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SELECTION THEOREMS WITH n-CONNECTEDNESS

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ABSTRACT. We give a generalization of the selection theorem of Ben-El-Mechaiekh and Oudadess to complete LD-metric spaces with the aid of the notion of n-connectedness. Our new selection theorem is used to obtain new results on fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values consisting of $\mathcal{D}$-sets in a complete LD-metric space.

0. Introduction

In 1991 Horvath [3] extended Michael's selection theorem [4] for closed convex valued lower semicontinuous maps to nonconvex values. In 1995 Ben-El-Mechaiekh and Oudadess [1] gave a generalized selection theorem by combining the result in [3] with [5] related to sets of topological dimension ≤ 0. Using the concept of n-connectedness, we introduce LD-metric spaces which are more general than *l.c.* metric spaces given in [3]. The purpose in this paper is first to extend the selection theorem in [1] to closed valued lower semicontinuous maps with $\mathcal{D}$-set values in a complete LD-metric space except possibly on a set of topological dimension ≤ 0 and then to give new results on fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values consisting of $\mathcal{D}$-sets in a complete LD-metric space.

1. Preliminaries

Let X and Y be topological spaces. A *set-valued map* (simply, a *map*) $T : X \multimap Y$ is a function from X into the set 2^Y of all nonempty

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subsets of Y; the map $T^- : Y \multimap X$ is defined by $T^-y := \{x \in X : y \in Tx\}$ whenever T is surjective. A map $T : X \multimap Y$ is said to be *compact* if its range $\bigcup_{x \in X} Tx$ is relatively compact in Y; and *lower semicontinuous* if $\{x \in X : Tx \cap V \neq \emptyset\}$ is open in X for every open set V in Y. A continuous function $f : X \to Y$ is called a *selection* of $T : X \multimap Y$ whenever $f(x) \in Tx$ for every x in X.

If Z is a subset of a topological space X, then $\dim_X Z \leq 0$ means that $\dim E \leq 0$ for every set $E \subset Z$ which is closed in X, where $\dim E$ denotes the covering dimension of E.

A topological space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f : S^k \to X$ for $k \leq n$ has a continuous extension over B^{k+1}, where S^k is the unit sphere and B^{k+1} the closed unit ball in $\mathbb{R}^{k+1}$. Note that a contractible space is n-connected for every $n \geq 0$.

Given a set Y, let $\langle Y \rangle$ denote the collection of all nonempty finite subsets of Y. Let $\Delta_n = \mathrm{co}\{e_0, \cdots, e_n\}$ be the standard simplex of dimension n, where $\{e_0, \cdots, e_n\}$ is the canonical basis of $\mathbb{R}^{n+1}$.

We introduce the following geometric structure as a generalization of convex sets with the aid of the notion of n-connectedness.

Let Y be a topological space. A *D-structure* on Y is a map $\mathcal{D} : \langle Y \rangle \to 2^Y$ such that it satisfies the following conditions:

(1) for each $A \in \langle Y \rangle$, $\mathcal{D}(A)$ is nonempty and n-connected for all $n \geq 0$;

(2) for each $A, B \in \langle Y \rangle$, $A \subset B$ implies $\mathcal{D}(A) \subset \mathcal{D}(B)$.

The pair $(Y, \mathcal{D})$ is called a *D-space*; a subset Z of Y is said to be a *$\mathcal{D}$-set* if $\mathcal{D}(A) \subset Z$ for each $A \in \langle Z \rangle$. A D-space $(Y, \mathcal{D})$ is called an *LD-metric space* if (Y, d) is a metric space such that for each $\epsilon > 0$,

$$B(E, \epsilon) = \{y \in Y : d(y, z) < \epsilon \quad \text{for some } z \in E\}$$

is a $\mathcal{D}$-set whenever $E \subset Y$ is a $\mathcal{D}$-set and open balls are $\mathcal{D}$-sets.

A D-space is a generalization of c-spaces in the sense of Horvath [3]. A simple example of a D-space but not a c-space is the space Y, obtained by forming the disjoint union of the comb space X and another copy X' of X and identifying a point $x_0 = (0, 1) \in X$ with the corresponding point $x_0' \in X'$, by setting $\mathcal{D}(A) := Y$ for every $A \in \langle Y \rangle$.

It can be shown that any D-space becomes a generalized convex space introduced by Park and Kim [8].

A *generalized convex space* (Y, Γ) consists of a topological space Y and a map $\Gamma : \langle Y \rangle \to 2^Y$ such that the following conditions are satisfied:

(1) for each $A, B \in \langle Y \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$;
(2) for each $A \in \langle Y \rangle$ with $|A| = n+1$, there exists a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $\Phi_A(\Delta_J) \subset \Gamma(J)$ for every $J \in \langle A \rangle$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

LEMMA 0. *A D-space $(Y, \mathcal{D})$ is a generalized convex space.*

Proof. Since $(Y, \mathcal{D})$ is a D-space, it suffices to show that for each $A \in \langle Y \rangle$ with $|A| = n+1$, there exists a continuous function $f : \Delta_n \to \mathcal{D}(A)$ such that $f(\Delta_J) \subset \mathcal{D}(J)$ for every $J \in \langle A \rangle$. Let $A = \{a_0, a_1, \cdots, a_n\} \in \langle Y \rangle$ be given such that $e_i \in \Delta_{\{a_i\}}$. For each $i \in \{0, 1, \cdots, n\}$, there exists a $y_i \in \mathcal{D}(\{a_i\})$. Define a function $f^0 : \Delta_n^0 \to \mathcal{D}(A)$ on the 0-skeleton of Δ_n by $f^0(e_i) := y_i$. Then the function f^0 is continuous and $f^0(\Delta_{\{a_i\}}) \subset \mathcal{D}(\{a_i\})$ for $i = 0, 1, \cdots, n$.

Assume that a continuous function $f^k : \Delta_n^k \to \mathcal{D}(A)$ on the k-skeleton of Δ_n has been constructed such that $f^k(\Delta_J) \subset \mathcal{D}(J)$ for all $J \in \langle A \rangle$ with $|J| \le k+1$.

Now let Δ_J be a face of dimension $k+1$ of Δ_n and let $J_i := J \setminus \{a_i\}$ for each $a_i \in J$. Let $\partial\Delta_J$ be the boundary of Δ_J. Then $\partial\Delta_J = \bigcup_{a_i \in J} \Delta_{J_i}$ is contained in the k-skeleton of Δ_n and we have

$$f^k(\partial\Delta_J) \subset \bigcup_{a_i \in J} f^k(\Delta_{J_i}) \subset \bigcup_{a_i \in J} \mathcal{D}(J_i) \subset \mathcal{D}(J).$$

Note that there is a homeomorphism $h : E^{k+1} \to \Delta_J$ such that $h(S^k) = \partial\Delta_J$. Since $f^k \circ h|_{S^k} : S^k \to \mathcal{D}(J)$ is continuous and $\mathcal{D}(J)$ is k-connected, the function $f^k \circ h|_{S^k}$ has a continuous extension $g^{k+1} : E^{k+1} \to \mathcal{D}(J)$. Thus, $f_J^{k+1} := g^{k+1} \circ h^{-1} : \Delta_J \to \mathcal{D}(J)$ is continuous and $f_J^{k+1}|_{\partial\Delta_J} = f^k|_{\partial\Delta_J}$.

If Δ_J and $\Delta_{J'}$ are $(k+1)$-dimensional faces of Δ_n, $\Delta_J \neq \Delta_{J'}$ and $\Delta_J \cap \Delta_{J'} \neq \emptyset$, then it is clear that

$$f_J^{k+1}|_{\Delta_J \cap \Delta_{J'}} = f^k|_{\Delta_J \cap \Delta_{J'}} = f_{J'}^{k+1}|_{\Delta_J \cap \Delta_{J'}}.$$

Therefore, on the $(k+1)$-skeleton of Δ_n we obtain a continuous function $f^{k+1} : \Delta_n^{k+1} \to \mathcal{D}(A)$ which has the property $f^{k+1}(\Delta_J) \subset \mathcal{D}(J)$ for all $J \in \langle A \rangle$ with $|J| \leq k+2$. It follows by the induction on $k \leq n$ that a continuous function $f : \Delta_n \to \mathcal{D}(A)$ has been constructed such that

$$f(\Delta_J) \subset \mathcal{D}(J) \quad \text{for every } J \in \langle A \rangle.$$

This completes the proof. □

2. Selection theorems

In this paper, paracompact spaces are assumed to be Hausdorff. The following proposition is a basic statement for the new selection theorem presented in this section.

PROPOSITION 1. *Let X be a paracompact space, $\mathcal{R}$ a locally finite open covering of X, $(Y, \mathcal{D})$ a D-space, and $\eta : \mathcal{R} \to Y$ a function. Then there exists a continuous function $g : X \to Y$ such that*

$$g(x) \in \mathcal{D}(\{\eta(U) : x \in U \text{ and } U \in \mathcal{R}\}) \quad \textit{for each } x \in X.$$

Proof. For any $k \geq 1$, (B^{k+1}, S^k) is homeomorphic to $(s, \partial s)$, where s is a $(k+1)$-simplex and ∂s is its boundary (cf. [10], 3.1.22). Therefore, under the weak condition of n-connectedness instead of contractibility, we can verify our result along the lines of proof of Theorem 3.1 in [3].□

Having established Proposition 1, we now turn to the selection theorem. It begins with the following lemma on ϵ-approximate selections.

LEMMA 2. *Let X be a paracompact space, $(Y, \mathcal{D})$ an LD-metric space, Z a subset of X with $\dim_X Z \leq 0$, and $T : X \multimap Y$ a lower semicontinuous map such that Tx is a $\mathcal{D}$-set for all $x \notin Z$. Then for every $\epsilon > 0$, T admits an ϵ-approximate selection, that is, a continuous* *single-valued function* $g_\epsilon : X \to Y$ *such that* $g_\epsilon(x) \in B(Tx, \epsilon)$ *for every* $x \in X$.

The proof of Lemma 2 proceeds in precisely the same fashion as Lemma 2 in [1], except that all c-sets in an $l.c.$ metric space is replaced by $\mathcal{D}$-sets in an LD-metric space.

The following main theorem is a generalization of Ben-El-Mechaiekh and Oudadess [1, Theorem 3] which generalizes Michael and Pixley [5, Theorem 1.1].

THEOREM 3. *Let* X *be a paracompact space,* $(Y, \mathcal{D})$ *a complete* LD*-metric space,* Z *a subset of* X *with* $\dim_X Z \leq 0$, *and* $T : X \multimap Y$ *a lower semicontinuous map with closed values such that* Tx *is a* $\mathcal{D}$*-set for all* $x \notin Z$. *Then* T *admits a selection* $g : X \to Y$.

Proof. Set $T_1 := T$. By Lemma 2, there is a continuous function $g_1 : X \to Y$ such that

$$g_1(x) \in B(T_1 x, \frac{1}{2}) \qquad \text{for every } x \in X.$$

Hence, a map $T_2 : X \multimap Y, x \mapsto T_1 x \cap B(g_1(x), \frac{1}{2})$, is lower semicontinuous(cf. [4, Proposition 2.4]) and $T_2 x$ is a $\mathcal{D}$-set for all $x \notin Z$.

Assume that for $k = 1, \cdots, n$, a lower semicontinuous map $T_k : X \multimap Y$ has been defined and a continuous function $g_k : X \to Y$ has been chosen such that

$$T_1 x = Tx$$

$$T_k x = T_{k-1} x \cap B(g_{k-1}(x), \frac{1}{2^{k-1}}) \quad \text{for } k = 2, \cdots, n$$

are nonempty $\mathcal{D}$-sets for all $x \notin Z$ and

$$g_k(x) \in B(T_k x, \frac{1}{2^k}) \qquad \text{for every } x \in X.$$

Hence, a map $T_{n+1} : X \multimap Y$, $T_{n+1} x := T_n x \cap B(g_n(x), \frac{1}{2^n})$, is lower semicontinuous and $T_{n+1} x$ is a $\mathcal{D}$-set for all $x \notin Z$. By Lemma 2, there exists a continuous function $g_{n+1} : X \to Y$ such that

$$g_{n+1}(x) \in B(T_{n+1} x, \frac{1}{2^{n+1}}) \qquad \text{for every } x \in X.$$

It follows by induction that there is a sequence of functions $g_n : X \to Y$ which has the above properties for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary. Then there is a $y \in Y$ such that $y \in T_{n+1}x \cap B(g_{n+1}(x), \frac{1}{2^{n+1}})$ for all $x \in X$, hence we have

$$d(g_{n+1}(x), g_n(x)) \leq d(g_{n+1}(x), y) + d(y, g_n(x)) < \frac{1}{2^{n+1}} + \frac{1}{2^n}.$$

It is also clear that the sequence (g_n) is a uniformly Cauchy sequence. Since Y is complete, (g_n) converges uniformly on X.

Define a map $g : X \to Y$ by

$$g(x) := \lim_{n\to\infty} g_n(x) \qquad \text{for } x \in X.$$

Then g is continuous and $g(x) \in Tx$ for every $x \in X$ since Tx is closed. This completes the proof. □

Using Theorem 3, we give a sufficient condition for a lower semicontinuous set-valued map with closed values to have the selection extension property.

COROLLARY 4. *Let $(Y, \mathcal{D})$ be a complete LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. Let X be a paracompact space, Z a subset of X with $\dim_X Z \leq 0$, and $T : X \multimap Y$ a lower semicontinuous map with closed values such that Tx is a $\mathcal{D}$-set for all $x \notin Z$. If A is closed in X, then every selection g for $T|_A$ extends to a selection for T. Here $T|_A$ denotes the restriction of T to A.*

Proof. Let $g : A \to Y$ be a selection for $T|_A$. We define a map $T_g : X \multimap Y$ by

$$T_g x := \begin{cases} \{g(x)\} & \text{for } x \in A \\ Tx & \text{for } x \notin A \end{cases}.$$

Then T_g is a lower semicontinuous map with closed values and $T_g x$ is a $\mathcal{D}$-set for all $x \notin Z$. By Theorem 3, T_g has a selection $f : X \to Y$, which is a selection for T that extends g because $g : A \to Y$ is a selection for $T|_A$. □

COROLLARY 5. *Let $(Y,\mathcal{D})$ be a complete LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. Let X be a paracompact space, A a closed subset of X and $g : A \to Y$ a continuous function. Then there is a continuous function $f : X \to Y$ which extends g.*

Proof. A map $T : X \multimap Y$, defined by

$$Tx := \begin{cases} \{g(x)\} & \text{for } x \in A \\ Y & \text{for } x \notin A \end{cases}.$$

is lower semicontinuous and its values are closed $\mathcal{D}$-sets. By Theorem 3, T has a continuous selection $f : X \to Y$. Since $f(x) \in Tx$ for all $x \in X$, we obtain $f|_A = g$. □

3. Applications to fixed points and coincidence points

We need the following theorem due to Park [7, Theorem 2].

THEOREM 6. *Let X be a compact Hausdorff space, (Y,Γ) a generalized convex space and $T : X \multimap Y$ a map with the property that there is a map $S : X \multimap Y$ such that the following conditions are satisfied:*

(1) *for each $x \in X$, $A \in \langle Sx \rangle$ implies $\Gamma(A) \subset Tx$; and*
(2) *$X = \bigcup\{\operatorname{int} S^- y : y \in Y\}$, where int denotes the interior.*

Then T has a continuous selection $f : X \to Y$. More precisely, there exist a simplex Δ_n and two continuous functions $p : X \to \Delta_n$ and $q : \Delta_n \to Y$ such that $f = q \circ p$ and $f(X) \subset \Gamma(A)$ for some $A \in \langle Y \rangle$ with $|A| = n + 1$.

An immediate consequence of Theorem 6 and Brouwer's fixed point theorem is in connection with fixed points and coincidence points for set-valued maps. Since D-spaces are generalized convex spaces by Lemma 0, Theorem 6 works for D-spaces.

THEOREM 7. *Let X be a compact Hausdorff space, $(Y,\mathcal{D})$ a D-space, $S,T : X \multimap Y$ two maps such that the following conditions are satisfied:*

(1) *$A \in \langle Sx \rangle$ implies $\mathcal{D}(A) \subset Tx$ for every $x \in X$;*
(2) *$X = \bigcup\{\operatorname{int} S^- y : y \in Y\}$.*

Then

(a) For any continuous function $g : Y \to X$ there is a $y_0 \in Y$ such that $y_0 \in Tg(y_0)$.

(b) If $R : X \multimap Y$ is a set-valued map such that $R^- : Y \multimap X$ has a continuous selection, then there is an $x_0 \in X$ such that $Rx_0 \cap Tx_0 \neq \emptyset$.

Proof. (a) Let $g : Y \to X$ be a continuous function. By Theorem 6, T has a continuous selection $f : X \to Y$ and there exist continuous functions $p : X \to \Delta_n$ and $q : \Delta_n \to Y$ such that $f = q \circ p$. The continuous function $\varphi : \Delta_n \to \Delta_n, z \mapsto p \circ g \circ q(z)$, has a fixed point z_0, by Brouwer's fixed point theorem. Setting $y_0 = q(z_0)$, we have

$$y_0 = (q \circ p \circ g \circ q)(z_0) = (f \circ g)(y_0) \in Tg(y_0).$$

(b) Let $h : Y \to X$ be a continuous selection for R^-. By (a), there is a $y_0 \in Y$ such that $y_0 \in Th(y_0)$ and also $h(y_0) \in R^- y_0$. If $x_0 := h(y_0)$, then $Rx_0 \cap Tx_0 \neq \emptyset$. This completes the proof. □

Using the selection theorems above, we establish the existence of fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values in a complete LD-metric space.

THEOREM 8. *Let $(Y, \mathcal{D})$ be an LD-metric space and suppose that for every $\epsilon > 0$ there are two maps $S, T : Y \multimap Y$ such that the following conditions are satisfied:*

(1) *$A \in \langle Sy \rangle$ implies $\mathcal{D}(A) \subset Ty$ for every $y \in Y$;*
(2) *$Y = \bigcup \{\text{int}\, S^- y : y \in Y\}$; and*
(3) *$y \in B(Ty, \epsilon)$ for all $y \in Y$.*

Then any compact continuous function $g : Y \to Y$ has a fixed point.

Proof. Let $\epsilon > 0$. Applying Theorem 7 to $T|_{\overline{g(Y)}}$, there is a point y_ϵ in Y such that $y_\epsilon \in Tg(y_\epsilon)$, hence by (3), $d(g(y_\epsilon), y_\epsilon) < \epsilon$. Since $g(Y)$ is relatively compact in Y and g is continuous, it is easy to verify that there exists a $y_0 \in Y$ such that $g(y_0) = y_0$. □

REMARK. Theorem 8 remains true if Y is a Hausdorff uniform space with a D-structure $\mathcal{D}$ on Y.

COROLLARY 9. *Let $(Y, \mathcal{D})$ be an LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. Then any compact continuous function $g : Y \to Y$ has a fixed point.*

Proof. Apply Theorem 8 with $S = T$ and $Tx := Y$ for every $x \in X$. □

THEOREM 10. *Let $(Y, \mathcal{D})$ be a complete LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$ and let Z be a subset of Y with $\dim_Y Z \leq 0$. Then any compact lower semicontinuous map $T : Y \multimap Y$ with closed values such that Ty is a $\mathcal{D}$-set for all $y \notin Z$ has a fixed point.*

Proof. By Theorem 3, T has a continuous selection $g : Y \to Y$. Since g is compact, by Corollary 9, $g : Y \to Y$ has a fixed point. Thus, $y_0 = g(y_0) \in Ty_0$ for some $y_0 \in Y$. □

COROLLARY 11. *Let $(Y, \mathcal{D})$ be a complete LD-metric space, and Z a subset of Y with $\dim_Y Z \leq 0$ such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. Let $T : Y \multimap Y$ be a compact map with closed values such that Tx is a $\mathcal{D}$-set for all $x \notin Z$ and $T^- y$ is open for all $y \in Y$. Then T has a fixed point.*

COROLLARY 12. *Let X be a paracompact space, $(Y, \mathcal{D})$ a complete LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$, Z a subset of X with $\dim_X Z \leq 0$, and let $S, T : Y \multimap Y$ be two maps such that the following conditions are satisfied.*

(1) *T is a compact lower semicontinuous map with closed values such that Tx is a $\mathcal{D}$-set for all $x \notin Z$;*
(2) *$S^- : Y \multimap X$ has a continuous selection.*

Then there is an $x_0 \in X$ such that $Sx_0 \cap Tx_0 \neq \emptyset$.

Proof. Let $g : Y \to X$ be a continuous selection for S^-. The composition $T \circ g : Y \multimap Y$ is compact, lower semicontinuous. By Theorem 10, there is a $y_0 \in Y$ such that $y_0 \in Tg(y_0)$. Since $g(y_0) \in S^- y_0$, we have $Sg(y_0) \cap Tg(y_0) \neq \emptyset$. □

With the help of $\mathcal{D}$-functions, we give a fixed point theorem which is a generalization of a result in [2].

Let $(X, \mathcal{D})$ be a D-space. A continuous function $f : X \times X \to \mathbb{R}$ is said to be a *$\mathcal{D}$-function* if it has the following properties:

(1) For every $x \in X$ and every $\lambda \in \mathbb{R}$, $\{y \in X : f(x,y) > \lambda\}$ is a $\mathcal{D}$-set.

(2) $f(x,x) \geq 0$ for all $x \in X$.

THEOREM 13. *Let $(X, \mathcal{D})$ be a compact Hausdorff D-space. Suppose that for any $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$ there is a $\mathcal{D}$-function $f : X \times X \to \mathbb{R}$ such that $f(x_1, x_2) < 0$. Then any compact continuous function $g : X \to X$ has a fixed point.*

Proof. For $\lambda < 0$ and $\mathcal{D}$-function f, let

$$T_\lambda(f) = \{(x,y) \in X \times X : f(x,y) > \lambda\}.$$

Then $T_\lambda(f)$ is a graph of the multimap $x \mapsto \{y \in X : f(x,y) > \lambda\}$ having open inverses and $\mathcal{D}$-set values.

For $\lambda_i < 0$ and $\mathcal{D}$-functions $f_i, i = 1, \cdots, n$, $\bigcap_{i=1}^n T_{\lambda_i}(f_i)$ is a graph of the multimap $x \mapsto \{y \in X : f_i(x,y) > \lambda_i \text{ for all } i\}$ having open inverses and $\mathcal{D}$-set values. Since Y is compact, there exists a unique uniform structure on Y (cf. [9], II 3.6 Satz 1).

Now let V be an open entourage and $(x_1, x_2) \in (X \times X) \setminus V$. By assumption, there is a $\mathcal{D}$-function f and a number $\lambda < 0$ such that $f(x_1, x_2) < \lambda$. Therefore, we have $(x_1, x_2) \notin \overline{T_\lambda(f)}$. The collection

$$\{(X \times X) \setminus \overline{T_\lambda(f)} : \lambda < 0 \text{ and } f \text{ is a } \mathcal{D}\text{-function}\}$$

covers the closed set $(X \times X) \setminus V$. By the compactness of $X \times X$, there are finitely many $\mathcal{D}$-functions $f_1, \cdots, f_n$ and numbers $\lambda_1, \cdots, \lambda_n < 0$ such that

$$(X \times X) \setminus V \subset (X \times X) \setminus \bigcap_{i=1}^{n} \overline{T_{\lambda_i}(f_i)}$$

hence $\bigcap_{i=1}^n T_{\lambda_i}(f_i) \subset V$. By Theorem 8, any compact continuous function $g : X \to X$ has a fixed point. □

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