J. Korean Math. Soc. **35** (1998), No. 1, pp. 149–164

HOMOLOGY OF THE TRIPLE LOOP SPACE OF THE EXCEPTIONAL LIE GROUP F_4

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ABSTRACT. We study the homology of the triple loop space of the exceptional Lie group F_4 by exploiting the spectral sequences and the homology operations.

Introduction

The homology of the iterated loop space is one of the main topics in topology. Besides its own interest it has many applications to other branches. Especially the homology of the triple loop space is very interesting from the gauge theoretic view point.

Let G be a compact connected simple Lie group. Since $\pi_3(G) = \pi_4(BG) = Z$, the principal G bundles P_k over S^4 are classified by the integer k in Z. For a given P_k, the orbit space of connections up to the based gauge equivalence is homotopy equivalent to the triple loop space of G [1]. Then there is a natural inclusion map $i : \mathcal{M}_k \to \mathcal{C}_k \simeq \Omega^3_k G$ where $\mathcal{M}_k$ is the moduli space of G instantons. Moreover the inclusion map $i : \mathcal{M}_\infty \to \mathcal{C}_\infty$ induces a homotopy equivalence [11] where $\mathcal{M}_\infty$ and $\mathcal{C}_\infty$ are the direct limits under the inclusions. So the homology of the triple loop space is a cornerstone for getting information about the homology of the instanton space [2], [3], [4].

In this paper we study the homology of the triple loop space of the exceptional Lie group F_4 by computing the Eilenberg–Moore spectral sequence and the Serre spectral sequence with the aid of the Dyer–Lashof operations.

Received August 28, 1997.

1991 Mathematics Subject Classification: 55R20, 55S12.

Key words and phrases: exceptional Lie group F_4, Serre spectral sequence, Eilenberg–Moore spectral sequence, triple loop space, Dyer–Lashof operation.

Supported by KOSEF Research Grant 971-0103-014-1 and GARC.

G is called p-regular if G is p-equivalent to a product of spheres, i.e.,

$$G \simeq_p \Pi S^{2m_i+1}$$

and G is called quasi p-regular if G is p-equivalent to a product of spheres and sphere bundles over spheres of type $B(2n+1, 2n+2p-1)$, i.e.,

$$G \simeq_p \Pi S^{2m_i+1} \times \Pi B(2n_j+1, 2n_j+2p-1).$$

Here $B(2n+1, 2n+2p-1)$ is a mod p H-space defined as an S^{2n+1}-bundle over $S^{2n+1+2(p-1)}$ with characteristic element α_p where α_p is the generator of the p-primary component of $\pi_{2n+2(p-1)-1}(S^{2n})$.

It is well-known that F_4 is p-regular if and only if $p \geq 13$ and F_4 is quasi p-regular if and only if $p \geq 5$ [10]. Hence for $p \geq 5$, the homology of the triple loop space of F_4 is simply determined by the homologies of the triple loop spaces of spheres and the homologies of the triple loop spaces of $B(2n_j+1, 2n_j+2p-1)$'s. So we first determine the case $p \geq 5$ and then we concentrate on the cases $p = 2$ and $p = 3$.

1. Preliminaries

Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ be the divided power algebra on x which is free over $\gamma_i(x)$ as a $\mathbb{F}_p$ module with the product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$. In this paper the subscript of the element always denotes the degree of that element.

We have homology operations $Q_{i(p-1)}$ on the $(n+1)$-loop space $\Omega^{n+1}X$

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; \mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^{n+1}X; \mathbb{F}_p)$$

for $0 \leq i \leq n$ which are natural with respect to $(n+1)$-loop maps. In particular, we have $Q_0 x = x^p$. The iterated power Q_i^a denotes the composition of Q_i's a times, i.e., $Q_i^a = \underbrace{Q_i \circ \cdots \circ Q_i}_{a\ times}$.

These operations satisfy the following properties.

PROPOSITION 1.1. [6] *In the path-loop fibration*

$$\Omega^{n+2}X \to P\Omega^{n+1}X \to \Omega^{n+1}X,$$

we have the following;
(1) *if* $x \in H_q(\Omega^{n+1}X;\mathbb{F}_p)$ *is transgressive in the Serre spectral sequence, then so is* $Q_i x$ *and* $\tau \circ Q_{i(p-1)}x = Q_{(i+1)(p-1)} \circ \tau x$ *for each* i, $0 \le i \le n$ *where* τ *is the transgression,*
(2) *for* $p > 2$ *and* $n \ge 1$, $d^{|x|(p-1)}(x^{p-1} \otimes \tau(x)) = -\beta Q_{(p-1)}\tau(x)$,
(3) *for* $p = 2$, $Sq^1_* Q_i x = Q_{i-1}x$ *if* $x \in H_q(\Omega^{n+1}X;\mathbb{F}_2)$ *and* $q+i$ *is even.*

Let $P^*(\Omega X;\mathbb{F}_p)$ be the primitives of $H^*(\Omega X;\mathbb{F}_p)$ and $Q^*(\Omega X;\mathbb{F}_p)$ be the indecomposables of $H^*(\Omega X;\mathbb{F}_p)$.

PROPOSITION 1.2. [7] *Let* X *be a simply connected* H*-space. Then the following is true:*
(1) *The suspension* $\sigma : Q^{odd}(X;\mathbb{F}_p) \to P^{even}(\Omega X;\mathbb{F}_p)$ *is injective,*
(2) *the suspension* $\sigma : Q^{even}(X;\mathbb{F}_p) \to P^{odd}(\Omega X;\mathbb{F}_p)$ *is onto,*
(3) *the quotient* $P^{even}(\Omega X;\mathbb{F}_p)/\sigma(Q^{odd}(X;\mathbb{F}_p))$ *is obtained by transpotence,*
(4) *the elements in* $\ker\sigma$ *are dual to elements in the image of the homology transpotence.*

For $n > 1$, as an algebra the homology of the triple loop space of the odd sphere is determined by [6]

$$\begin{aligned}
H_*(\Omega^3 S^{2n+1};\mathbb{F}_2) &= \mathbb{F}_2[Q_1^a Q_2^b u_{2n-2} : a \ge 0, b \ge 0].\\
H_*(\Omega^3 S^{2n+1};\mathbb{F}_p) &= \mathbb{F}_p[Q^a_{2(p-1)}u_{2n-2} : a \ge 0]\\
&\quad \otimes E(Q^a_{(p-1)}\beta Q^b_{2(p-1)}u_{2n-2} : a \ge 0, b > 0)\\
&\quad \otimes \mathbb{F}_p[\beta Q^a_{(p-1)}\beta Q^b_{2(p-1)}u_{2n-2} : a > 0, b > 0],\ p\text{: odd.}
\end{aligned}$$

The exceptional Lie group F_4 when localized at p splits as follows:

$$\begin{aligned}
F_4 &\simeq_{(p)} B(3,11) \times B(15,23), && p = 5,\\
F_4 &\simeq_{(p)} B(3,15) \times B(11,23), && p = 7,\\
F_4 &\simeq_{(p)} B(3,23) \times S^{11} \times S^{15}, && p = 11,\\
F_4 &\simeq_{(p)} \Pi_{k=0}^{1} S^{8k+3} \times \Pi_{k=0}^{1} S^{8k+15}, && p \ge 13.
\end{aligned}$$

Hence $H_*(\Omega^3 F_4;\mathbb{F}_p)$ is a tensor product of $H_*(\Omega^3 S^{2n+1};\mathbb{F}_p)$'s for $p \ge 13$, and $H_*(\Omega^3 F_4;\mathbb{F}_p)$ a tensor product of $H_*(\Omega^3 S^{2n+1};\mathbb{F}_p)$'s and $H_*(\Omega^3 B(2n+1, 2n+2p-1);\mathbb{F}_p)$'s for $p \ge 5$.

The space $B(2n+1, 2n+2p-1)$ is equivalent to a direct factor of the p-localization of $SU(n+p)/SU(n)$ [10] and its cohomology ring is

$$H^*(B(2n+1, 2n+2p-1);\mathbb{F}_p) = E(x_{2n-1}, x_{2n+2p-1})$$

with $\mathcal{P}^1 x_{2n+1} = x_{2n+2p-1}$.

So from the computation of the homology of the triple loop space of the special unitary group $SU(m)$[4, 12], we get directly the homology of the triple loop space of $B(2n+1, 2n+2p-1)$.

THEOREM 1.3. *For an odd prime p, as an algebra the homologies of $\Omega^3 B(2n+1, 2n+2p-1)$ are given by:*

$$H_*(\Omega^3 B(2n+1, 2n+2p-1); \mathbb{F}_p) = H_*(\Omega^3 S^{2n+1}; \mathbb{F}_p) \otimes H_*(\Omega^3 S^{2n+2p-1}; \mathbb{F}_p) \quad \text{for } n > 1,$$

$$\begin{aligned} H_*(\Omega_0^3 B(3, 2p+1); \mathbb{F}_p) = & \mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0] \\ & \otimes E(Q^a_{p-1} Q^b_{3(p-1)} u_{2p^2-3} : a \geq 0, b \geq 0) \\ & \otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} u_{2p^2-3} : a \geq 0, b > 0]. \end{aligned}$$

where $\Omega_0^3 B(3, 2p+1)$ is the zero component of $\Omega^3 B(3, 2p+1)$.

2. Homology of the double loop space of F_4

From now we turn to the cases $p = 2$ and $p = 3$. The following is well–known.

THEOREM 2.1. $H^*(F_4; \mathbb{F}_2) = \mathbb{F}_2(x_3)/(x_3^4) \otimes E(Sq^2 x_3, x_{15}, Sq^8 x_{15})$, $H^*(F_4; \mathbb{F}_3) = \mathbb{F}_3(\beta\mathcal{P}^1 x_3)/((\beta\mathcal{P}^1 x_3)^3) \otimes E(x_3, \mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11})$.

THEOREM 2.2. *The cohomology of the loop space ΩF_4 is*

$$H^*(\Omega F_4; \mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_8, y_{10}, y_{14}, y_{22}).$$

Proof. In the the Eilenberg–Moore spectral sequence converging to $H^*(\Omega F_4; \mathbb{F}_2)$,

$$\begin{aligned} E_2 &= Tor_{H^*(X;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \\ &= \Gamma(\sigma(Sq^2 x_3), \sigma x_{15}, \sigma(Sq^8 x_{15})) \otimes (E(\sigma x_3)) \otimes \Gamma(\phi_2(x_3)). \end{aligned}$$

Since E_2 is even dimensional, we have $E_2 = E_\infty$. Since the Eilenberg–Moore spectral sequence is a spectral sequence of Steenrod modules, we have

$$Sq^2(\sigma x_3) = (\sigma x_3)^2 = \sigma(Sq^2 x_3).$$

So we get the conclusion. □

THEOREM 2.3. *The cohomology of the loop space ΩF_4 is*

$$H^*(\Omega F_4;\mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^9)\otimes\Gamma(y_{10},y_{14},y_{18},y_{22}).$$

Proof. In the EMSS converging to $H^*(\Omega F_4; Z/(3))$

$$\begin{aligned} E_2 &= Tor_{H^*(X;Z/(p))} \\ &= \Gamma(\sigma x_3,\sigma\mathcal{P}^1x_3,\sigma x_{11},\sigma\mathcal{P}^1x_{11})\otimes(E(\sigma\beta\mathcal{P}^1x_3))\otimes\Gamma(\phi_1(\beta\mathcal{P}^1x_3)).\end{aligned}$$

Since the cohomology of the loop space of a compact simple Lie group is even–dimensional and torsion free from the Morse theory, we have that $d_2(\gamma_3(\sigma x_3)) = \sigma\beta\mathcal{P}^1x_3$. So

$$E_3 = \mathbb{F}_3^1[\sigma x_3]\otimes\Gamma(\phi_1(\beta\mathcal{P}^1x_3)))\otimes\Gamma(\sigma\mathcal{P}^1x_3,\sigma x_{11},\sigma\mathcal{P}^1x_3).$$

Since E_3 is even dimensional, we have $E_3 = E_\infty$. Let $y_2\in H^*(\Omega F_4;\mathbb{F}_3)$ represent σx_3 and $y_{22}\in H^*(\Omega F_4;\mathbb{F}_3)$ represent $\phi_1(\beta\mathcal{P}^1x_3)$. Hence as a coalgebra

$$H^*(\Omega F_4;\mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^3)\otimes\Gamma(y_{22})\otimes\Gamma(\sigma\mathcal{P}^1x_3,\sigma x_{11},\sigma\mathcal{P}^1x_{11}).$$

The action of the Steenrod operators on E_∞ induces the algebra structure of $H^*(\Omega F_4;\mathbb{F}_3)$. In E_∞, we have $(\sigma x_3)^3 = \mathcal{P}^1\sigma x_3 = \sigma\mathcal{P}^1x_3$ which in turn gives that as an algebra

$$H^*(\Omega F_4;\mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^9)\otimes\Gamma(y_{10},y_{14},y_{18},y_{22}). \qquad \square$$

As in the case of the cohomology, we can compute the Eilenberg–Moore spectral sequence converging to $H_*(\Omega F_4;\mathbb{F}_p)$ with

$$E_2 = \mathrm{Ext}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p).$$

COROLLARY 2.4. *As an algebra the homologies of the loop space ΩF_4 are*

$$H_*(\Omega F_4;\mathbb{F}_2) = E(y_2)\otimes\mathbb{F}_2[y_4,y_{10},y_{14},y_{22}],$$

$$H_*(\Omega F_4;\mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^3)\otimes\mathbb{F}_3[y_6,y_{10},y_{14},y_{22}].$$

Now consider the path–loop fibration

$$\Omega^2F_4 \dashrightarrow * \longrightarrow \Omega F_4$$

THEOREM 2.5. *The cohomology suspension map*

$$\sigma : Q^*(\Omega F_4; \mathbb{F}_p) \to P^{*-1}(\Omega^2 F_4; \mathbb{F}_p)$$

is a monomorphism for $p = 2$ *or* $p = 3$.

Proof. We divide the proof in two cases.

(Case 1) $p = 2$
Assume that $x \in (\ker \sigma)_{2q}$. From Proposition (1.2,4), $2q = n \times 2^s + 2$ for some $n \geq 1$ and $s \geq 2$. Hence $2q \equiv 2 \pmod 4$. Let $\tilde{F}_4$ is the 3-connected cover of F_4. Then $\Omega^2 F_4 \simeq S^1 \times \Omega^2 \tilde{F}_4$. Since $\tilde{F}_4$ is in fact 7-connected, we have $n \geq 6$.

Then $2q \geq 6 \times 4 + 2 = 26$ and there is no indecomposable element of dimension d with $d \geq 26$ and $d \equiv 2 \pmod 4$. Hence $\ker \sigma = 0$ and σ is a monomorphism.

(Case 2) $p = 3$
Since ΩF_4 has torsion free cohomology we have $H^{odd}(\Omega F_4; \mathbb{F}_3) = 0$.

Assume that $x \in (\ker \sigma)_{2q}$. From Proposition (1.2,4) we must have $2q = 2n3^s + 2$ for some $n \geq 1$ and $s \geq 1$. Since $Q_{2n}(\Omega^2 F_4; \mathbb{F}_3)$ is dual to $P^{2n}(\Omega^2 F_4; \mathbb{F}_3)$ and $H^{odd}(\Omega F_4; \mathbb{F}_3) = 0$, from Proposition (1.2,3) $P^{2n}(\Omega^2 F_4; \mathbb{F}_3)$ is obtained by the transpotence ϕ_k for some k. Hence $2n = (2m3^r - 2)$ for some $m \geq 1$ and $r \geq 1$. Then $2q = (2m3^r - 2)3^s + 2 \geq (2 \cdot 3 - 2)3 + 2 = 14$ and $2q \equiv 2 \pmod 3$.

By inspecting $H^*(\Omega F_4; \mathbb{F}_3)$ we can see that $y_{14} \in H^*(\Omega F_4; \mathbb{F}_3)$ is the only possible elements in $\ker \sigma$. But y_2 is of height greater than 3. Therefore y_{14} is not in $\ker \sigma$. Hence σ is a monomorphism. □

From the above theorem the Eilenberg–Moore spectral sequence converging to $H^*(\Omega^2 F_4; \mathbb{F}_2)$ (or $H_*(\Omega^2 F_4; \mathbb{F}_2)$) collapses from the E_2 term and we get the following theorem.

THEOREM 2.6. *As an algebra, the homology rings of the double loop space of* F_4 *are as follows:*

(a) $H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_1^a z_i : a \geq 0, i = 7, 9, 13, 21]$,

(b) $H_*(\Omega^2 F_4; \mathbb{F}_3) = E(z_1) \otimes \mathbb{F}_3[z_{16}] \otimes E(Q_2^a z_{17}; a \geq 0)$
$\otimes \mathbb{F}_3[\beta Q_2^a z_{17} : a > 0] \otimes E(Q_2^a z_9 : a \geq 0) \otimes \mathbb{F}_3[\beta Q_2^a z_9 : a > 0]$
$\otimes E(Q_2^a z_{13} : a \geq 0) \otimes \mathbb{F}_3[\beta Q_2^a z_{13} : a > 0] \otimes E(Q_2^a z_{21} : a \geq 0)$
$\otimes \mathbb{F}_3[\beta Q_2^a z_{21} : a > 0]$.

Proof. (a) Consider the path loop fibration: $\Omega^2 F_4 \to * \to \Omega F_4$. Then the elements $y_2, y_4^2, y_{10}, y_{14}, y_{22}$ in $H_*(\Omega F_4; \mathbb{F}_2)$ are dual to generators in cohomology. Hence by Theorem 2.5, they are transgressive and let $z_1, z_7, z_9, z_{13}, z_{21}$ be the corresponding images in $H_*(\Omega^2 F_4; \mathbb{F}_2)$. Since y_4 is not primitive, the element y_4 is not transgresive and $d^2(y_4 \otimes 1) = y_2 \otimes z_1$. Since $d^2(y_4^2 \otimes 1) = 0$, $y_2 y_4 \otimes z_1$ hits $z_6 \in H_6(\Omega^2 F_4; \mathbb{F}_2)$ and y_4^2 transgresses to $z_7 \in H_7(\Omega^2 F_4; \mathbb{F}_2)$. By Proposition (1.1), $y_4^{2^{a+1}} = Q_0^a y_4^2$ transgresses to $Q_1^a z_7$ and $Q_0^a y_4^2 \otimes Q_1^a z_7$ hits $(Q_1^a z_7)^2$. For $i = 10, 14, 22$, y_i transgresses to $z_{i-1} \in H_7(\Omega^2 F_4; \mathbb{F}_2)$. So we have the following;

(1) $\tau(Q_0^a y_i) = Q_1^a z_{i-1}, a \geq 0, \quad i = 10, 14, 22$

(2) $d(y_2 y_4 \otimes z_1 z_6^k) = 1 \otimes z_6^{k+1}, k \geq 0,$

(3) $\tau(y_4^{2^{a+1}}) = \tau(Q_0^a y_4^2) = Q_1^a z_7, a \geq 0.$

Therefore

$H_*(\Omega^2 F_4; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_1^a z_i : a \geq 0, i = 7, 9, 13, 21].$

(b) The generators $y_2, y_{10}, y_{14}, y_6^3, y_{22}$ in $H_*(\Omega F_4; \mathbb{F}_3)$ are dual to the generators in $H^*(\Omega F_4; \mathbb{F}_3)$. Hence in the Serre spectral sequence corresponding to the path–loop fibration, they are transgressive by Theorem 2.5 and let z_i for $i = 1, 9, 13, 17, 21$ be the corresponding trangressive images in $H_*(\Omega^2 F_4; \mathbb{F}_3)$ Since y_6 is not primitive, the element y_6 can not be transgresive and $d^2(y_6 \otimes 1) = y_2^2 \otimes z_1$. We also have $d^2(y_6^3 \otimes 1) = 0$ and $E^3_{r,s} = 0$ for $1 \leq r \leq 15$, $r \neq 10, 14$ and $2 \leq s \leq 9$. $y_2^2 y_6^2 \otimes z_1$ hits $z_{16} \in H_{16}(\Omega^2 F_4; \mathbb{F}_3)$ Therefore we have the following in the Serre spectral sequence corresponding the path–loop fibration;

(1) $\tau(Q_0^a y_i) = Q_2^a z_{i-1}, a \geq 0, \quad i = 10, 14, 22,$

$d((Q_0^a y_i)^2 \otimes (Q_2^a z_{i-1})(\beta Q_2^{a+1} z_{i-1})^k)$

$= 1 \otimes (\beta Q_2^{a+1} z_{i-1})^{k+1}, a \geq 0, k \geq 0,$

(2) $d(y_2^2 y_6^2 \otimes z_1 z_{16}^k) = 1 \otimes z_{16}^{k+1}, k \geq 0,$

(3) $\tau(y_6^{3^{a+1}}) = \tau(Q_0^a y_6^3) = Q_2^a z_{17}, a \geq 0,$

$d((y_6)^{3^{a+1}} \otimes (Q_2^a z_{17})(\beta Q_2^{a+1} z_{17})^k)$

$= 1 \otimes (\beta Q_2^{a+1} z_{17})^{k+1}, a \geq 0, k \geq 0.$

Therefore the conclusion follows. □

REMARK. In fact we can get the relations such that $\beta z_7 = z_6$ and $\beta z_{17} = z_{16}$ from the homology of the double loop space of the 3-connected cover $\tilde{F}_4$ [13].

3. Homology of the triple loop space of $F_4/Spin(9)$ and F_4

We have the following fibration

$$Spin(9) \xrightarrow{i} F_4 \xrightarrow{p} F_4/Spin(9)$$

where $F_4/Spin(9) = S^8 \cup e^{16}$ is the Cayley projective plane such that

$$H^*(F_4/Spin(9);\mathbb{F}_2) = \mathbb{F}_2[x_8]/(x_8^3).$$

So we have a sequence of fibrations:

$$\cdots \to \Omega^n Spin(9) \xrightarrow{\Omega^n i} \Omega^n F_4 \xrightarrow{\Omega^n p} \Omega^n F_4/Spin(9) \to \cdots$$

$$\cdots \to \Omega F_4/Spin(9) \xrightarrow{\gamma} Spin(9) \xrightarrow{i} F_4 \xrightarrow{p} F_4/Spin(9)$$

In this section we first study the homology of the triple loop space of the Cayley projective plane and study $H_*(\Omega^3 F_4;\mathbb{F}_p)$ later. We do the case of $p = 2$ first.

LEMMA 3.1. *As an algebra,*

$$H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_2)=\mathbb{F}_2[Q_1^a u_5 : a \geq 0] \otimes \mathbb{F}_2[Q_1^a Q_2^b u_{20} : a \geq 0, b \geq 0].$$

Proof. In the Eilenberg–Moore spectral sequence converging to $H^*(\Omega F_4/Spin(9);\mathbb{F}_2)$, we have

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(F_4/Spin(9);\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \\ &= E(y_7) \otimes \Gamma(y_{22}). \end{aligned}$$

By the degree reason, there can not be non–trivial differential. Hence $E_2 = E_\infty$ and we get

$$H^*(\Omega F_4/Spin(9);\mathbb{F}_2) = E(y_7) \otimes \Gamma(y_{22}).$$

Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 F_4/Spin(9);\mathbb{F}_2)$ with

$$\begin{aligned} E_2 &= \mathrm{Ext}^{H^*(\Omega F_4/Spin(9);\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \\ &= \mathbb{F}_2[z_6] \otimes \mathbb{F}_2[Q_1^a z_{21} : a \geq 0]. \end{aligned}$$

This is a spectral seqence of a Hopf algebra. So the source of the first non-trivial differential is an indecomposable element and the target of the first non-trivial differential is a primitive element. In E_2, all generators are of bidegree $(-1,7)$ or of bidegree $(-1,22\times 2^n)$ for $n\geq 0$. So the bidegree of the target of the first differential is of the form $(-1-m,6+m)$ or $(-1-m,22\times 2^n+m-1)$. In the other hand every primitive element in the E_2-term is of bidegree $(-2^m,7\times 2^m)$ or $(-2^m,22\times 2^{m+n})$ for any $m,n\geq 0$. But $22\times 2^n-2=2(11\times 2^n-1)$ can not be of the form $7\times 2^m-2^m=2(2^m\times 3)$ for any $m,n\geq 0$. Therefore there is no non trivial differential. Hence the spectral sequence collapses from the E^2-term and we have

$$H_*(\Omega^2F_4/Spin(9);\mathbb{F}_2)=\mathbb{F}_2[z_6]\otimes\mathbb{F}_2[Q_1^a z_{21}:a\geq 0].$$

Consider the following fibration

$$\Omega^3F_4/Spin(9)\longrightarrow\Omega^2Spin(9)\longrightarrow\Omega^2F_4$$

and the corresponding Serre spectral sequence with

$$E^2=H_*(\Omega^2F_4;\mathbb{F}_2)\otimes H_*(\Omega^3F_4/Spin(9);\mathbb{F}_2).$$

From [5] we know that

$$\begin{aligned}H_*(\Omega^2Spin(9);\mathbb{F}_2)=&E(z_1)\otimes\mathbb{F}_2[z_6]\\&\otimes\mathbb{F}_2[Q_1^a z_i:a\geq 0,i=5,7,9,13].\end{aligned}$$

From Theorem 2.6,

$$\begin{aligned}H_*(\Omega^2F_4;\mathbb{F}_2)=&E(z_1)\otimes\mathbb{F}_2[z_6]\\&\otimes\mathbb{F}_2[Q_1^a z_i:a\geq 0,i=7,9,13,21].\end{aligned}$$

Since the homology of the total space has a 5-dimensional generator and has no 21-dimensional generator, the homology of the fibre space must have a 5-dimensional generator, say u_5, and a 20-dimensional element, say u_{20}, which should be the target of the differential from z_{21}. Since the homology of the total space contains $Q_1^a z_5$ $a\geq 0$, by the naturality of the action of the Dyer- Lashof operators, the homology of the fibre space also has $Q_1^a u_5$. Now we consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_2))$ with

$$\begin{aligned} E^2 =& \mathrm{Cotor}_{H_*(\Omega^2 F_4/Spin(9);\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \\ =& \mathbb{F}_2[Q_1^a u_5 : a \geq 0] \otimes \mathbb{F}_2[Q_1^a Q_2^b u_{20} : a, b \geq 0]. \end{aligned}$$

Then the spectral sequence collapses from E^2 and we get the conclusion. □

Since $\pi_3(F_4) = Z$, $\pi_0(\Omega^3 F_4) = Z$. Let $\Omega_0^3 F_4$ be the zero component of $\Omega^3 F_4$.

THEOREM 3.2. *As an algebra,*

$$\begin{aligned} H_*(\Omega_0^3 F_4;\mathbb{F}_2) =& \mathbb{F}_2[Q_1^a u_5 : a \geq 0] \\ & \otimes \mathbb{F}_2[Q_1^a Q_2^b u_i : i = 6, 8, 12, 20, a \geq 0, b \geq 0]. \end{aligned}$$

Proof. Consider the following sequence of fibrations:

$$\Omega^3 Spin(9) \xrightarrow{\Omega^3 i} \Omega^3 F_4 \xrightarrow{\Omega^3 p} \Omega^3 F_4/Spin(9) \xrightarrow{\Omega^2 \gamma} \Omega^2 Spin(9)$$

and the Serre spectral sequence converging to $H_*(\Omega_0^3 F_4;\mathbb{F}_2)$ with

$$E^2 = H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_2) \otimes H_*(\Omega_0^3 Spin(9);\mathbb{F}_2).$$

From [5] we know that

$$\begin{aligned} H_*(\Omega_0^3 Spin(9);\mathbb{F}_2) =& \mathbb{F}_2[Q_1^a u_5 : a \geq 0] \\ & \otimes \mathbb{F}_2[Q_1^a Q_2^b u_i : i = 4, 6, 8, 12, a \geq 0, b \geq 0]. \end{aligned}$$

From the proof of Lemma 3.1 we know that

$$(\Omega^2 \gamma)_*(Q_1^a(u_5)) = Q_1^a(z_5), a \geq 0.$$

Since $\Omega^2 \gamma \circ \Omega^3 p$ is null-homotopic, $Q_1^a(u_5)$ can not survive to E^∞. This means that we have the following differentials

$$d(Q_0^a Q_1^b(u_5)) = Q_1^a Q_2^b(u_4), \quad a \geq 0, b \geq 0$$

from the properties of the Dyer-Lashof operations in Proposition (1.1, (1)). Note that u_4 is the unique 4-dimensional primitive element in $H_*(\Omega_0^3 Spin(9); \mathbb{F}_2)$. Since this is a spectral seqence of a Hopf algebra, we have

$$\Delta_*(d(u_{20})) = d(\Delta_*(u_{20})).$$

Hence $d(u_{20})$ should be primitive since u_{20} is primitive. But there is no other 19-dimensional primitive element than $Q_1^2 u_4$ in $H_*(\Omega_0^3 Spin(9); \mathbb{F}_2)$, so that $d(u_{20}) = 0$ up to choices of generators. Hence by the properties of the Dyer-Lashof operations, we have

$$d(Q_1^a Q_2^b(u_{20})) = 0,$$

so that $Q_1^a Q_2^b(u_{20})$ survives permanently for each $a, b \geq 0$. Hence the result follows. □

Now we turn to the case of the homology with $\mathbb{F}_3$ coefficients.

LEMMA 3.3. [4,12] *As an algebra,*

$$\begin{aligned}
H_*(\Omega^2 SU(9); \mathbb{F}_3) = {} & E(Q_2^a z_{2i-1}; i = 1,2,4,5,7,8\ a \geq 0) \\
& \otimes \mathbb{F}_3[Q_4^a z_i : i = 16, 34, a \geq 0] \otimes \mathbb{F}_3[\beta Q_2^a z_{2i-1} : i = 4,5,7,8, a > 0]. \\
H_*(\Omega_0^3 SU(9); \mathbb{F}_3) = {} & \mathbb{F}_3[Q_4^a(Q_4[1] * [-3]) : a \geq 0] \\
& \otimes \mathbb{F}_3[Q_4^a u_{2i-2} : i = 2,4,5,7,8, a \geq 0] \\
& \otimes E(Q_2^a Q_6^b u_{6i-3} : i = 3, 6, a \geq 0, b \geq 0) \\
& \otimes \mathbb{F}_3[\beta Q_2^a Q_6^b u_{6i-3} : i = 3, 6, a > 0, b \geq 0] \\
& \otimes E(Q_2^a \beta Q_4^b u_{2i-2} : i = 4,5,7,8, a \geq 0, b > 0) \\
& \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{2i-2} : i = 4,5,7,8, a > 0, b > 0],
\end{aligned}$$

where $[1] \in H_*(\Omega_1^3 SU(9); \mathbb{F}_3)$ is the image of the generator in $\tilde{H}_0(S^0)$ for the map: $S^0 \to \Omega^3 SU(9)$.

By the Harris splitting [9],

$$SU(2n+1) \simeq_{(3)} (SU(2n+1)/SO(2n+1)) \times SO(2n+1).$$

Hence from the cases of the homologies of the double and triple loop spaces of $SU(9)$[12], we easily get the following corollary.

COROLLARY 3.4. *As an algebra,*

$$\begin{aligned}
&H_*(\Omega^2 Spin(9);\mathbb{F}_3) = H_*(\Omega^2 SO(9);\mathbb{F}_3)\\
&= E(Q_2^a z_1; a \geq 0) \otimes \mathbb{F}_3[Q_4^a z_{16} : a \geq 0] \otimes E(Q_2^a z_9 : a \geq 0)\\
&\otimes \mathbb{F}_3[\beta Q_2^a z_9 : a > 0] \otimes E(Q_2^a z_{13} : a \geq 0) \otimes \mathbb{F}_3[\beta Q_2^a z_{13} : a > 0],
\end{aligned}$$

$$\begin{aligned}
&H_*(\Omega_0^3 Spin(9);\mathbb{F}_3) = H_*(\Omega_0^3 SO(9);\mathbb{F}_3)\\
&= \mathbb{F}_3[Q_4^a(Q_4[1] * [-3]) : a \geq 0] \otimes \mathbb{F}_3[Q_4^a u_i : 8, 12, a \geq 0]\\
&\otimes E(Q_2^a Q_6^b u_{15} : a \geq 0, b \geq 0) \otimes \mathbb{F}_3[\beta Q_2^a Q_6^b u_{15} : a > 0, b \geq 0]\\
&\otimes E(Q_2^a \beta Q_4^b u_i : i = 8, 12, a \geq 0, b > 0)\\
&\otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_i : i = 8, 12, a > 0, b > 0].
\end{aligned}$$

Similar to the case of the coefficients $\mathbb{F}_2$, by computing Eilenberg–Moore spectral sequnce twice we get the following lemma.

LEMMA 3.5. *As an algebra,*

$$\begin{aligned}
H_*(\Omega^2 F_4/Spin(9);\mathbb{F}_3) = &\mathbb{F}_3[z_6] \otimes E(Q_2^a z_{21} : a \geq 0)\\
&\otimes \mathbb{F}_3[\beta Q_2^a z_{21} : a > 0].
\end{aligned}$$

From Theorem 2.6, we can analyze the Serre spectral sequence converging to $H_*(\Omega^2 F_4;\mathbb{F}_3)$ for the following fibration

$$\Omega^2 Spin(9) \xrightarrow{\Omega^2 i} \Omega^2 F_4 \xrightarrow{\Omega^2 p} \Omega^2 F_4/Spin(9).$$

Then we have the following differentials

$$d(Q_0^a z_6) = Q_2^{a+1} z_1, \quad a \geq 0.$$

And $(Q_0^a z_6)^2 Q_2^{a+1} z_1$ survives eventually and becomes $E(Q_2^a z_{17})$ in $H_*(\Omega^2 F_4;\mathbb{F}_3)$. Then we have

$$\begin{aligned}
&\Omega^2 i_*(Q_2^a z_j) = Q_2^a z_j, a \geq 0, j = 9, 13, 21\\
&\Omega^2 i_*((\beta Q_2^a z_j)^k) = (\beta Q_2^a z_j)^k, a > 0, j = 9, 13, 21, k \geq 0,\\
&\Omega^2 i_*(z_{16}) = z_{16},\\
&\Omega^2 i_*((Q_4^a z_{16})^k) = (\beta Q_2^a z_{17})^k, a > 0, k \geq 0.
\end{aligned}$$

LEMMA 3.6. *As an algebra,*

$$\begin{aligned} H_*(\Omega^3 F_4/Spin9;\mathbb{F}_3) = E(Q_2^a u_5 : a \geq 0) \otimes \mathbb{F}_3[Q_4^a u_{16} : a \geq 0] \\ \otimes\mathbb{F}_3[Q_4^a u_{20} : a \geq 0] \otimes E(Q_2^a \beta Q_4^b u_{20} : a \geq 0, b > 0) \\ \otimes\mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{20} : a > 0, b > 0]. \end{aligned}$$

Proof. Consider the following morphisms of fibrations:

$$\begin{array}{ccccc} \Omega^3 F_4 & \longrightarrow & * & \longrightarrow & \Omega^2 F_4 \\ \downarrow & & \downarrow & & \| \\ \Omega^3 F_4/Spin(9) & \xrightarrow{\Omega^2\gamma} & \Omega^2 Spin(9) & \xrightarrow{\Omega^2 i} & \Omega^2 F_4 \end{array}$$

Now we study the Serre spectral sequence converging to $H_*(\Omega^2 Spin(9);\mathbb{F}_3)$ with

$$E^2 = H_*(\Omega^2 F_4;\mathbb{F}_3) \otimes H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_3).$$

$H_*(\Omega^2 Spin(9);\mathbb{F}_3)$ has the generator $Q_2 z_1$ but $H_*(\Omega^2 F_4;\mathbb{F}_3)$ has no generator of degree 5. So the homology of the fiber space must contain the element of degree 5, say u_5, and by the natuality of the action of the Dyer–Lashof operators the homology of the fiber space contains $Q_2^a u_5$ for $a \geq 0$. In $H_*(\Omega^2 Spin(9);\mathbb{F}_3)$ there are no generators of the same degree as $Q_2^a z_{17}, a \geq 0$. So we have the differentials from $Q_2^a z_{17}, a \geq 0$. Since the element $Q_2^a z_{17}$ is odd primitive for each $a \geq 0$, $Q_2^a z_{17}$ is transgressive in the path–loop fibration by Proposition 1.2. So by naturality the homology of $\Omega^3 F_4/Spin(9)$ have $Q_4^a u_{16}, a \geq 0$ such that

$$d(Q_2^a z_{17}) = Q_4^a u_{16}, a \geq 0.$$

From the facts

$$\begin{aligned} &\Omega^2 p_*(Q_2^a z_{21}) = Q_2^a z_{21}, a \geq 0, \\ &\Omega^2 p_*((\beta Q_2^a z_{21})^k) = (\beta Q_2^a z_{21})^k, a > 0, k \geq 0, \end{aligned}$$

we have differentials from $Q_2^a z_{21}, a \geq 0$ and $Q_0^a \beta Q_2^b z_{21}, a \geq 0, b > 0$ such that

$$\begin{aligned} &d(Q_2^a z_{21}) = Q_4^a u_{20}, \\ &d(Q_0^a \beta Q_2^b z_{21}) = Q_2^a \beta Q_4^b u_{20}, \\ &d((Q_0^a \beta Q_2^b z_{21}))^2 Q_2^a \beta Q_4^b u_{20}) = -\beta Q_2^{a+1} \beta Q_4^b u_{20}. \end{aligned}$$

Now we consider the Eilenberg–Moore spectral sequence for the above fibration converging to $H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_3)$ with

$$\begin{aligned}E^2=&\mathrm{Cotor}^{H_*(\Omega^2 F_4;\mathbb{F}_3)}(\mathbb{F}_3, H_*(\Omega^2 Spin(9);\mathbb{F}_3))\\ =&\mathrm{Cotor}^{H_*(\Omega^2 F_4;\mathbb{F}_3)//\Omega^2 i_*}(\mathbb{F}_3,\mathbb{F}_3)\otimes E(Q_2^a u_5 : a\geq 0)\\ =&E(Q_2^a u_5 : a\geq 0)\otimes \mathbb{F}_3[Q_4^a u_{16} : a\geq 0]\otimes \mathbb{F}_3[Q_4^a u_{20} : a\geq 0]\\ &\otimes E(Q_2^a\beta Q_4^b u_{20} : a\geq 0, b>0)\otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{20} : a>0, b>0].\end{aligned}$$

The above information about the Serre spectral sequence implies that the Eilenberg–Moore spectral sequence collapses from E^2 and we get the conclusion. □

THEOREM 3.7. *As an algebra,*

$$\begin{aligned}&H_*(\Omega_0^3 F_4;\mathbb{F}_3)=\mathbb{F}_3[Q_4^a u_{16} : a\geq 0]\otimes E(Q_2^a Q_6^b u_{15} : a\geq 0, b\geq 0)\\ &\otimes\mathbb{F}_3[\beta Q_2^a Q_6^b u_{15} : a>0, b\geq 0]\otimes E(Q_2^a\beta Q_4^b u_i : i=8,12,20, a\geq 0, b>0)\\ &\otimes\mathbb{F}_3[Q_4^a u_i : i=8,12,20, a\geq 0]\otimes \mathbb{F}_3[\beta Q_2^a\beta Q_4^b u_i : i=8,12,20, a>0, b>0].\end{aligned}$$

Proof. We have the following sequence of fibrations

$$\Omega^3 Spin(9)\longrightarrow \Omega^3 F_4\longrightarrow \Omega^3 F_4/Spin(9)\xrightarrow{\Omega^2\gamma}\Omega^2 Spin(9).$$

Consider the Serre spectral sequence converging to $H_*(\Omega_0^3 F_4;\mathbb{F}_3)$ with

$$E^2=H_*(\Omega^3 F_4/Spin(9);\mathbb{F}_3)\otimes H_*(\Omega_0^3 Spin(9);\mathbb{F}_3).$$

By Corollary 3.4 we have

$$\begin{aligned}&H_*(\Omega_0^3 Spin9;\mathbb{F}_3)=H_*(\Omega_0^3 SO(9);\mathbb{F}_3)\\ &=\mathbb{F}_3[Q_4^a(Q_4[1]*[-3]) : a\geq 0]\otimes\mathbb{F}_3[Q_4^a u_i : 8,12, a\geq 0]\\ &\otimes E(Q_2^a Q_6^b u_{15} : a\geq 0, b\geq 0)\otimes\mathbb{F}_3[\beta Q_2^a Q_6^b u_{15} : a>0, b\geq 0]\\ &\otimes E(Q_2^a\beta Q_4^b u_i : i=8,12, a\geq 0, b>0)\\ &\otimes\mathbb{F}_3[\beta Q_2^a\beta Q_4^b u_i : i=8,12, a>0, b>0].\end{aligned}$$

From Lemma 3.6, we have

$$\Omega^2\gamma_*(Q_2^a u_5)=Q_2^{a+1} z_1, a\geq 0.$$

So from the uniqueness of the 4-dimensional primitive element we have the following differential

$$d(Q_2^a u_5)=Q_4^a(Q_4[1]*[-3]).$$

Since there is no 19 dimensional primitive element in $H_*(\Omega_0^3 Spin9;\mathbb{F}_3)$, the element u_{20} survives. Now we should determine whether there is a differential from u_{16} to the 15-dimensional primitive element u_{15}. Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega_0^3 F_4;\mathbb{F}_3)$ with

$$E^2 = \mathrm{Cotor}^{H_*(\Omega^2 F_4;\mathbb{F}_3)}(\mathbb{F}_3,\mathbb{F}_3).$$

Then the 16-dimensional indecomposable element lies in (-1, 17) and the 15-dimensional primitive element lies in (-1, 16). Therefore there can not be non trivial differential. Hence the differential from x_{16} is trivial. So we get the conclusion. □

For each odd prime p, there exist a simply connected finite CW complex X whose localization $X(p)$ at p is an H-space with

$$H^*(X(p);\mathbb{F}_3) = \mathbb{F}_3[\beta\mathcal{P}^1 x_3]/((\beta\mathcal{P}^1 x_3)^3)\otimes E(x_3,\mathcal{P}^1 x_3).$$

Though F_4 is not quasi 3-regular, it can be decomposed as a product [8];

$$F_4 \simeq_{(3)} X(3)\times B(11,15).$$

So from the homology of triple loop space of F_4, we get that

COROLLARY 3.8. *As an algebra,*

$$\begin{gathered}H_*(\Omega_0^3 X(3);\mathbb{F}_3) = \mathbb{F}_3[Q_4^a u_{16} : a \geq 0]\\ \otimes E(Q_2^a Q_6^b u_{15} : a \geq 0, b \geq 0)\\ \otimes \mathbb{F}_3[\beta Q_2^a Q_6^b u_{15} : a > 0, b \geq 0]\\ \otimes E(Q_2^a \beta Q_4^b u_{20} : a \geq 0, b > 0)\\ \otimes \mathbb{F}_3[\beta Q_4^a u_{20} : a \geq 0] \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b u_{20} : a > 0, b > 0].\end{gathered}$$

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