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OPTIMAL PROBLEM OF REGULAR COST FUNCTION FOR RETARDED SYSTEM

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ABSTRACT. We study the optimal control problem of system governed by retarded functional differential

$$x'(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + B_0u(t)$$

in Hilbert space H. After the fundamental facts of retarded system and the description of condition so called a weak backward uniqueness property are established, the technically important maximal principle and the bang-bang principle are given. its corresponding linear system.

1. Introduction

In this paper we deal with the control problem for retarded functional differential equation:

(1.1)
$$\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + B_0u(t),$$

(1.2)
$$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0)$$

in Hilbert space H. We investigate the optimization of control functions appearing as the cost function with particular objective.

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We solve the optimization problem by introducing the structural operator F and the transposed dual system in the sense of S. Nakagiri [7].

In section 2, we consider some the regurality and a formular representation for functional differential equations in Hilbert spaces. We establish a form of the mild solution which is described by the integral equation in terms of fundamental solution using structural operator. In section 3, we shall give a cost function, which is called the feedback control law for regulator problem and consider results on the existence and uniqueness of optimal control on some admissible set. After considering the relation between the operator A_1 and the structural operator F, we will give the condition so called a weak backward uniqueness property. Maximal principle and bang-bang principle for the given technologically important cost function are also derived.

2. Functional differential equation with time delay

Let H be a Hilbert spaces and V be a dense subspace in H. The norm on V(resp. H) will be denoted by $\|\cdot\|$ (resp. $|\cdot|$) and the corresponding scalar products will be denoted by $((\cdot,\cdot))$(resp. $(\cdot,\cdot)$). Assume that the injection of V into H is continuous. The antidual of V is denoted by V^*, and the norm of V^* by $\|\cdot\|^*$. Identifying H with its antidual we may consider that H is embedded in V^*. Hence we have $V \subset H \subset V^*$ densely and continuously. If the operator A_0 is a bounded linear operator from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T \|A_0 e^{tA_0} x\|_*^2 dt < \infty\}, \tag{2.1}$$

for the time $T > 0$ where $\|\cdot\|_*$ is the norm of the element of V^*. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. Therefore, in terms of the interpolation theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H \tag{2.2}$$

and hence we can also replace the interpolation space F in the paper G. Blasio, K. Kunisch and A. Sinestrari [2] with the space H. Hence, from now on we derive the same results of [2]. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,v) \geq c_0||u||^2 - c_1|c|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A_0 be the operator associated with a sesquilinear form

$$(A_0 u, v) = -a(u,v), \quad u,\ v \in V.$$

Then A_0 generates an analytic semigroup in both H and V^* and so the equation (1.1) and (2.2) may be considered as an equation in both H and V^*.

Let the operators A_1 and A_2 be a bounded linear operators from V to V^*. The function $a(\cdot)$ is assume to be a real valued Hölder continuous in $[-h,0]$ and the controller operator B_0 is a bounded linear operator from some Hilbert space U to H. Under these conditions, from (2.2) and Theorem 3.3 of [2] we can obtain following result.

Let Z denote the product reflexive space $H \times L^2(-h,0;V)$ with the norm

$$||g||_Z = (|g^0| + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

The adjoint space Z^* of Z is identified with the product space $H \times L^2(0,T;V^*)$ via duality pairing

$$(g,h)_Z = (g^0, h^0) + \int_{-h}^{0} (g^1(s), h^1(s)) ds, \quad g \in Z,\ f \in Z^*$$

where $(\cdot,\cdot)$ denotes the duality pairing.

PROPOSITION 2.1. *Let $g = (g^0, g^1) \in Z$ and $u \in L^2(0,T;U)$. Then for each $T > 0$, a solution x of the equation (1.1) and (1.2) belongs to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

where $W^{1,2}(0,T;V^*)$ denotes the Sobolev space of V^*-valued measurable functions on $(0,T)$ such that itself and its distributional derivative belong to $L^2(0,T;V^*)$.

Let $x(t;g,u)$ be the solution of (1.1) and (1.2) with initial value $g=(g^0,g^1)\in Z$ and control $u\in L^2(0,T;U)$. According to S. Nakagiri [7], we define the fundamental solution $W(t)$ for (1.1) and (1.2) by

$$W(t)g^0=\begin{cases} x(t;(g^0,0),0), & t\geq 0\\ 0 & t<0\end{cases}$$

for $g^0\in H$. Here, we note that

$$x(t;(g^0,0),0)=G(t)+\int_0^t G(t-s)(A_1W(s-h)+\int_{-h}^0 a(\tau)A_2W(\tau+s)d\tau)ds,\quad t\geq 0$$

where $G(t)$ is an analytic semigroup generated by A_0. Since we assume that $a(\cdot)$ is Hölder continuous, the fundamental solution exists as seen in [11]. It is also known that $W(t)$ is strongly continuous and $A_0W(t)$ and $dW(t)/dt$ are strongly continuous except at $t=nh,\ n=0,\ 1,\ 2,\ \cdots$.

For each $t>0$, we introduce the structural operator $F(\cdot)$ from $H\times L^2(0,T;V)$ to $H\times L^2(0,T;V^*)$ defined by

$$Fg=(g^0,F_1g^1),$$

$$F_1g^1=A_1g^1(-h-s)+\int_{-h}^s a(\tau)A_2g^1(\tau-s)d\tau$$

for $g=(g^0,g^1)\in Z$. The solution $x(t)=x(t;g,u)$ of (1.1) and (1.2) is represented by

$$x(t)=W(t)g^0+\int_{-h}^0 U_t(s)g^1(s)ds+\int_0^t W(t-s)B_0u(s)ds$$

where

$$U_t(s)=W(t-s-h)A_1+\int_{-h}^s W(t-s+\tau)a(\tau)A_2d\tau$$

for $t\geq 0$.

PROPOSITION 2.2. *If $A_1 : V \longrightarrow V^*$ is an isomorphism, then $F : Z \longrightarrow Z^*$ is an isomorphism.*

Proof. For $f \in Z^*$ the element $g \in Z$ satisfying $g^0 = f^0$ and

$$g^1(-h-s) + \int_{-h}^{s} a(\tau)A_1^{-1}A_2 g^1(\tau - s)d\tau = A_1^{-1}f^1(s)$$

is the unique solution of $Fg = f$. The integral equation mentioned above is of Volterra type, and so it can be solved by successive approximation method. □

The following result is obtained from Lemma 5.1 in [8].

LEMMA 2.1. *Let $f \in L^p(0,T;H)$, $1 \le p \le \infty$. If*

$$\int_0^t W(t-s)f(s)ds = 0, \quad 0 \le t \le T,$$

then $f(t) = 0$ a.e. $0 \le t \le T$.

THEOREM 2.1. *Let A_1 be an isomorphism. Then the solution $x(t;g,0)$ is identically zero on a positive measure containing zero in $[-h,T]$ for $T \ge h$ if and only if $g^0 = 0$ and $g^1 \equiv 0$.*

Proof. With the change of variable and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-h}^{0} U_t(s)g^1(s)ds \\
&\quad = \int_{-h}^{0} W(t-s-h)A_1 g^1(s)ds \\
&\quad + \int_{-h}^{0}\Big(\int_{-h}^{s} W(t-s+\tau)a(\tau)A_2 d\tau\Big)g^1(s)ds \\
&\quad = \int_{-h}^{0} W(t+s)\{A_1\chi_{(-h,0)}(s)g^1(-h-s) \\
&\quad + \int_{-h}^{s} a(\tau)A_2(\tau)g^1(\tau-s)d\tau\}ds \\
&\quad = \int_{-h}^{0} W(t+s)[F_1 g^1](s)ds.
\end{aligned}$$

Thus the mild solution $x(t;g,0)$ is represented by

$$x(t) = W(t)g^0 + \int_{-h}^{0} W(t+s)[F_1 g^1](s)ds.$$

Thus, we have that $x(0) = W(0)g^0 = g^0 = 0$ in H. Because that A_1 is an isomorphism and, we obtain that F_1 is isomorphism from Proposition 2.2. Therefore from Lemma 2.1 $x(t;g,0) = 0$ if and only if $g^0 = 0$ and $g^1 \equiv 0$. □

Let $I = [0,T]$, $T > 0$ be a finite interval. We introduce the transposed system which is exactly same as in S. Nakagiri[8]. Let $q_0^* \in X^*$, $q_1^* \in L^1(I;H)$. The retarded transposed system in H is defined by

(2.3)
$$\frac{dy(t)}{dt} + A_0^* y(t) + A_1^* y(t+h) + \int_{-h}^{0} a(s)A_2 y(t-s)ds + q_1^*(t) = 0 \quad \text{a.e.} \quad t \in I,$$

(2.4)
$$y(T) = q_0^*, \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h].$$

Let $W^*(t)$ denote the adjoint of $W(t)$. Then as proved in S. Nakagiri [8], the mild solution of (2.3) and (2.4) is defined as follows:

$$y(t) = W^*(T-t)(q_0^*) + \int_t^T W^*(\xi - t)q_1^*(\xi)d\xi,$$

for $t \in I$ in the weak sense. The transposed system is used to present a concrete form of the optimality conditions for control optimization problems.

COROLLARY 2.1. *The solution $y(t)$ is identically zero on a positive measure containing T in $[T, T+h]$ if and only if $q_0^* = 0$ and $q_1^* = 0$.*

If the equation (2,3) and (2.4) satisfies the result in Corollary 2.1, the equation (2.3) and (2.4) is said to have a weak backward uniqueness property.

3. Optimality for the regular cost function

In this section, the optimal control problem is to find a control u which minimizes the cost function

$$J(u) = (Gx(T), x(T))_H + \int_0^T ((D(t)x(t), x(t))_H + (Q(t)u(t), U(t))_U)dt$$

where $x(\cdot)$ is a solution of (1.1) and (1.2), $G \in B(H)$ is self adjoint and nonnegative, and $D \in \mathcal{B}_\infty(0,T;H,H)$ which is a set of all essentially bounded operators on $(0,T)$ and $Q \in \mathcal{B}_\infty(0,T;U,U)$ are self adjoint and nonnegative, with $Q(t) \geq m$ for some $m > 0$, for almost all t. Let us assume that there exists no admissible control which satisfies $Gx(T;g,u) \neq 0$.

THEOREM 3.1. *Let U_{ad} be closed convex in $L^2(0,T;U)$. Then there exists a unique element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{3.1}$$

Moreover, it is holds the following inequality:

$$\int_0^T (B_0^* y(s) + Q(s)u(s), v(s) - u(s))ds \geq 0$$

where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx_u(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $D(t)x_u(t)$. That is, $y(t)$ satisfies the following transposed system:

(3.2)
$$\frac{dy(t)}{dt} + A_0^* y(t) + A_1^* y(t+h) + \int_{-h}^0 a(s)A_2 y(t-s)ds + D(t)x_u(t) = 0 \quad \text{a.e.} \quad t \in I,$$

(3.3)
$$y(T) = Gx_u(T), \quad y(s) = 0 \quad \text{a.e. } s \in (T, T+h]$$

in the weak sense.

Proof. Let $x(t) = x(t; g, 0)$. Then it holds that

$$J(v) = \pi(v, v)$$

where

$$\pi(u, v) = (Gx_u(T), x_v(T))_H + \int_0^T ((D(t)x_u(t), x_v(t))_H + (Q(t)u(t), v(t))_U)dt$$

The form $\pi(u, v)$ is a continuous bilinear form in $L^2(0, T; U)$ and from assumption of the positive definite of the operator Q we have

$$\pi(v, v) \geq c||v||^2 \quad v \in L^2(0, T; U).$$

Therefore in virtue of Theorem 1.1 of Chapter 1 in [6] there exists a unique $u \in L^2(0, T; U)$ such that (3.1). If u is an optimal control (cf. Theorem 1.3. Chapter 1 in [6]), then

$$J'(u)(v - u) \geq 0 \quad u \in U_{ad}, \tag{3.4}$$

where $J'(u)v$ means the Fréchet derivative of J at u, applied to v. It is easily seen that

$$\begin{aligned} x'_u(t)(v - u) &= (v - u, x'_u(t)) \\ &= x_v(t) - x_u(t). \end{aligned}$$

Since

$$\begin{aligned} J'(u)(v - u) = {} & 2(Gx_u(T), x_v(T) - x_u(T)) \\ & + 2\int_0^T (D(t)x_u(t), x_v(t) - x_u(t)) \\ & + 2(Q(t)u(t), v(t) - u(t))dt, \end{aligned}$$

(3.4) is equivalent to that

$$\begin{aligned} & \int_0^T (B_0^* W^*(T - s)(Gx_u(T), v(s) - v(s))ds + \\ & \int_0^T (B_0^* \int_s^T W^*(t - s)D(t)x_u(t)dt + Qu(s), v(s) - u(s))ds \\ & \geq 0. \end{aligned}$$

Hence

$$y(s) = W^*(T-s)Gx_u(T) + \int_s^T W^*(t-s)D(t)x_u(t)dt$$

is solves (3.2) and (3.3). □

Let the admissible set U_{ad} be

$$U_{ad} = \{u \in L^2(0,T;U) : u(t) \in W\}$$

where W is a bounded subset in U.

COROLLARY 3.1 (MAXIMAL PRINCIPLE). *Let W be bounded and $Q = 0$. If u be an optimal solution for J then*

$$\max_{v \in W}(v, \Lambda_U^{-1}B_0^*q(t)) = (u(t), \Lambda_U^{-1}B_0^*q(t))$$

almost everywhere in $0 \leq t \leq T$ where $q(t) = -y(t)$ and $y(t)$ is given by in Theorem 3.1.

Proof. We note that if U_{ad} is bounded then the set of elements $u \in U_{ad}$ such that (3.1) is a nonempty, closed and convex set in U_{ad}. Let t be a Lebesgue point of u, $v \in U_{ad}$ and $t < t + \epsilon < T$. Further, put

$$v_\epsilon(s) = \begin{cases} v, & \text{if} \quad t < s < t + \epsilon \\ u(s), & \text{otherwise.} \end{cases}$$

Then Substituting v_ϵ for v in (3.4) and dividing the resulting inequality by ϵ, we obtain

$$\frac{1}{\epsilon}\int_t^{t-\epsilon} (\Lambda_U^{-1}B_0^*q(s), u(s) - v(s))ds \geq 0.$$

Thus by letting $\epsilon \to 0$, the proof is complete. □

Thus from Theorem 3.1 the result is obtained.

THEOREM 3.2 (BANG-BANG PRINCIPLE). *Let B_0^* be one to one mapping. Then the optimal control $u(t)$ is a bang-bang control, i.e, $u(t)$ satisfies $u(t) \in \partial W$ for almost all t where ∂W denotes the boundary of U_{ad}.*

Proof. On account of Corollary 3.1 it is enough to show that $B_0^*q(t) \neq 0$ for almost all t. If $B_0^*q(t) = 0$ on a set e of positive measure containing T, then $q(t) = 0$ for each $t \in e$. By Corollary 2.1, we have $Gx_u(T) = 0$, which is a contraction. □

From now on, we consider the case where $U_{ad} = L^2(0,T;U)$. Let $x_u(t) = x(t;g,0) + \int_0^t W(t-s)B_0u(s)ds$ be solution of (1.1) and (1.2). Define $T \in B(H, L^2(0,T;H))$ and $T_T \in B(L^2(0,T;H),H)$ by

$$(T\phi)(t) = \int_0^t W(t-s)\phi(s)ds,$$
$$T_T\phi = \int_0^T W(T-s)\phi(s)ds.$$

Then we can write the cost function as

(3.5)
$$\begin{aligned} J(u) =& (G(x(T;g,0) + T_TB_0u), (x(T;g,0) + T_TB_0u))_H \\ &+ (D(x(\cdot;g,0) + TB_0u), x(\cdot;g,0) + TB_0u)_{L^2(0,T;H)} \\ &+ (Qu,u)_{L^2(0,T;U)}. \end{aligned}$$

The adjoint operators T^* and T_T^* are given by

$$(T^*\phi)(t) = \int_t^T W^*(s-t)\phi(s)ds,$$
$$(T_T^*\phi)(t) = W^*(T-t)\phi.$$

THEOREM 3.3. *Let $U_{ad} = L^2(0,T;U)$. Then there exists a unique control u such that (4.1) and*

$$u(t) = -A^{-1}B_0^*y(t)$$

for almost all t, where $A = Q + B_0^*T^*DTB_0 + B_0^*T_T^*GT_T^*B_0$ and where $y(t)$ is a solution of (2.3) and (2.4) for initial condition that $y(T) = Gx(T)$ and $y(s) = 0$ for $s \in (T, T+h]$ substituting $q_1^*(t)$ by $Dx(t)$.

Proof. The optimal control for J is unique solution of

$$J'(u)v = 0. \tag{3.6}$$

From (3.5) we have

$$\begin{aligned} J'(u)v =& 2(G(x(T;g,0) + T_T B_0 u), T_T B_0 v)) \\ &+ 2(D(x(\cdot;g,0) + TB_0 u), TB_0 v) \\ &+ 2(Qu, v) \\ =& 2((Q + B_0^* T^* D T B_0^* + B T_T^* G T_T B_0)u, v) \\ &+ 2(B_0^* T^* Dx(\cdot;g,0) + B_0^* T_T^* Gx(T;g,0), v). \end{aligned}$$

Hence (3.6) is equivalent to that

$$(Au + B_0^* T^* Dx(t;g,0) + B_0^* T_T^* Gx(T;g,0), v) = 0$$

since $A^{-1} \in \mathcal{B}_\infty(0,T;H,U)$ (see Appendix of [3]). Hence from The definitions of T and T_T it follows that

$$\begin{aligned} y(t) &= T^* Dx(t;g,0) + T_T^* Gx(T;g,0) \\ &= W^*(T-t)Gx(T) + \int_t^T W^*(s-t)Dx(t)ds. \end{aligned}$$

Therefore, the proof is complete. □

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