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A BASIS OF THE SPACE OF MEROMORPHIC QUADRATIC DIFFERENTIALS ON RIEMANN SURFACES

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ABSTRACT. It is proved [6] that there exists a basis of L^Γ (the space of meromorphic vector fields on a Riemann surface, holomorphic away from two fixed points) represented by the vector fields which have the expected zero or pole order at the two points. In this paper, we carry out the same task for the quadratic differentials. More precisely, we compute a basis of Q^Γ (the space of meromorphic quadratic differentials on a Riemann surface, holomorphic away from two fixed points). This basis consists of the quadratic differentials which have the expected zero or pole order at the two points. Furthermore, we show that Q^Γ has a Lie algebra structure which is induced from the Krichever-Novikov algebra L^Γ.

0. Introduction

The Riemann-Roch Theorem is central in the theory of compact Riemann surfaces. It tells us there are how many linearly independent meromorphic functions satisfying certain restrictions on their poles. Moreover, we can translate the results on the existence of meromorphic functions to the existence of holomorphic sections of a certain line bundle.

A special choice of basis for meromorphic sections of line bundles, in which all poles lie at the punctures, allows the decomposition of field operators (which are sections of bundles) into modes analogous to the

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standard decomposition on the sphere. Many of the calculational techniques used on the sphere can be reproduced for higher genus surfaces in this basis.

A very interesting development in this direction can be found in three papers by Krichever and Novikov [3, 4, 6]. The starting point of this paper is the construction of the Krichever-Novikov algebra.

To state the result we have to introduce 2-Weierstraβ points. Let Γ be a compact Riemann surface with genus $g \geq 2$. For a point $P \in \Gamma$ to be a 2-Weierstraβ point it is equivalent that there exists a quadratic differential with a zero of order $\geq 3g-3$ at the point P (The classical Weierstraβ points are with regard to differentials. There the condition is : there exists a holomorphic differential with a zero of order $\geq g$ at the point P). There are only finitely many 2-Weierstraβ points on Γ [1,p.84].

THEOREM 1 (3,4,6). *Let $P_\pm$ be two fixed points on a Riemann surface Γ with genus $g \geq 2$, none of them is a 2-Weierstraβ point. Let L^Γ be the space of meromorphic vector fields on the Riemann surface, holomorphic away from the points $P_\pm$. Then there exists a basis of L^Γ represented by the vector fields e_i which are defined and uniquely determined up to a constant by the followng behaviour near $P_\pm$:*

$$e_i = a_i^\pm z_\pm^{\pm i - g_0 + 1}(1 + O(z_\pm))\frac{\partial}{\partial z_\pm},$$

where $g_0 = \frac{3}{2}g$, i takes on integral values ($i = \cdots, -1, 0, 1, \cdots$) for even g and half-integral values ($i = \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots$) for odd g, and z_+ (resp. z_-) the local coordinate around P_+ (resp. P_-).

With the Lie bracket of vector fields, the space L^Γ carries a natural structure of a Lie algebra. This algebra is called the Krichever-Novikov algebra (see section 3).

In this paper, we carry out the same task for the quadratic differentials. More precisely, we prove the following.

THEOREM 2. *Let $P_\pm$ be two fixed points on a Riemann surface Γ with genus $g \geq 2$, none of them is a 2-Weierstraβ point. Let Q^Γ be the space of meromorphic quadratic differentials holomorphic away from the points $P_\pm$. Then there exists a basis $\{E_i\}$ of Q^Γ with the following behaviour near $P_\pm$:*

$$E_i = c_i^\pm z_\pm^{\mp i + g_0 - 2}(1 + O(z_\pm))(dz_\pm)^2, \quad c_i^\pm \in \mathbb{C},$$

where g_0 and $z_\pm$ are the same as in Theorem 1.

The space of Q^Γ has a Lie algebra structure which is induced from the Krichever-Novikov algebra L^Γ

1. Preliminary

The following result follows from Riemann-Roch Theorem.

PROPOSITION 3. *Let X be a compact Riemann surface of genus g and L be a holomorphic line bundle on X. Then*

$$\dim H^0(X,L)\begin{cases} = 0, & \deg L < 0 \\ \geq 1 - g + \deg L, & 0 \leq \deg L < 2g-1 \\ = 1 - g + \deg L, & \deg L \geq 2g-1\ . \end{cases}$$

Suppose Γ is a compact Riemann surface with genus $g \geq 2$. Let p_i be points and n_i be integers ($i = 1, 2, \ldots, k$). We set

$$R = \bigotimes_i L_{p_i}^{\otimes n_i},$$

where L_{p_i} is the line bundle which has a section with exactly one zero at the point p_i and vanishes nowhere else.

LEMMA 4. *Suppose L is a line bundle on the Riemann surface Γ and $M = L \otimes R$. Then the space $H^0(\Gamma, M)$ of holomorphic sections of M is isomorphic to the space of meromorphic sections of the bundle L which are holomorphic outside the points p_i and have at most a pole of order n_i at the point p_i (have a zero of order at least $-n_i$ if $n_i < 0$. $i = 1, 2, \ldots, k$).*

LEMMA 5. *Let D be a divisor with $\dim H^0(\Gamma, \mathcal{O}_D) = \gamma$. Then*

$$\gamma \leq \dim H^0(\Gamma, \mathcal{O}_{D+P}) \leq \gamma + 1.$$

LEMMA 6. *For a positive integer n, if a point $P \in \Gamma$ is not a 2-Weierstraß point then*

$$\dim H^0(\Gamma, \mathcal{O}_{2K-nP}) = \begin{cases} (3g-3) - n, & \text{if } n \leq 3g-3 \\ 0, & \text{if } n \geq 3g-3 \end{cases}$$

where $2K$ *is the divisor for the quadratic differentials of* Γ.
Inparticular

$$\dim H^0(\Gamma, \mathcal{O}_{-K+nP}) = n - (3g-3) if n \geq 3g-3$$

Proof. $\dim H^0(\Gamma, \mathcal{O}_{2K-nP})$ is the dimension of the space of quadratic differentials having a zero of order $\geq n$ at P. Since the point P is not 2-Weierstraß point,

$$\dim H^0(\Gamma, \mathcal{O}_{2K-nP}) = 0 \quad \text{for} \quad n \geq 3g-3.$$

Now

$$\deg(2K - nP) = 2(2g-2) - n.$$

If $n \leq 2g-3$ then $\deg(2K - nP) \geq 2g-1$, hence

$$\begin{aligned} \dim H^0(\Gamma, \mathcal{O}_{2K-nP}) &= 1 - g + \deg(2K - nP) \\ &= 1 - g + 4g - 4 - n \\ &= (3g-3) - n \quad (\geq g). \end{aligned}$$

By Lemma 5, the dimension will drop by one from $n = 2g-3$ to $n = 3g-3$

n	0	$\cdots$	$2g-3$	$\cdots$	$3g-3$	$\cdots$
$\dim H^0$	$3g-3$	$\cdots$	g	$\cdots$	0	0

Hence

$$\dim H^0(\Gamma, \mathcal{O}_{2K-nP}) = (3g-3) - n \quad \text{for} \quad n \leq 3g-3.$$

2. Proof of Theorem 2

Let us take the bundles

$$M_i = \omega^2 \otimes L_P^{i-g_0+2} \otimes L_P^{-i-g_0+2},$$

where $\omega^2 = \omega \otimes \omega$ is the bundle of quadratic differentials.
Then

$$\begin{aligned} \deg M_i &= 2(2g-2) + (i - g_0 + 2) + (-i - g_0 + 2) \\ &= g < 2g - 1. \end{aligned}$$

By Proposition 3,

$$\dim H^0(\Gamma, M_i) \geq 1 - g + \deg M_i = 1.$$

We will show that

$$\dim H^0(\Gamma, M_i) = 1.$$

and the corresponding (up to a scalar multiple) unique meromorphic section of ω^2 has exactly the required zero or ploe order at $P_\pm$ (see Lemma 4). Let $i - g_0 + 2 = -n$, i.e. $-i - g_0 + 2 = -3g + 4 + n$.

Then

$$H^0(\Gamma, \mathcal{O}_{2K-nP_+-(3g-4-n)P_-}) \cong H^0(\Gamma, M_i).$$

To prove this, we use the Riemann Roch theorem

$$\begin{aligned}\dim H^0(\Gamma, \mathcal{O}_{2K-nP_+-(3g-4-n)P_-}) &- \dim H^0(\Gamma, \mathcal{O}_{-K+nP_++(3g-4-n)P_-}) \\ &= 1 - g + \deg M_i = 1.\end{aligned}$$

Case 1. $n \geq 3g - 3$.
If $n = 3g - 3$, then

$$\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_++P_-}) - \dim H^0(\Gamma, \mathcal{O}_{-K+(3g-3)P_+-P_-}) = 1.$$

By Lemma 6,

$$\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_+}) = 0$$

hence

$$\begin{aligned}\dim H^0(\Gamma, \mathcal{O}_{-K+(3g-3)P_+}) &= -g + 1 + \deg(-K + (3g-3)P_+) \\ &= 0.\end{aligned}$$

So

$$\dim H^0(\Gamma, \mathcal{O}_{-K+(3g-3)P_+-P_-}) = 0.$$

Now we get

$$\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_++P_-}) = 1.$$

Since $\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_+}) = 0$, the generator f_1 of this space has the right pole order at P_-. Since $\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-2)P_++P_-}) = 0$, the generator f_1 has the right zero order at P_+.
Let me deduce this for $n = 3g - 2$. The proof for the general case is essentially the same and goes by induction.
Since

$$\dim H^0(\Gamma, \mathcal{O}_{-K+(3g-3)P_+-2P_-}) = 0.$$

By Riemann-Roch we get

$$\dim H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_++2P_-}) = 2.$$

Let f_1 be as above and let f be a second element, such that f_1, f is a basis of this vector space. We can solve the equation

$$af_1^{(3g-3)}(P_+) + cf^{(3g-3)}(P_+) = 0, \quad c \neq 0.$$

$f_2 = af_1 + cf$ is a vector field such that f_1, f_2 is again a basis. f_2 has at least a zero of order $3g-2$ at P_+. We do not want f_2 to have a higher order zero. For this we have to make sure that

$$af_1^{(3g-2)}(P_+) + cf^{(3g-2)}(P_+) \neq 0$$

by choosing P_+ suitable. Now f_2 generates $H^0(\Gamma, \mathcal{O}_{2K-(3g-2)P_+ + 2P_-})$. It has the right zero order at P_+. If we assume that it has not the full pole order 2 at P_- it would be an element of $H^0(\Gamma, \mathcal{O}_{2K-(3g-3)P_+ + P_-})$, hence it would be a multiple of f_1 in contradiction to its construction.

Case 2. $0 < n \leq 3g-4$.
If we take P_- general enough, then

$$((*)) \quad \dim H^0(\Gamma, \mathcal{O}_{2K-nP_+-kP_-}) = 3g-3-n-k \text{ for } 0 \leq k \leq 3g-3-n.$$

So we obtain

$$\dim H^0(\Gamma, \mathcal{O}_{2K-nP_+-(3g-4-n)P_-}) = 1.$$

The generator again has the minimal zero order at $P_\pm$. If it has a zero order at P_+ more than the expected, it would be an element of

$$H^0(\Gamma, \mathcal{O}_{2K-(n+1)P_+-(3g-4-n)P_-}).$$

But this space is zero by (*). The argument for P_- is the same.
Now we get

$$\dim H^0(\Gamma, M_i) = 1$$

and the corresponding meromorphic section is given by quadratic differentials E_i with the following behaviour near $P_\pm$:

$$E_i = c_i^\pm z_\pm^{\mp i + g_0 - 2}(1 + O(z_\pm))(dz_\pm)^2, \quad c_i^\pm \in \mathbb{C}.$$

3. A Lie Algebra Structure of Q^Γ

PROPOSITION 7. *For the basis e_i of the vector fields L^Γ, the space L^Γ is g_0-graded algebra with respect to the Lie bracket*

$$[e_i, e_j] = \sum_{k=-g_0}^{g_0} c_{ij}^k e_{i+j-k}, \quad c_{ij}^k \in \mathbb{C},$$

where the sum is over the integers for even g and half integers for odd g.

Proof. We know that the vector field given by the Lie bracket can again be developed in these $e_i's$. To get its coefficients, we have to calculate its order of the poles and zeros at the point $P_\pm$. We start with P_+. e_i and e_j may be given locally as above. Then we calculate ($a_i^+ = 1$):

$$\begin{aligned}
[e_i, e_j] &= z_+^{i-g_0+1}(1 + O(z_+))\frac{\partial}{\partial z_+}\left\{z_+^{j-g_0+1}(1 + O(z_+))\frac{\partial}{\partial z_+}\right\} \\
&\quad -z_+^{j-g_0+1}(1 + O(z_+))\frac{\partial}{\partial z_+}\left\{z_+^{i-g_0+1}(1 + O(z_+))\frac{\partial}{\partial z_+}\right\} \\
&= z_+^{(i+j-g_0)-g_0+1}((j - i) + O(z_+))\frac{\partial}{\partial z_+}.
\end{aligned}$$

The order of the zero of this vector field is $\geq (i + j - g_0) - g_0 + 1$. Hence only $e_r's$ with

$$r \geq (i + j - g_0)$$

are involved.

At the point P_-, we calculate

$$\begin{aligned}
[e_i, e_j] &= a_i^- z_-^{-i-g_0+1}(1 + O(z_-))\frac{\partial}{\partial z_-}\left\{a_j^- z_-^{-j-g_0-1}(1 + O(z_-))\frac{\partial}{\partial z_-}\right\} \\
&\quad -a_j^- z_-^{-j-g_0+1}(1 + O(z_-))\frac{\partial}{\partial z_-}\left\{a_i^- z_-^{-i-g_0-1}(1 + O(z_-))\frac{\partial}{\partial z_-}\right\} \\
&= a_i^- a_j^- z_-^{-(i+j+g_0)-g_0+1}((i - j) + O(z_-))\frac{\partial}{\partial z_-}.
\end{aligned}$$

The order of the pole is $\leq (i + j - g_0) + g_0 - 1$. Hence $e_r's$ with

$$r \leq (i + j + g_0)$$

are involved.

Combining these two facts, we get

$$[e_i, e_j] = \sum_{k=-g_0}^{g_0} c_{ij}^k e_{i+j-k}, \quad c_{ij}^k \in \mathbb{C}.$$

This Lie algebra is called the Krichever-Novikov algebra. In our proof we are even able to give the exact formula for the coefficients in the extremal cases.

$$c_{ij}^{g_0} = j - i, \quad c_{ij}^{-g_0} = (i - j)\frac{a_i^- a_j^-}{a_{i+j+g_0}^-}.$$

Using the Lie algebra structure of L^Γ, one can define a Lie algebra structure on the space Q^Γ as follows ;

PROPOSITION 8. *The space Q^Γ is a g_0-graded algebra with respect to the Lie bracket*

$$[E_i, E_j] = \sum_{k=-g_0}^{g_0} d_{ij}^k E_{i+j-k}, \quad d_{ij}^k \in \mathbb{C},$$

where the sum is over the integers for even g and half integers for odd g.

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