|^2)(1 - \| w \|^2)}{|1 - < w, z > |^2} \right)^q \left(\frac{(1 - \| z \|^2)}{|1 - < w, z > |^2} \right)^{n+1} d\nu(w) \\
&= c_q (1 - \| z \|^2)^{n+q+1} \int_B f(w) \frac{(1 - \| w \|^2)^q}{|1 - < w, z > |^{2(n+q+1)}} d\nu(w) \\
&= c_q (1 - \| z \|^2)^{n+q+1} \times \\
& \int_B f(w) \frac{(1 - \| w \|^2)^q}{(1 - < w, z >)^{n+q+1} (1 - < z, w >)^{n+q+1}} d\nu(w).
\end{aligned}$$

Replace $f(w)$ again by $f(w)(1-<w,z>)^{n+q+1}$,

$$f(z)(1-\| z \|^2)^{n+q+1}$$

$$= c_q(1-\| z \|^2)^{n+q+1} \int_B f(w) \frac{(1-\| w \|^2)^q}{(1-<z,w>)^{n+q+1}} d\nu(w).$$

$$f(z) = c_q \int_B \frac{(1-\| w \|^2)^q}{(1-<z,w>)^{n+q+1}} f(w) d\nu(w).$$

□

THEOREM 3. *Suppose* $q>0, z \in B$, *and* $f \in \mathcal{B}_q$. *Then*

$$f(z) = f(0) + \frac{c_q}{n+q} \int_B \frac{(1-\| w \|^2)^q \nabla f(w) \cdot z}{<z,w>(1-<z,w>)^{n+q}} d\nu(w).$$

Proof. Taking the line integral from 0 to z, we get

$$f(z) - f(0) = \int_0^1 \nabla f(tz) \cdot z dt.$$

Applying Theorem 2 to a holomorphic function $\nabla f(w) \cdot z, z \in B$, and integrating $\nabla f(\zeta) \cdot z, \zeta = tz$, in t over [0, 1], we get

$$\int_0^1 \nabla f(\zeta) \cdot z dt$$

$$= \int_0^1 c_q \int_B \frac{(1-\| w \|^2)^q \nabla f(w) \cdot z}{(1-<\zeta,w>)^{n+q+1}} d\nu(w) dt,$$

$$= c_q \int_B (1-\| w \|^2)^q \nabla f(w) \cdot z \left[\int_0^1 \frac{1}{(1-t<z,w>)^{n+q+1}} dt \right] d\nu(w)$$

$$= c_q \int_B (1-\| w \|^2)^q \nabla f(w) \cdot z \times$$

$$\left[\frac{1}{(n+q)<z,w>(1-<z,w>)^{n+q}} - \frac{1}{(n+q)<z,w>} \right] d\nu(w)$$

$$= \frac{c_q}{n+q} \int_B (1-\| w \|^2)^q \nabla f(w) \cdot z \frac{1}{<z,w>(1-<z,w>)^{n+q}} d\nu(w)\cdot$$

$$- \frac{c_q}{n+q} \int_B (1-\| w \|^2)^q \nabla f(w) \cdot z \frac{1}{<z,w>} d\nu(w).$$

It can be seen that the second integral vanishes. Indeed, let $w = r\eta (0 < r < 1, \eta \in S)$ be the polar form of $w \in B$. Then

$$\int_B \frac{(1- \| w \|^2)^q \nabla f(w) \cdot z}{< z, w >} d\nu(w)$$

$$= \int_0^1 \int_S \frac{(1-r^2)^q \nabla f(r\eta) \cdot z}{r < z, \eta >} 2nr^{2n-1} dr d\eta$$

$$= 2n \int_0^1 (1-r^2)^q r^{2n-2} dr \int_S \frac{\nabla f(r\eta) \cdot z}{< z, \eta >} d\sigma(\eta).$$

By the homogeneous polynomial expansion of a holomorphic function,

$$\nabla f(r\eta) \cdot z = \sum_{k=0}^{\infty} P_k(r\eta) \cdot z.$$

By [7, proposition 1.4.7 (1)],

$$\int_S \frac{\nabla f(r\eta) \cdot z}{< z, \eta >} d\sigma(\eta) = \int_S d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\nabla f(r\zeta e^{i\theta}) \cdot z.}{< z, \zeta e^{i\theta} >} d\theta$$

$$= \int_S \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P_k(r\zeta) \cdot z}{< z, \zeta >} e^{(k+1)i\theta} d\theta d\sigma(\zeta).$$

But the inner integrals vanish for all $\zeta \in S$ and all $k = 0, 1, 2, \cdots$ by the periodicity of exponential function. Namely,

$$\frac{c_q}{n+q} \int_B (1 - \| w \|^2)^q \nabla f(w) \cdot z \frac{1}{< z, w >} d\nu(w) = 0$$

and, we have

$$f(z) - f(0) = \frac{c_q}{n+q} \int_B \frac{(1 - \| w \|^2)^q \nabla f(w) \cdot z}{< z, w > (1- < z, w >)^{n+q}} d\nu(w). \qquad \square$$

3. Some properties of weighted Bloch spaces

Let us define norm on $\mathcal{B}_q$ as follows;

$$\| f \|_q = |f(0)| + \sup\{(1 - \| w \|^2)^q \| \nabla f(w) \| \mid w \in B\}.$$

LEMMA 1. *If $f \in \mathcal{B}_q$, $q > 0$, then*

$$|f(z)| \leq |f(0)| + \| f \|_q (1 - \| z \|^2)^{-q}.$$

Proof.

$$f(z) - f(0) = \int_0^1 \nabla f(tz) \cdot z dt.$$

$$\begin{aligned} |f(z) - f(0)| &\leq \int_0^1 \| \nabla f(tz) \| \| z \| dt \\ &\leq \int_0^1 \frac{\| \nabla f(tz) \| (1 - \| tz \|^2)^q}{(1 - \| tz \|^2)^q} dt \\ &\leq \| f \|_q \int_0^1 \frac{1}{(1 - t^2 \| z \|^2)^q} dt \\ &\leq \| f \|_q \frac{1}{(1 - \| z \|^2)^q}. \end{aligned}$$

□

THEOREM 4. *For each $q > 0$, $\mathcal{B}_q$ is a Banach space with norm $\| \|_q$.*

Proof. Let (f_n) be a Cauchy sequence in $\mathcal{B}_q$. By Lemma 1,

$$|(f_n - f_m)(z) - (f_n - f_m)(0)| \leq M \| f_n - f_m \|_q (1 - \| z \|^2)^{-q}.$$

It follows that the sequence (f_n) is a Cauchy sequence in the topology of uniform convergence on compact sets. Thus there exists holomorphic function $f : B \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of B as $n \to \infty$.

Since $f_n \to f$ uniformly on compact subsets of B as $n \to \infty$, it follows that $\nabla f_n(z) \to \nabla f(z)$ uniformly on compact subsets of B as $n \to \infty$.

Thus, for each n, as $m \to \infty$

$$(1-\| z \|^2)^q \| \nabla(f_n - f_m)(z) \| \to (1-\| z \|^2)^q \| \nabla(f_n - f)(z) \|$$

for each $z \in B$. Therefore, for all $n \geq N$

$$(1-\| z \|^2)^q \| \nabla(f_n - f)(z) \| \leq \epsilon.$$

Namely, $\| f_n - f \|_q \leq \epsilon$. □

THEOREM 5. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1-\| w \|^2)^t}{|1-<z,w>|^{n+1+c+t}} d\nu(w), \quad z \in B$$

we have

(i) *$I_{c,t}(z)$ is bounded in B if $c < 0$;*
(ii) *$I_{0,t}(z) \sim -log(1-\| z \|^2)$ as $\| z \| \to 1^-$;*
(iii) *$I_{c,t}(z) \sim (1-\| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.*

Proof. See [7 Proposition 1.4.10]. □

THEOREM 6. *Suppose $q > 1$. Then f is in $\mathcal{B}_q$ if and only if $(1-\| z \|^2)^{q-1}|f(z)|$ is bounded on B.*

Proof. First assume that f is in $\mathcal{B}_q$. By Theorem 3,

$$f(z) = f(0) + \frac{c_q}{n+q} \int_B \frac{(1-\| w \|^2)^q \nabla f(w) \cdot z}{<z,w>(1-<z,w>)^{n+q}} d\nu(w).$$

It follows that

$$|f(z) - f(0)| \leq \frac{c_q}{n+q} \| f \|_q \int_B \frac{\| z \|}{|<z,w>||1-<z,w>|^{n+q}} d\nu(w).$$

The factor $|<z,w>|$ in the denominator does not change the growth rate of the integral for z near the boundary. Thus Theorem 5 implies that there is a constant $C > 0$ such that

$$|f(z) - f(0)| \leq C \| f \|_q (1-\| z \|^2)^{-(q-1)}, \ z \in B.$$

This shows that $(1-\| z \|^2)^{q-1} f(z)$ is bounded on B.

Conversely, if $(1-\| z \|^2)^{q-1}|f(z)| \leq M$ for some constant $M > 0$, then

$$f(z) = c_q \int_B \frac{(1-\| w \|^2)^{q-1}}{(1-<z,w>)^{n+q}} f(w) d\nu(w)$$

by Theorem 2.

Differentiating under the integral sign, we obtain

$$\nabla f(z)$$

$$= c_q \int_B \frac{(n+q)(1-<z,w>)^{n+q-1}(-\bar{w})(1-\| w \|^2)^{q-1} f(w)}{(1-<z,w>)^{2(n+q)}} d\nu(w).$$

$$\| \nabla f(z) \| \leq c_q(n+q)M \int_B \frac{1}{|1-<z,w>|^{n+q+1}} d\nu(w).$$

By Theorem 5 , there exists a constant $C > 0$ such that

$$\| \nabla f(z) \| \leq CM(1-\| z \|^2)^{-q}$$

for all $z \in B$. This clearly shows that f is in $\mathcal{B}_q$. □

4. Little Bloch space

LEMMA 2. *$f_n \in \mathcal{B}_{q,0}, f \in \mathcal{B}_q$ and $\| f_n - f \|_q \to 0$ if and only if*
(1) $f_n(z) \to f(z)$ as $n \to \infty$, locally uniformly in B.
(2) $(1-\| z \|^2)^q \| \nabla f_n(z) \| \to 0$ as $|z| \to 1$, uniformly in n.

Proof. Applying Lemma 1 to $f_n - f$, we get

$$|(f_n - f)(z)| \leq |(f_n - f)(0)| + \| f_n - f \|_q \frac{1}{(1-r^2)^q}, \quad |z| \leq r.$$

Since $\| f_n - f \|_q \to 0, \sup_{\| z \| \leq r} |f_n(z) - f(z)| \leq \epsilon$.
Thus, $f_n(z) \to f(z)$ as $n \to \infty$, locally uniformly in B which proves (1).
Since $\| f_n - f \|_q \to 0$, there exists $n_o(\epsilon)$ such that if $m, n \geq n_o(\epsilon)$, then $\| f_n - f_m \|_q \leq \epsilon$. Since $(1-\| z \|^2)^q \| \nabla(f_n - f_{n_o})(z) \| \leq \| f_n - f_{n_o} \|_q$,

$$(1-\| z \|^2)^q \| \nabla f_n(z) \| \leq (1-\| z \|^2)^q \| \nabla f_{n_o}(z) \| + \| f_n - f_{n_o} \|_q$$

$$\leq (1-\| z \|^2)^q \| \nabla f_{n_o}(z) \| - \epsilon.$$

Since $f_{n_o} \in \mathcal{B}_{q,0}$, we deduce that, for some $\varrho < 1$,

$$(1-\| z \|^2)^q \| \nabla f_n(z) \| < 2\epsilon \quad (n > n_0 \ \varrho < |z| < 1).$$

Thus

$$(1-\| z \|^2)^q \| \nabla f_n(z) \| \rightarrow 0$$

as $|z| \rightarrow 1$, uniformly in n.
Conversely, by (2), $f_n \in \mathcal{B}_{q,0}$. Furthermore, $\nabla f_n(z) \rightarrow \nabla f(z) \quad (n \rightarrow \infty)$ for each $z \in B$ by (1). Together with (2), this shows that

$$(1-\| z \|^2)^q \| \nabla f(z) \| \rightarrow 0 \quad (|z| \rightarrow 1).$$

Therefore, $f \in \mathcal{B}_{q,0}$.
By (2), we can choose $\varrho < 1$ shch that

$$(1-\| z \|^2)^q \| \nabla f_n(z) - \nabla f(z) \| < \epsilon \quad (n = 1, 2, , \varrho < |z| < 1).$$

We use (1) to estimate this difference for $|z| \leq \varrho$ and large n and deduce that

$$\| f_n - f \|_q \rightarrow 0 \quad (n \rightarrow \infty)$$

□

LEMMA 3. *$f \in \mathcal{B}_{q,0}$ if and only if $\| f(z) - f(z\zeta) \|_q \rightarrow 0$* as $\zeta \rightarrow 1, |\zeta| \leq 1$.

Proof. We have already proved that $f_n \in \mathcal{B}_{q,0}$, $\| f_n - f \|_q \rightarrow 0 (n \rightarrow \infty)$ implies $f \in \mathcal{B}_{q,0}$. Thus $\mathcal{B}_{q,0}$ is closed. If $f \in \mathcal{B}_q$, then $f(z\zeta) \in \mathcal{B}_{q,0}$ for every $\zeta \in B$. Since $\mathcal{B}_{q,0}$ is closed and $\| f(z) - f(z\zeta) \|_q \rightarrow 0, f \in \mathcal{B}_{q,0}$. Conversely, let $f \in \mathcal{B}_{q,0}$ and $\zeta_n \rightarrow 1 \, (n \rightarrow \infty), |\zeta_n| \leq 1$. It is clear that the functions $f(\zeta_n z) = f_n(z)$ in $\mathcal{B}_{q,0}$ satisfy
(1) $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$, locally uniformly in B.
(2) $(1-\| z \|^2)^q \| \nabla f_n(z) \| \rightarrow 0$ as $|z| \rightarrow 1$, uniformly in n.
By Lemma 2, $\| f(z) - f(z\zeta) \|_q \rightarrow 0$ as $\zeta \rightarrow 1, |\zeta| \leq 1$. □

THEOREM 7. *For $q \geq 1$, $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_{q,0}$. In particular, $\mathcal{B}_{q,0}$ is a separable Banach space by itself.*

Proof. Since every polynomial belongs to $\mathcal{B}_{q,0}$ and $\mathcal{B}_{q,0}$ is closed, it follows that the closure of the polynomials is contained in $\mathcal{B}_{q,0}$.
If we choose $\zeta_n = 1 - \frac{1}{n}$, then $f_n(z) = f(\zeta_n z)$ is analytic in $\| z \| \leq 1$. Hence we can find a polynomial $P_n(z)$ such that

$$|f(\zeta_n z) - P_n(z)| < \frac{1}{n} \quad for \quad \| z \| < 1.$$

Since $\| f \|_q \leq 2\sup_{z\in B} |f(z)|$ for $q \geq 1$ by the Schwarz-Pick lemma,

$$\| f - P_n \|_q \leq \| f - f_n \|_q + \frac{2}{n} \to 0 \quad (n \to \infty).$$ □

COROLLARY 8. *Suppose $q > 1$. f is in $\mathcal{B}_{q,0}$ if and if only if $(1- \| z \|^2)^{q-1} f(z) \to 0$ as $|z| \to 1^-$.*

Proof. It is clear from Theorem 6. □

References

[1] J. Anderson, *Bloch functions: The Basic theory, operators and function theory*, S. Power. editor, D. Reidel (1985).

[2] J. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12-37.

[3] J. Arazy, S. D. Fisher, J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), 989-1054.

[4] S. Axler, *The Bergman spaces, the Bloch space and commutators of multiplication operators*, Duke Math. J. **53** (1986), 315-332.

[5] K. T. Hahn, *Holomorphic mappings of the hyperbolic space into the complex Euclidean space and Bloch theorem*, Canadian J. Math. **27** (1975), 446-458.

[6] K. T. Hahn, E. H. Youssfi, *M-harmonic Besov p-spaces and Hankel operators in the Bergman space on the unit ball in $\mathbb{C}^n$*, Manuscripta Math **71** (1991), 67-81.

[7] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer Verlag, New York (1980).

[8] R. M. Timoney, *Bloch functions of several variables*, J. Bull. London Math. Soc. **12** (1980), 241-267.

[9] K. H. Zhu, *Duality and Hankel operators on the Bergman spaces of bounded symmetric domains*, J. Funct. Analysis **81** (1988), 262-278.

[10] K. H. Zhu, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math **23** (1993), 1143-1177.

Kyong Taik Hahn
Department of Mathmatics
The Pennsylvania State University
University Park, PA 16802
USA

and

Department of Mathematics
Konyang university
NonSan 320-800, Korea

Ki Seong Choi
Department of Mathematics
Konyang university
NonSan 320-800, Korea
E-mail: ksc@kytis.konyang.ac.kr