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THE RIEMANN PROBLEM FOR A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE (I)

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ABSTRACT. We prove the existence of solutions of the Riemann problem for a system of conservation laws of mixed type using the method of vanishing viscosity term.

0. Introduction

In this paper we study the existence of solutions of the Riemann Problem for a 2×2 system of conservation laws of the mixed type

$$\begin{aligned} u_t - f(v)_x &= 0, \\ v_t - g(u)_x &= 0 \end{aligned} \tag{0.1}$$

with the initial data

$$(u,v)(x,0) = \begin{cases} (u_+, v_+) & x > 0, \\ (u_-, v_-) & x < 0. \end{cases} \tag{0.2}$$

Here we assume
(I) $f \in C^2(\mathbb{R})$ is a strictly increasing convex function.
(II) $g \in C^2(\mathbb{R})$ and there exist α, β, η with $\alpha < \eta < \beta$ such that

$$\begin{gathered} g'(u) \geq 0 \text{ if } u \notin (\alpha, \beta) \text{ and } g'(u) < 0 \text{ for } u \in (\alpha,\beta), \\ g''(u) < 0 \text{ if } u < \eta \text{ and } g''(u) > 0 \text{ if } u > \eta. \end{gathered}$$

(III) $g(u) \to \pm\infty$ as $u \to \pm\infty$.

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If $f(v) = v$, then the typical model of this equation (0.1) is the one-dimensional isothermal motion of a compressible elastic fluid or solid in the Lagrangian coordinates. In this case the existence of solutions to the Riemann problem (0.1), (0.2) has been studied by Dafermos[1], Dafermos and DiPerna[2], Fan[3], James[4], Slemrod[6]. These approach was based on a vanishing "viscosity" term pursued by Kalashnikov[5], Tupchiev[7][8]. Their idea is to replace (0.1) with the system

$$\begin{aligned} u_t - f(v)_x &= \epsilon t u_{xx}, \\ v_t - g(u)_x &= \epsilon t v_{xx} \end{aligned} \tag{0.3}$$

for $x \in \mathbb{R}$, $t > 0$ and construct solutions as the limit of the solutions of (0.3), (0.2) as $\epsilon \to 0+$. Since the system is invariant under the transformation $(x, t) \to (ax, at)$ where $a > 0$, (0.3) and (0.2) admit solutions of the form $(u_\epsilon(\xi), v_\epsilon(\xi))$,where $\xi = \frac{x}{t}$. A simple computation shows that $u = u_\epsilon(\xi), v = v_\epsilon(\xi)$ is a solution of (0.3), (0.2) if it satisfies

$$\begin{aligned} -\xi u' - f(v)' &= \epsilon u'', \\ -\xi v' - g(u)' &= \epsilon v'' \end{aligned} \tag{0.4}$$

with the boundary condition

$$(u, v)(\pm\infty) = (u_\pm, v_\pm) \tag{0.5}$$

where $' = \frac{d}{d\xi}$ and $'' = \frac{d^2}{d\xi^2}$. We shall call the boundary value problem (0.4) and (0.5) the problem (P_ϵ). Similarly the initial value problem (0.1) and (0.2) are called the Riemann problem (P). This paper consists of two parts. The first part carried out in Section 1 and 2 establishes that if the data are in different phases there is solution of P_ϵ which exhibits one change of phase. In order to proof the results, we use the arguments of Dafermos[1] and Slemrod[6]. In second part in Section 3 and 4 we prove the existence of solution to the Riemann problem to give conditions on which solutions of P_ϵ possess limits. Throughout this paper we always assume Assumptions (I) and (II) unless other mentions it.

1. The existence theorem of the problem (P_ϵ)

In this section we will study the existence of solutions to the boundary value problem

$$(1.1)\qquad \begin{aligned} \epsilon u'' &= -\xi u' - \mu f(v)' \\ \epsilon v'' &= -\xi v' - \mu g(u)' \\ (u,v)(\pm L) &= (u_\pm, v_\pm) \end{aligned}$$

where $L > 1$, and $0 \leq \mu \leq 1$.

THEOREM 1.1. *Assume $u_- < \alpha$, $u_+ > \beta$ and there exists a constant M_0 such that every possible solution of* (1.1) *with $u'(\xi) > 0$ when $\alpha \leq u(\xi) \leq \beta$ satisfies the a priori estimate*

$$(1.2)\qquad \sup_{|\xi|<L} (|u(\xi)| + |u'(\xi)| + |v(\xi)| + |v'(\xi)|) \leq M_0$$

then P_ϵ has a solution with $u'(\xi) > 0$ if $\alpha \leq u(\xi) \leq \beta$.

Proof. Let $u_- < \alpha$, $u_+ > \beta$. Set $U(\xi) = u(\xi) - u_0(\xi)$ and $V(\xi) = v(\xi) - v_0(\xi)$, where $(u_0(\xi), v_0(\xi))$ is a unique solution of (1.1) with $\mu = 0$. Then $U(-L) = U(L) = V(L) = V(-L) = 0$. If u and v are solutions of (1.1), U, V satisfies

$$\begin{aligned} \epsilon U'' &= -\xi U' - \mu f(V + v_0)', \\ \epsilon V'' &= -\xi V' - \mu g(U + u_0)'. \end{aligned}$$

Define

$$Y(\xi) = \begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix}, F(\xi, Y) = \begin{pmatrix} -f(V + v_0) \\ -g(U + u_0) \end{pmatrix}.$$

Then

$$(1.3)\qquad \begin{aligned} \epsilon Y'' &= -\xi Y' - \mu F(\xi, Y)', \\ Y(-L) &= Y(L) = 0. \end{aligned}$$

Let $Z \in C^1([-L, L]; \mathbb{R}^2)$. Define T to be the solution map that carries Z into Y where Y solves

$$(1.4)\qquad \begin{aligned} \epsilon Y'' &= -\xi Y' + F(\xi, Z)', \\ Y(-L) &= Y(L) = 0. \end{aligned}$$

The integral formula of (1.4) is of the form

$$\begin{aligned} Y(\xi) =& c\int_{-L}^{\xi} \exp\left(-\frac{\zeta^2}{2\epsilon}\right) d\zeta + \frac{1}{\epsilon}\int_{-L}^{\xi} F(\zeta, Z(\zeta))\, d\zeta \\ &+ \frac{1}{\epsilon^2}\int_{-L}^{\xi}\int_{0}^{\zeta} \tau F(\tau, Z(\tau)) \exp\left(\frac{\tau^2 - \zeta^2}{2\epsilon}\right) d\tau d\zeta \end{aligned}$$

where

$$\begin{aligned} c\int_{-L}^{L} \exp\left(-\frac{\zeta^2}{2\epsilon}\right) d\zeta = & -\frac{1}{\epsilon}\int_{-L}^{L} F(\zeta, Z(\zeta))\, d\zeta \\ &+ \frac{1}{\epsilon^2}\int_{-L}^{L}\int_{0}^{\zeta} \tau F(\tau, Z(\tau)) \exp\left(\frac{\tau^2 - \zeta^2}{2\epsilon}\right) d\tau d\zeta \end{aligned}$$

Then $T : C^1([-L, L]; \mathbb{R}^2) \to C^1([-L, L]; \mathbb{R}^2)$ is continuous and compact. Define Ω by the set of pairs U, V in $C^1([-L, L]; \mathbb{R}^2)$ such that

$$U(-L) + u_0(-L) < \alpha, \quad U(L) + u_0(L) > \beta$$

$$U'(\xi) + u_0'(\xi) > 0 \text{ if } \alpha \le U(\xi) + u_0(\xi) \le \beta$$

$$\begin{aligned} \sup_{|\xi|<L} & \left(|U(\xi) + u_0(\xi)| + |U'(\xi) + u_0'(\xi)| + |V(\xi) + v_0(\xi)| + |V'(\xi) + v_0'(\xi)|\right) \\ &\le M + 1 \end{aligned}$$

Then Ω is open and $0 \in \text{int}\Omega$

We note that $\phi \in \partial\Omega$, $\phi = \mu T\phi$, $\mu \in (0, 1)$ if and only if there is a solution $(u(\xi), v(\xi))$ of (1.1) satisfying $u'(\xi) \ge 0$ if $\alpha \le u(\xi) \le \beta$ and either

(i) $u'(\xi_0) = 0$, $\alpha \le u(\xi_0) \le \beta$ for some $\xi_0 \in (-L, L)$

or

(ii) $\sup_{-L<\xi<L}\{|u(\xi)| + |v(\xi)| + |u'(\xi)| + |v'(\xi)|\} = M_0 + 1$

or both (i) and (ii).

The following lemma proved by Dafermos[1] is often useful.

LEMMA 1.2. *The initial value problem for* (1.3), *with fixed* $\epsilon > 0$, $\mu \in [0,1]$, *has a unique solution.*

In order to use the Leray-Schauder fixed theorem, we take the Banach space $X = C^1([-L,L];\mathbb{R}^2)$.

Let us consider the case (i): either $\alpha < u(\xi_0) < \beta$, $u(\xi_0) = \alpha$, or $u(\xi_0) = \beta$.

Case 1. $\alpha < u(\xi_0) < \beta$, $u(\xi_0) = \alpha$, $u(\xi_0) = \beta$. Using Lemma 1.2 and the same method of Slemrod's proof[6], we can not satisfy (1.1), $u_- < \alpha$, $u_+ > \beta$.

Case 2. $u(\xi_0) = \alpha$, $u'(\xi_0) = 0$. In this case there are the three possibilities, $u''(\xi_0) > 0$, $u''(\xi_0) = 0$, or $u''(\xi_0) < 0$. The first and second cases are same as Case 1. So we need only consider $u''(\xi_0) < 0$. In this case $u(\xi_0) = \alpha$ is a local maximum. Hence if $u(L) = u_+ > \beta$, the local maximum of u occurs at $\xi_1 > \xi_0$, i.e. $u(\xi_1) < \alpha$, $u'(\xi_1) = 0$, $u''(\xi_1) \geq 0$; $u(\xi) < \alpha$, $u'(\xi) < 0$, $\xi_0 < \xi \leq \xi_1$. The case $u''(\xi_1) = 0$ is impossible because of $v'(\xi_1) = 0$ and the Lemma 1.2. Thus we only consider $u''(\xi_1) > 0$. From (1.1) and the assumption(I) of f we see that $v(\xi_1) < 0$ and $v(\xi_0) > 0$ which implies v has a local maximum at a point $\xi_0 < \zeta < \xi_1$, $u(\zeta) \leq 0$, and again Lemma 1.2 shows that $v''(\zeta) > 0$. Since $g'(u) > 0$ for $u < \alpha$ this implies by use of (1.1) that $u'(\zeta) > 0$ which contradicts the fact that u is decreasing on (ξ_0, ξ_1).

Case 3. $u(\xi_0) = \beta$, $u'(\xi_0) = 0$. This case is similar to Case 1.

From Case 1, 2, 3 of (i) there is no solution of (1.1), $\mu \in (0,1)$, $(u(\xi) - u_0(\xi), v(\xi) - v_0(\xi))$ in Ω for which (i) can hold. Thus all solutions of (1.1),$\mu \in (0,1)$ in $\bar{\Omega}$ must satisfy $u'(\xi) > 0$ in $\alpha \leq u(\xi) \leq \beta$. But the hypothesis of our theorem, (ii) cannot hold either. Thus from Leray-Schauder fixed point theorem, (1.1) possesses a solution for which $(u(\xi) - u_0(\xi),\ v(\xi) - v_0(\xi))$ is in $\bar{\Omega}$. To extend the domain of u, v as follows: Set

$$u(\xi;L) = u_+, v(\xi;L) = v_+ \text{ if } \xi > L,$$
$$u(\xi;L) = u_-, v(\xi;L) = v_- \text{ if } \xi < -L.$$

The extended pair $(u(\cdot;L), v(\cdot;L))$ form a sequence in $C^0((-\infty,\infty);\mathbb{R}^2)$ and by virtue of the hypothesis of theorem we know $\sup_{|\xi|<L}\{|u'(\xi;L)| + |v'(\xi;L)|\} \leq M$. Thus the sequence $\{(u(\xi;L), v(\xi;L))\}$ is precompact in $C^0((-\infty,\infty);\mathbb{R}^2)$ and so there is a subsequence $L_n \to \infty$ as $n \to \infty$

since that $(u(\xi; L), v(\xi; L)) \to (u(\xi), v(\xi))$ uniformly as $n \to \infty$ on $(-\infty, \infty)$. Thus $(u(\xi), v(\xi))$ is a solution of P_ϵ and by its construction $u'(\xi) \geq 0$ if $\alpha \leq u(\xi) \leq \beta$. But by the same reason used in Cases 2 and 3 $u'(\xi) > 0$ if $\alpha \leq u(\xi) \leq \beta$. This completes the proof of Theorem 1.1. □

REMARK 1.3. The conclusion of Theorem 1.1 remains valid if (1.2) is replaced by the a priori estimate

$$\sup_{|\xi|<L} (|u(\xi)| + |v(\xi)|) \leq M_1$$

where $M_1 = M_1(u_-, v_-, u_+, v_+, \epsilon, f, g)$ but is independent of μ and L.

REMARK 1.4. Assume $v_- > v_+$ and $u_-, u_+ < \alpha(v_- < v_+$ and $u_-, u_+ > \beta)$ and there exist a constant M_2 such that every possible solution of (1.1) satisfies the a priori estimate

$$\sup_{|\xi|<L} (|u(\xi)| + |v(\xi)|) \leq M_2$$

Here $M_2 = M_2(v_-, v_+, u_-, u_+, \epsilon, f, g)$ but not independent of μ and L. Then there exist solutions of (P_ϵ) which satisfy the constraints $u(\xi) < \alpha(u(\xi) > \beta)$.

2. The a priori estimates

In this section we derive the *a priori* estimates needed to apply Theorem 1.1 and Remark 1.3 and 1.4. We give a series of Lemmas which is useful. Lemma 2.1 is a result of Dafermos[1].

LEMMA 2.1. *Let $(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$. Then on any subinterval (l_1, l_2) for which $g'(u(\xi)) > 0$ one of the following holds:*

(i) *$u(\xi)$ and $v(\xi)$ are constant on (l_1, l_2).*

(ii) *$v(\xi)$ is a strictly increasing(or decreasing) function with no critical points in (l_1, l_2); $u(\xi)$ has, at most, one critical point in (l_1, l_2) that necessarily must be a maximum(or minimum).*

(iii) *$u(\xi)$ is a strictly increasing (or decreasing) function with no critical point in (l_1, l_2); $v(\xi)$ has, at most, one critical point in (l_1, l_2) that necessarily must be a maximum(or minimum).*

LEMMA 2.2. *$(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$. Then on any subinterval (l_1, l_2) for which $g'(u(\xi)) < 0$ the graph of $v = v(u)$ is convex(or concave) at points where $u'(\xi) > 0$(or $u'(\xi) < 0$).*

Proof. Denote by $\frac{dv}{du} = \frac{v'(\xi)}{u'(\xi)}$. Then

$$\epsilon \frac{d^2v}{du^2} = \frac{\mu}{u'}(f'(v)(\frac{dv}{du})^2 - g'(u)).$$

The result follows from the above identity. □

LEMMA 2.3. *$(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$ with $u'(\xi) > 0$ if $\alpha \le u(\xi) \le \beta$. Then u and v can have no local maxima or minima at ξ for which $u(\xi) = \alpha$ or $u(\xi) = \beta$.*

Proof. Since $u'(\xi) > 0$ if $\alpha \le u(\xi) \le \beta$, u has no local maxima or a local minima at points where $u(\xi) = \alpha$. On the other hand if $v(\xi)$ has a local maximum or minimum at such a point, then $v'(\xi) = 0$ there and hence by (1.1) $v''(\xi) = 0$ as well. Differentiating (1.1) with respect to ξ, $g''(\alpha) < 0$, $g''(\beta) > 0$ implies that $u''(\xi) = 0$ at such points, so u could not have taken on a local maximum or minimum. □

Lemma 2.4 is the same result as Slemrod[6]. The proof is similar to his Lemma 2.4.

LEMMA 2.4. *Assume that $u_- < \alpha$, $u_+ > \beta$ and let $u(\xi)$, $v(\xi)$ be a solution of (1.1) with $\mu > 0$ for which $u'(\xi) > 0$ when $\alpha \le u(\xi) \le \beta$. Then one of the following holds: (0) No extreme points: $u(\xi)$, $v(\xi)$ have no local maxima or minima on $[-L, L]$. They are non-constant and monotone, u being monotone increasing.*

(i) *One extreme point: (a) $u(\xi)$ has a minimum at some ξ_-, $u(\xi_-) < u_-$; $v(\xi)$ is decreasing on $[-L, L]$. (b) $u(\xi)$ has a maximum at some ξ_+, $u(\xi_+) > u_+$; $v(\xi)$ is decreasing on $[-L, L]$. (c) $v(\xi)$ has a maximum at some η_-(or η_+); $u(\eta_-) < \alpha$ (or $u(\eta_+) > \beta$) and $u(\xi)$ is increasing on $[-L, L]$. (d) $v(\xi)$ has a minimum at some η; $\alpha < u(\eta) < \beta$ and $u(\xi)$ is increasing on $[-L, L]$.*

(ii) *Two extreme points:* (a) *$v(\xi)$ has a local maximum at η_- (or η_+) and a local minimum at η, $u(\xi)$ is increasing on $[-L, L]$ and $u_- < u(\eta_-) < \alpha$ (or $u_+ > u(\eta_+) > \beta$), $\alpha < u(\eta) < \beta$.* (b) *$u(\xi)$ has a* *minimum at ξ_-, $u(\xi_-) < u_-$; $v(\xi)$ has a local minimum at η, $\eta > \xi_-$, $\alpha < u(\eta) < \beta$.* (c) *$u(\xi)$ has a maximum at ξ_+, $u(\xi_+) > u_+$; $v(\xi)$ has a local minimum at η, $\eta < \xi_+$, $\alpha < u(\eta) < \beta$.*

(iii) *Three extreme points:* (a) *$v(\xi)$ has local maxima at η_-, η_+ and a local minimum at η, $\eta_- < \eta < \eta_+$; $u(\xi)$ is increasing with $u_- < u(\eta_-) < \alpha$, $\alpha < u(\eta) < \beta$, $\beta < u(\eta_+) < u_+$.* (b) *$u(\xi)$ has a minimum at ξ_-, $u(\xi_-) < u_-$ and maximum at ξ_+, $u(\xi_+) > u_+$ and $v(\xi)$ has a local minimum at η, $\xi_- < \eta < \xi_+$, $\alpha < u(\eta) < \beta$.* (c) *$u(\xi)$ has a minimum at ξ_-, $u(\xi_-) < u_-$, $v(\xi)$ has a local minimum at η, $\alpha < u(\eta) < \beta$ and a local maximum at η_+, $\eta < u(\eta_+) < u_+$, $\xi_- < \eta < \eta_+$.* (d) *$u(\xi)$ has a maximum at ξ_+, $u(\xi_+) > u_+$, $v(\xi)$ has a local maximum at η_-, $u_- < u(\eta_-) < \alpha$, and a local minimum at η, $\alpha < u(\eta) < \beta$.*

THEOREM 2.5. *Assume $u_- < \alpha$, $u_+ > \beta$ ($u_- > \beta$, $u_+ < \alpha$). Then there exist constant M_1 such that every possible solution of (1.1), $0 \le \mu \le 1$, with $u'(\xi) > 0$ ($u'(\xi) < 0$) when $\alpha \le u(\xi) \le \beta$ satisfies*

$$\sup_{|\xi|<L} (|u(\xi)| + |v(\xi)|) \le M_1$$

where M_1 depends at most on u_-, u_+, v_-, v_+, ϵ, f, g and is independent of μ and L.

Proof. We will prove the case $u_- < \alpha$, $u_+ > \beta$. The proof for $u_- > \beta$, $u_+ < \alpha$ is similar.

The case (0) is nothing to prove.

The case (ia) Since v is decreasing, $v_+ \le v(\xi) \le v_-$. Since u has a minimum at ξ_-, we need only bound u from below. Assume $\xi_- \ge 0$. In case $\xi_- \le 0$ will be similarly proved. Integrating (1.1) from ξ_- to L and use $u'(\xi_-) = 0$, we have

$$\epsilon u'(L) + \int_{\xi_-}^{L} \xi u'(\xi)\, d\xi \le -\mu f(v_+) + \mu f(v(\xi_-)).$$

Since $u'(L) > 0$, we have

$$\int_{\xi_-}^{L} \xi u'(\xi)\, d\xi \le -\mu f(v_+) + \mu f(v(\xi_-)).$$

If $\zeta \geq \max\{1, \xi_-\}$, then $u'(\xi) \leq \xi u'(\xi)$ on (ζ, L) so that

$$u(L) - u(\zeta) \leq -\mu f(v_+) + \mu f(v(\xi_-)).$$

and hence

$$u(\zeta) \geq u_+ + \mu f(v_+) - \mu f(v(\xi_-)). \tag{2.1}$$

Since $v_+ \leq v(\xi_-) \leq v_-$, $0 \leq \mu \leq 1$, we have

$$u(\zeta) \geq u_+ + f(v_+) - f(v_-) \text{ if } \xi_- \geq 1.$$

If $0 \leq \xi_- < 1$, integrate (1.1) from ξ_- to ξ where $\xi_- < \xi < 1$, then

$$\epsilon u'(\xi) + \int_{\xi_-}^{\xi} \zeta u'(\zeta)\, d\zeta = -\mu f(v(\xi)) + \mu f(v(\xi_-)).$$

Since $u'(\xi) > 0$ on (ξ_-, L), we obtain $\zeta u'(\zeta) > 0$ and

$$\epsilon u'(\xi) \leq -\mu f(v(\xi)) + \mu f(v(\xi_-)), \quad \xi_- < \xi < 1. \tag{2.2}$$

Integrate (2.2) from ξ_- to 1, We see that

$$\epsilon u(1) - \epsilon u'(\xi_-) \leq -\mu \int_{\xi}^{1} (f(v(\xi)) + \mu f(v(\xi_-)))\, d\xi. \tag{2.3}$$

Since $v_+ \leq v(\xi) \leq v_-$ and $u(1)$ is bounded from below by (2.1), (2.3) implies that $u(\xi_-)$ is bounded from below when $0 \leq \xi_- < 1$.

The cases (ib) and (ic) are proven similarly.

The case (id): Since $u(\xi)$ is increasing so $u_- \leq u(\xi) \leq u_+$. Assume that $\eta \geq 0$. In case $\eta < 0$ is similar. First integrate (1.1) from η to L, this implies

$$\epsilon v'(L) + \int_{\eta}^{L} \xi u'(\xi)\, d\xi = -\mu g(u_+) + \mu g(u(\eta)).$$

Since $v'(L) > 0$ this implies

$$\int_{\eta}^{L} \xi u'(\xi)\, d\xi \leq -\mu g(u_+) + \mu g(u(\eta)).$$

If $\zeta \geq \max\{1, \eta\}$, since $v'(\xi) > 0$ on (ζ, L) we find $v'(\xi) \leq \xi v'(\xi)$ on (η, L) and

$$v_+ - v(\zeta) = \int_\zeta^L v'(\xi)\, d\xi \leq \int_\eta^L \xi u'(\xi)\, d\xi \leq -\mu g(u_+) + \mu g(u(\eta))$$

Thus we have

$$v(\zeta) \geq v_+ + \mu g(u_+) - \mu g(u(\eta)). \tag{2.4}$$

Since $\alpha < u(\eta) < \beta$, we see for $\eta \geq 1$

$$v(\eta) \geq v_+ + \mu g(u_+) - \mu g(\alpha)) \geq v_+ - g(\alpha).$$

Again if $0 \leq \eta < 1$, integrate (1.1) from η to ξ where $\eta < \xi < 1$. Then we have

$$\epsilon v'(\xi) + \int_\eta^\xi \zeta u'(\zeta)\, d\zeta = -\mu g(u(\xi)) + \mu g(u(\eta)).$$

Since $\zeta v'(\zeta) > 0$ on (η, ξ), we find

$$\epsilon v'(\xi) \leq -\mu g(u(\xi)) + \mu g(u(\eta)).$$

and integrate it from η to 1 we have

$$\epsilon v(1) - \epsilon v(\eta) \leq -\mu \int_\eta^1 (g(u(\xi)) - g(u(\eta)))\, d\xi.$$

and

$$\epsilon v(1) + \mu \int_\eta^1 (g(u(\xi)) - g(u(\eta)))\, d\xi \leq \epsilon v(\eta). \tag{2.5}$$

We know $\max(v_-, v_+) \geq v(\xi)$ and so v is bounded from above. Since $u(\xi)$ is bounded, (2.4) and (2.5) imply that $v(\xi)$ is bounded from below on $[-L, L]$ independently of μ and L.

The case (iia) : Assume v has a local maximum at η_-, $u(\eta_-) < \alpha$. The case $u(\eta_+)$ is similar. Then the local minimum is at η, $\eta_- < \eta$,

$\alpha < u(\eta) < \beta$. For u we know $u_- \leq u(\xi) \leq u_+$. In case there are two cases $\eta \geq 0$ and $\eta < 0$. If $\eta \geq 0$, the same method of the proof of (id) implies the boundedness of v. If $\eta < 0$, then $\eta_- < 0$. We will show $u(\eta_-)$ is bounded from below. We consider first $\eta_\leq -1$ and then $-1 \leq \eta_- \leq 0$. In the first case we use (ic) on $-L \leq \xi\eta$ to bound $v(\eta_-)$ from above; in the second case we use (id) on $\eta_- \leq \xi \leq L$ to bound $u(\xi)$ from below. These bounds is independent of μ and L.

The case (iib): If $\eta \geq 0$, the argument of (id) says that $v(\eta)$ is bounded from below. Since $v(\eta)$ is bounded from above by $\max(u_-, u_+)$, $v(\xi)$ is bounded from above and below. Use (ia) on $[-L, \eta]$, u is bounded from below at $\xi_- \in (-L, \eta)$. If $\eta < 0$, then argument of (id) implies

$$v(\zeta) \geq v_- + \mu g(u_-) - \mu g(u(\eta))$$

if $\zeta \leq \min\{-1, \eta\}$. But $\alpha < u(\eta) < \beta$ so $u(\eta)$ is bounded from below if $\eta \leq -1$. If $-1 < \eta \leq 0$, argument (id) can be used again. First integrate (1.1) from η to ξ where $\xi \in (-1, \eta$. This implies

$$\epsilon v'(\xi) + \int_\eta^\xi \zeta v'(\zeta)\, d\zeta = \mu g(u(\eta)) - \mu g(u(\xi)).$$

On (ξ, η), $\zeta v'(\zeta) > 0$ so

$$\epsilon v'(\xi) \geq \mu g(u(\eta)) - \mu g(u(\xi)) \tag{2.6}$$

Now integrate (2.6) from -1 to η,

$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu \int_{-1}^{\eta} (g(u(\eta)) - g(u(\xi)))\, d\xi. \tag{2.7}$$

Now $u(\xi) \leq u(\eta)$ on $(-1, \eta)$ since $\alpha < u(\eta) < \beta$,

$$g(u(\eta)) - g(u(\xi)) \geq g(\beta) - g(\alpha). \tag{2.8}$$

Insert (2.8) into (2.7) we have

$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu(\eta + 1)(g(\beta) - g(\alpha))$$

and hence

$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu(g(\beta) - g(\alpha)).$$

Thus $v(\eta)$ is bounded if $\eta \leq 0$. Now use (ia) on $(-L, \eta)$ $u(\xi_-)$ is bounded from below.

The case ii(c): This case is proved the same method of ii(b).

The case iii(a): Since u is monotone increase, $u_- \leq u(\xi) \leq u_+$ on $[-L, L]$. As to v, either $\eta_+ \geq 0$ or not. If $\eta \geq 0$, using the method of (ic) $v(\eta_+)$ is bounded from above. If $\eta_+ < 0$, then $\eta_- < 0$ and again using the same method of (ic) $v(\eta_-)$ is bounded from above. thus if $\eta_+ \geq 0$, $u_+ \leq u(\eta_+) \leq M_1$; if $\eta_+ < 0$ then $u_- \leq u(\eta_-) \leq M_2$. This case is reduced to the case (iia).

The case iii(b) : If $\eta \geq 0$, then ii(c) implies that for $\eta \geq 1$

$$v(\eta) \geq v_+ - \mu g(u(\eta)) + \mu g(u_+) \tag{2.9}$$

Since $\alpha \leq u(\eta) \leq \beta$, (2.9) shows that $u(\eta)$ is bounded from below. If $0 \leq \eta < 1$, ii(c) shows

$$\epsilon v(\eta) \geq \epsilon v(1) + \mu(g(\beta) - g(\alpha))$$

Thus $v(\eta)$ is bounded from below. If $\eta < 0$, ii(a) show $u(\eta)$ is bounded from below. Thus $v(\eta)$ is bounded from above and below.

The case iii(c) : If $\eta \leq 0$, then the proof is same as the method of ii(b).If $\zeta \leq \min\{-1, \eta\}$, then

$$v(\zeta) \geq v_- - \mu(g(u_-) - g(u(\eta))).$$

Since $\alpha \leq u(\eta) < \beta$, $v(\eta)$ is bounded from below if $\eta \leq -1$. If $-1 < \eta \leq 0$ we have

$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu \int_{-1}^{\eta} g(u(\eta)) - g(u(\xi))\, d\xi$$

where $u(\xi) \leq u(\eta)$, $-1 \leq \xi\eta$. In this case

$$g(u(\eta)) - g(u(\xi)) \geq g(\beta) - g(\alpha)$$

and so

$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu(g(\beta) - g(\alpha))$$

and $u(\eta)$ is bounded from above for $\eta \leq 0$. If $\eta \geq 0$, then $\eta_+ \geq 0$. The same argument of i(c) yields $v(\eta_+)$ is bounded from above. If $\zeta \geq \max\{\eta_+, 1\}$, we find

$$v(\zeta) \leq v_+ + \mu g(u_+) - \mu g(u(\eta_+)).$$

Since $\beta \leq u(\eta_+) \leq u_+$, $v(\eta_+)$ is bounded from above if $\eta_+ \geq 1$. If $0 \leq \eta_+ < 1$, we find

$$\epsilon v(\eta_+) \leq \epsilon v(-1) + \mu \int_{\eta_+}^{-1} g(u(\xi)) - g(u(\eta_+))\, d\xi.$$

But $\beta \leq u(\xi) \leq u_+$ for $\xi \in [\eta_+, 1]$, $v(\eta_+)$ is bounded from above. Then $\eta \leq 0$, $v(\eta)$ is bounded from above and below; if $\eta > 0$, then $v(\eta_+)$ is bounded from above and below.

The case iii(d) : The proof is similar of the proof of iii(c). □

THEOREM 2.6. *Assume $v_+ < v_-$ and $u_-, u_+ < \alpha$(or $v_- < v_+$ and u_-, $u_+ > \beta$). Then there is a constant M_2 such that every possible solution of* (1.1), $0 \leq \mu \leq 1$, *satisfies the a priori estimate*

$$\sup_{|\xi|<L} (|v(\xi)| + |u(\xi)|) \leq M_2$$

where M_2 depends at most on $u_-, u_+, v_-, v_+, \epsilon, f, g$ and is independent of μ and L.

COROLLARY 2.7. *If $u_- < \alpha, u_+ > \beta$ (or $u_- > \beta, u_+ < \alpha$), there are solutions of (P_ϵ) which satisfy the constants $u'(\xi) > 0 (u'(\xi) < 0)$ when $\alpha \leq u(\xi) \leq \beta$. If $v_+ < v_-$ and $u_-, u_+ < \alpha$ (or $v_- < v_+$ and $u_-, u_+ > \beta$) there are solutions of P_ϵ which satisfy the constraints $u(\xi) < \alpha (u(\xi) > \beta)$.*

3. Existence of Solutions of the Riemann problems assuming $\{(u_\epsilon, v_\epsilon)\}$ are uniformly bounded.

In this section we prove the existence of solutions to the Riemann problem assuming the set $\{(u_\epsilon, v_\epsilon)\}$ are uniformly bounded. Proposition 3.1 is a result of Dafermos[1].

PROPOSITION 3.1. *For fixed* $\epsilon > 0$, *let* (u_ϵ, v_ϵ) *denote a solution of* P_ϵ. *Suppose that the set* $\{(u_\epsilon, v_\epsilon) : 0 < \epsilon < 1\}$ *is of uniformly bounded variation. Then* $\{(u_\epsilon, v_\epsilon)\}$ *possesses a subsequence which converges almost everywhere on* $(-\infty, \infty)$ *of bounded variation. The pair* $u(\frac{x}{t})$, $v(\frac{x}{t})$ *provided a weak solution of* P.

Using Proposition 3.1, we have an existence theorem for the one phase case.

THEOREM 3.2. *If* $v_- > v_+$ *and* $u_-, u_+ < \alpha$*(or* $u_-, u_+ > \beta$*) and Assumption* (III) *holds. the sequence* $\{(u_\epsilon(\xi), v_\epsilon(\xi)); 0 < \epsilon < 1\}$ *as given by* Corollary 2.7 *possesses a subsequence which converges a.e. on* $(-\infty, \infty)$ *to function* $(u(\xi), v(\xi))$ *of bounded variation. The pair* $u(\frac{x}{t})$, $v(\frac{x}{t})$ *provides a solution to the Riemann problem (P) with* $u(\frac{x}{t}) < \alpha$*(or* $u(\frac{x}{t}) > \beta$*).*

LEMMA 3.3. *The list for* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *given in* Lemma 2.4 *is valid when* $L = \infty$.

LEMMA 3.4. *In case 0, i(a, b, c) of* Lemma 2.4 $(u_\epsilon(\xi), v_\epsilon(\xi))$ *are uniformly bounded independent of* ϵ *on* $(-\infty, \infty)$. *That is, there is a constant* N *dependent on* u_-, u_+, v_-, v_+, f, g *and independent of* ϵ, $0 < \epsilon < 1$ *such that*

$$\sup_{|\xi|<\infty} (|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq N. \tag{3.1}$$

Proof. Case 0: it is obvious. Case i(a): Since $v_\epsilon(\xi)$ is monotone decreasing, $v_+ \leq v_\epsilon(\xi) \leq v_-$ on $(-\infty, \infty)$. Denote $\frac{du}{dv}(\xi) = \frac{u'_\epsilon(\xi)}{v'_\epsilon(\xi)}$. We claim that

$$0 < \frac{du}{dv}(\xi) < \left(\frac{f'(v_\epsilon)}{g'(u_\epsilon)}\right)^{1/2} \qquad \text{on} \qquad (-\infty, \xi_-^\epsilon].$$

Indeed, if not, set

$$\xi_1 = \max\left\{\xi \in (-\infty, \xi_-^\epsilon] : \frac{du}{dv}(\xi) \geq \left(\frac{f'(v_\epsilon)}{g'(u_\epsilon)}\right)^{1/2}\right\}.$$

Since u_ϵ has its minimum at ξ_-^ϵ, $\frac{du}{dv}(\xi_1^\epsilon) = 0$ and so $\xi_1 < \xi_1^\epsilon$ must exist. A simple computation shows that

$$\epsilon\frac{d}{d\xi}\left(\frac{du}{dv}(\xi)\right) = -f'(v_\epsilon) + g'(u_\epsilon)\left(\frac{du}{dv}\right)^2$$

and so $\epsilon\frac{d}{d\xi}(\frac{du}{dv}(\xi)) = 0$ at $\xi = \xi_1$. By the definition of ξ_1 we have

$$0 < \frac{du}{dv}(\xi) < \left(\frac{f'(v_\epsilon)}{g'(u_\epsilon)}\right)^{1/2} \qquad \text{on} \qquad (\xi_1, \xi_-^\epsilon)$$

and thus $\frac{d}{d\xi}\frac{du}{dv}(\xi) < 0$ on (ξ_1, ξ_-^ϵ) and $\frac{d^2}{d\xi^2}\frac{du}{dv}(\xi_1) < 0$. On the other hand, differentiation of (3.2) shows that

$$\epsilon\frac{d^2}{d\xi^2}\left(\frac{du}{dv}(\xi)\right) = -f''(v_\epsilon)v_\epsilon'(\xi) + g''(u_\epsilon)u_\epsilon'(\xi)\left(\frac{du}{dv}\right)^2 \qquad \text{at} \quad \xi = \xi_1.$$

From Assumptions 1 and 2 it follows that

$$\frac{d^2}{d\xi^2}\left(\frac{du}{dv}(\xi)\right) > 0 \qquad \text{at} \quad \xi = \xi_1.$$

This contradicts the assumption. Thus we see $\frac{d}{d\xi}(\frac{du}{dv}(\xi)) \leq 0$ on $(-\infty, \xi_-^\epsilon]$. Hence for any $\xi \in (-\infty, \xi_-^\epsilon]$,

$$\frac{du}{dv}(\xi) < \frac{du}{dv}(-\infty) = \left(\frac{f'(v)}{g'(u)}\right)^{1/2}.$$

Now

$$\begin{aligned} u_\epsilon(\xi_-^\epsilon) - u_- &= \int_{v_-}^{v_\epsilon(\xi_-^\epsilon)} \frac{du}{dv}\,dv \\ &> -\int_{v_\epsilon(\xi_-^\epsilon)}^{v_-} \left(\frac{f'(v_-)}{g'(u_-)}\right)^{1/2} dv \\ &= -\left(\frac{f'(v_-)}{g'(u_-)}\right)^{1/2} (v_- - v_\epsilon(\xi_-^\epsilon)), \end{aligned}$$

which is bounded from below.

Case i(b) : The proof is similar to i(a).

Case i(c) : Let η_-^ϵ be a point such that $v_\epsilon(\xi)$ has its maximum value and $u_\epsilon(\eta_-^\epsilon) < \alpha$. Since $u_\epsilon(\xi)$ is increasing, $u_- \leq u_\epsilon(\xi) \leq u_+$ on $(-\infty, \infty)$. Denote by $\frac{dv}{du}(\xi) = \frac{v_\epsilon'(\xi)}{u_\epsilon'(\xi)}$. We claim that $0 < \frac{dv}{du}(\xi) < (\frac{g'(u_\epsilon)}{f'(v_\epsilon)})^{1/2}$ on $(-\infty, \eta_-^\epsilon]$. For if not, set

$$\xi_1 = \max\left\{\xi \in (-\infty, \eta_-^\epsilon] \mid \frac{dv}{du}(\xi) \geq \left(\frac{g'(u_\epsilon)}{f'(v_\epsilon)}\right)^{1/2}\right\}.$$

Since $\frac{dv}{du}(\xi) = 0$ at $\xi = \xi_1$, ξ_1 exist such that $\xi_1 < \eta_-^\epsilon$. A simple computation say

$$\epsilon \frac{d}{d\xi}\left(\frac{dv}{du}(\xi)\right) = -g'(u_\epsilon(\xi)) + f'(v_\epsilon(\xi))\left(\frac{dv}{du}\right)^{1/2} \tag{3.3}$$

implies $\frac{d}{d\xi}(\frac{dv}{du}(\xi_1)) = 0$. By the definition of ξ_1, $0 < \frac{dv}{du}(\xi) < \left(\frac{g'(u_\epsilon)}{f'(v_\epsilon)}\right)^{1/2}$ on $(\xi, \eta_-^\epsilon]$. Thus we have $\frac{d^2}{d\xi^2}(\frac{dv}{du}(\xi)) < 0$ at $\xi = \xi_1$. On the other hand, differentiation of (3.3) gives

$$\epsilon \frac{d^2}{d\xi^2}\left(\frac{dv}{du}(\xi)\right) = -g''(u_\epsilon)u_\epsilon'(\xi) + f''(v_\epsilon)v_\epsilon'(\xi)\left(\frac{dv}{du}\right)^2 > 0$$

at $\xi = \xi_1$, a contradiction. Thus we see that $\frac{d}{d\xi}(\frac{dv}{du}(\xi)) < 0$ on $(-\infty, \eta_-^\epsilon]$ and hence for any $\xi \in (-\infty, \eta_-^\epsilon]$,

$$0 < \frac{dv}{du}(\xi) < \frac{dv}{du}(-\infty) = \left(\frac{g'(u_-)}{f'(v_-)}\right)^{1/2}.$$

Then

$$v_\epsilon(\eta_-^\epsilon) - v_- = \int_{u_-}^{u_-(\eta_-^\epsilon)} \frac{dv}{du}\, du \leq \left(\frac{g'(u_-)}{f'(v_-)}\right)^{1/2} (u_\epsilon(\eta_-^\epsilon) - u_-).$$

Since $u_- \leq u(\eta_-^\epsilon) \leq u_+$, we see that $u_\epsilon(\eta_-^\epsilon)$ is bounded from above, independent of ϵ for $u(\eta_-^\epsilon) < \alpha$. Analogous computation shows that if $u_\epsilon(\eta_+^\epsilon) > \beta$ we have

$$v_\epsilon(\eta_+^\epsilon) \leq v_+ + \left(\frac{g'(u_+)}{f'(v_+)}\right)^{1/2} (u_\epsilon(\eta_-^\epsilon) - u_-)$$

and since $u_- \leq u(\eta_-^\epsilon) \leq u_+$, a bound on $v_\epsilon(\eta_-^\epsilon)$ independent of ϵ is provided. □

LEMMA 3.5. *Let η^ϵ denote the points such that $v_\epsilon(\xi)$ takes on its local minimum, $\alpha < u_\epsilon(\eta^\epsilon) < \beta$. If there is a subsequence $\{\eta^{\epsilon_n}\}$ of $\{\eta^\epsilon\}$, $\epsilon_n \to 0+$ such that either* (a) *$\eta^{\epsilon_n} \geq m > 0$ or $\eta^{\epsilon_n} \leq -m < 0$, m a constant independent of ϵ, or* (b) *$v_\epsilon(\eta^{\epsilon_n})$ is bounded from below independently of ϵ, then for Case i(d) $\{(u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)\}$ satisfies* (3.1).

Proof. Assume $\eta^{\epsilon_n} \leq m < 0$. Then $v'_{\epsilon_n}(\xi) \leq 0$ on $(-\infty, \eta^{\epsilon_n}]$ and $\xi v'_{\epsilon_n}(\xi) \geq -m v'_{\epsilon_n}(\xi)$ on $(-\infty, \eta^{\epsilon_n}]$. Now

$$\begin{aligned} -m(v_\epsilon(\eta^{\epsilon_n}) - v_-) &\leq \int_{-\infty}^{\eta^{\epsilon_n}} \xi v'_{\epsilon_n}(\xi)\, d\xi \\ &= \int_{-\infty}^{\eta^{\epsilon_n}} (g'(u) - \epsilon_n v'')\, d\xi \\ &= g(u(\eta^\epsilon)) - g(u_-) \end{aligned}$$

hence

$$\frac{1}{m}(g(u_-) - g(u(\eta^\epsilon))) + v_- \leq v(\eta^{\epsilon_n})$$

Since $u_\epsilon(\xi)$ is monotone, $u_- \leq u_\epsilon(\xi) \leq u_+$, we see that $v_\epsilon(\eta^{\epsilon_n})$ is bounded from below independently of ϵ. The case $\eta^{\epsilon_n}) \geq m > 0$ is similar. Thus in (a) or (b), $v(\eta^{\epsilon_n})$ is bounded for below and hence $\{(u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)) \,|\, 0 < \epsilon < 1\}$ satisfies (3.1). □

LEMMA 3.6. *In* case ii(a,b,c), iii(a,b,c,d) *assume $\{\eta^\epsilon\}$ satisfies the hypothesis of Lemma 3.4. Then $\{(u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi)) \,|\, 0 < \epsilon_n < 1\}$ satisfies* (3.1).

From Lemmas 3.4, 3.5, 3.6 and Prop 3.1 we have

THEOREM 3.7. *Assume* $u_- < \alpha$, $u_+ > \beta$(or $u_- > \alpha, u_+ < \beta$) *and let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *denote the solution of* P_ϵ *given by Corollary 2.7. Let Assumptions* (II) *and* (III) *and the hypothesis of* Lemma 3.4 *hold. Then* $\{(u_{\epsilon_n}(\xi), v_{\epsilon_n}(\xi))|0 < \epsilon_n < 1\}$ *possesses a subsequence which converges almost everywhere on* $(-\infty, \infty)$ *to a function* $(u(\xi), v(\xi))$ *of bounded variation. The pair* $u(\frac{x}{t})$, $v(\frac{x}{t})$ *provides a solution of the Riemann problem.*

REMARK 3.8. If the hypothesis of Lemma 3.5 does *not* hold then $\eta^\epsilon \to 0$, $v_\epsilon(\eta^\epsilon) \to -\infty$ as $\epsilon \to 0+$.

4. Existence of solutions to the Riemann problem: the case when $v_\epsilon(\eta^\epsilon) \to -\infty$ as $\eta^\epsilon \to 0$.

In this section we will prove the existence of solution to the Riemann problem in case when $v(\eta^\epsilon) \to -\infty$ as $\eta^\epsilon \to 0$. This situation was mentioned in Remark 3.8. First we must show that $u_\epsilon(\xi)$, $v_\epsilon(\xi)$ has a pointwise a.e. limit.

LEMMA 4.1. *Let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *be a solution of* P_ϵ *as given by Corollary 2.7 when* $u_- < \alpha, u_+ > \beta$. *Let* $\bar{v} = \min(v_-, v_+)$. *Then if* $v_\epsilon(\xi)$ *has a local minimum at* η^ϵ *with* $\alpha < u_\epsilon(\eta^\epsilon) < \beta$, *we have the estimate*

$$N_0(s_1 - s_2) \geq \int_{s_1}^{s_2} v_\epsilon(\xi)\, d\xi \geq \bar{v}(s_2 - s_1) + (g(\beta) - g(\alpha)) \tag{4.1}$$

$$\bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi - \eta^\epsilon|} \leq v_\epsilon(\xi) \leq N_0, \qquad -\infty < \xi < \infty \tag{4.2}$$

Here $(s_1, s_2) \subset (-\infty, \infty)$ *and* N_0 *is a constant independent of* ϵ.

Proof. The bound from above on $v_\epsilon(\xi)$ in (4.1), (4.2) follows from the proof of Lemma 3.3, 3.4, and 3.5. Thus we now proceed to get the bounds from below. i(d) Fix $l < \infty$ sufficiently large so that $u_\epsilon(-l) < \alpha, u_\epsilon(l) > \beta$. Assume for the moment $v_\epsilon(-l) \leq v_\epsilon(l)$,and let $\theta > -l$ be such that $v_\epsilon(\theta) = v_\epsilon(-l)$. Then we have $v_\epsilon(\xi) \leq v_\epsilon(-l)$ on $(-l, \theta)$, $v_\epsilon(\xi) \geq v_\epsilon(-l)$ on $\theta < \xi < l$ when $-l < \eta^\epsilon < \theta < l$. From (P_ϵ) we know that

$$\epsilon(v_\epsilon(\xi) - v_\epsilon(-l))'' + \xi(u_\epsilon(\xi) - u_\epsilon(-l))' = -g(u_\epsilon)' \tag{4.3}$$

and integration of (4.3) from $-l$ to θ shows that

$$\epsilon(v'_\epsilon(\theta) - v'_\epsilon(-l)) - \int_{-l}^{\theta} (v_\epsilon(\xi) - v_\epsilon(-l))\, d\xi = -g(u_\epsilon(\theta)) + g(u_\epsilon(-l))$$

But $v'(\theta) > 0$, $v'(-l) < 0$ and hence

$$\int_{-l}^{\theta} (v_\epsilon(-l) - v_\epsilon(\xi))\, d\xi \leq g(u_\epsilon(-l)) - g(u_\epsilon(\theta)) \tag{4.4}$$

Since $u_\epsilon(\theta) > u_\epsilon(-l)$, the right-hand side of (4.4) is bounded from above by $g(\alpha) - g(\beta)$. Then for any $(s_1, s_2) \subset (-l, \theta)$ we have

$$\int_{s_1}^{s_2} (v_\epsilon(-l) - v_\epsilon(\xi))\, d\xi \leq g(\alpha) - g(\beta) \tag{4.5}$$

and hence

$$v_\epsilon(-l)(s_2 - s_1) + (g(\beta) - g(\alpha)) \leq \int_{s_1}^{s_2} v_\epsilon(\xi))\, d\xi.$$

Letting $l \to -\infty$ we have

$$\bar{v}(s_2 - s_1) + (g(\beta) - g(\alpha)) \leq \int_{s_1}^{s_2} v_\epsilon(\xi))\, d\xi. \tag{4.6}$$

If $(s_1, s_2) \subset (\theta, l)$, then $v_\epsilon(\xi) \geq v_\epsilon(-l)$ and we see

$$\bar{v}(s_2 - s_1) \leq \int_{s_1}^{s_2} v_\epsilon(\xi)\, d\xi. \tag{4.7}$$

Finally if $-l < s_1 < \theta$, $\theta < s_2 < l$, we write

$$\int_{s_1}^{s_2} v_\epsilon(\xi)\, d\xi = \int_{s_1}^{\theta} v_\epsilon(\xi)\, d\xi + \int_{\theta}^{s_2} v_\epsilon(\xi)\, d\xi$$

and use (4.6) and (4.7) to obtain (4.1) again. To get the bound from below in (4.2), we observe that when $\eta^\epsilon < \xi < \theta$

$$(v_\epsilon(-l) - v_\epsilon(\xi))(\xi - \eta^\epsilon) \leq \int_{-l}^{\theta} (v_\epsilon(-l) - v_\epsilon(\xi))\, d\xi. \tag{4.8}$$

From (4.8) and (4.5) we see that

$$(v_\epsilon(-l) - v_\epsilon(\xi))(\xi - \eta^\epsilon) \leq g(\alpha) - g(\beta)$$

Now letting $l \to \infty$ we obtain (4.2). If $-l < \xi < \eta^\epsilon$ we again (4.2) and if $\theta \leq \xi \leq l$, we also obtain (4.2). The proof for $v_\epsilon(-l) > v_\epsilon(l)$ is analogous. □

LEMMA 4.2. *Let* $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon < 1\}$ *be a solution of* (P_ϵ) *as given by* Corollary 2.7 *when* $u_- < \alpha, u_+ > \beta$. *Then for any given compact subset* S *of* $(-\infty, 0)$ *or* $(0, \infty)$ *there exists constants* K *and* ϵ_0 *(depending at most on* $u_-, u_+, v_-, v_+, f, g, S$*) such that*

$$\sup_{\xi \in S}(|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq K \qquad \textit{for} \qquad 0 < \epsilon < \epsilon_0.$$

Proof. Let $S_+ \subset [a,b]$, $S_- \subset [-b,-a]$, $0 < a < b < \infty$. Then for ϵ sufficiently small $|\eta^\epsilon| \leq \frac{a}{2}$ and (4.2) yield $\sup_{\xi \in S_\pm} |v_\epsilon(\xi)| \leq K$. We now need to get a similar estimate on $u_\epsilon(\xi)$. In case i(a), i(b) of Lemma 2.4, the proof of Lemma 3.3, 3.4, 3.5 yields a uniform in ϵ and ξ, $(-\infty < \xi < \infty)$, bound on $u_\epsilon(\xi)$ where as in case 0, i(c), ii(a), iii(a), $u_\epsilon(\xi)$ is monotone so that trivially $u_- \leq u_\epsilon(\xi) \leq u_+$ for $\xi \in (-\infty, \infty)$. Hence the only cases left to search are ii(b),(c), iii(b),(c),(d).

Case ii(b). On S_+, $u_\epsilon(\xi)$ is uniformly bounded in ϵ, ξ and so we need only verify S_-. Let $\eta \in S_-$, $\zeta \in S_+$. For ϵ sufficiently small $\eta < \eta^\epsilon < \zeta$. Integrate (P_ϵ) from η to ζ to obtain

$$\epsilon v_\epsilon'(\zeta) - \epsilon v_('\eta) + \int_\eta^\zeta \xi v_\epsilon'(\xi)\, d\xi = g(u_\epsilon(\eta)) - g(u_\epsilon(\zeta)). \tag{4.9}$$

Since $v_\epsilon'(\zeta) > 0$ and $v_\epsilon'(\eta) < 0$, (4.9) implies

$$\int_\eta^\zeta \xi v_\epsilon'(\xi)\, d\xi \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\zeta)).$$

and integration by parts yields

$$\zeta v_\epsilon(\zeta) - \eta v_\epsilon(\eta) - \int_\eta^\zeta \xi v_\epsilon'(\xi)\, d\xi \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\zeta)). \tag{4.10}$$

Now use (4.1), (4.2) to bound the right-hand side of (4.10) from below

$$\zeta \bar{u} + \frac{\zeta(g(\beta) - g(\alpha))}{|\zeta - \eta^\epsilon|} - \eta N_0 - N_0(\zeta - \eta) \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\zeta)).$$

Since $\alpha \le u_\epsilon(\zeta) \le u_+$, we see $g(u_\epsilon(\zeta)) \le g(\beta)$. Hence this fact combined with $|\zeta - \eta^\epsilon| \ge \frac{a}{2}$ yields

$$-b|\bar{u}| + \frac{2b(g(\beta) - g(\alpha))}{a} - bN_0 + g(\beta) \le g(u_\epsilon(\eta)). \tag{4.11}$$

Since $u_\epsilon(\eta) \le \beta$, (4.11) and the fact that $g(u) \to -\infty$ as $u \to -\infty$ show $u_\epsilon(\eta)$ uniformly bounded in ϵ, η for ϵ sufficiently small, $\eta \in S_-$.

Case ii(c), iii(b). Proceed as for Case ii(b).

Case iii(c). From the mean value theorem there is $\zeta \in [1,2]$ such that $v_\epsilon'(\zeta) = v_\epsilon(2) - v_\epsilon(1)$ and so by (4.2) $\epsilon v_\epsilon'(\zeta)$ is uniformly bounded. Thus for this ζ and arbitrary $\eta \in S_-$ we again derive (4.9) and since $v_\epsilon'(\eta) < 0$ we find that

$$\begin{aligned}\epsilon v_\epsilon'(\zeta) - \int_\eta^\zeta v_\epsilon(\xi)\,d\xi &\le g(u_\epsilon(\eta)) - g(u_\epsilon(\zeta)) \\ &\le g(u_\epsilon(\eta)) - g(\alpha).\end{aligned}$$

The same argument as given above for case iii(b) shows $u_\epsilon(\eta)$ is uniformly bounded in ϵ, η for ϵ sufficiently small, $\eta \in S_-$.

Case iii(d). Proceed analogously as in Case iii(c). □

LEMMA 4.3. *Let $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon < 1\}$ be a solution of (P_ϵ) as given by* Corollary 2.7 *when $u_- < \alpha$, $u_+ > \beta$. Let ξ_-^ϵ, ξ_+^ϵ denote the points of local minima for $v_\epsilon(\xi)$ (when they exist). Define $\bar{u} =$* $\min(u_-, u_+)$,

$$B_\epsilon^- = u_- - \left(\frac{f'(v_-)}{g'(u_-)}\right)^{1/2} v_- + \left(\frac{f'(v_-)}{g'(u_-)}\right)^{1/2} \left(\bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi_-^\epsilon - \eta^\epsilon|}\right)$$

$$B_\epsilon^+ = u_+ - \left(\frac{f'(v_+)}{g'(u_+)}\right)^{1/2} v_+ - \left(\frac{f'(v_+)}{g'(u_+)}\right)^{1/2} \left(\bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi_+^\epsilon - \eta^\epsilon|}\right).$$

Then in the case of Lemma 2.4*(with $\mu = 1, L = \infty$) we have the following estimates:*

In cases 0, i(a),(b),(c), (3.1) *holds.*

In the remaining cases $v_\epsilon(\xi)$ satisfies (4.2) *and $u_\epsilon(\xi)$ satisfies*

$u_- \le u_\epsilon(\xi) \le u_+$ *in* case i(d), ii(a), iii(a).

$B_\epsilon^- \le u_\epsilon(\xi) \le u_+$ *in* case ii(b), iii(c).

$u_- \le u_\epsilon(\xi) \le B_\epsilon^+$ *in* case ii(c), iii(d).

$B_\epsilon^- \le u_\epsilon(\xi) \le B_\epsilon^+$ *in* case iii(b).

LEMMA 4.4. *Let* $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon < 1\}$ *be a solution of* (P_ϵ) *as given by* Corollary 2.7 *when* $u_- < \alpha$, $u_+ > \beta$. *Then on any semi-infinite interval* $(-\infty, -a]$ *or* $[a, \infty)$, $a > 0$ *there exist constants* k *and* ϵ_0 *(depending at most on* $u_-, u_+, v_-, v_+, f, g, a$*) such that*

$$\sup_{(-\infty,a]} (|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq k, \tag{4.12}$$
$$\sup_{[a,\infty)} (|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq k,$$

for $0 < \epsilon < \epsilon_0$.

LEMMA 4.5. *Let* $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon < 1\}$ *be a solution of* (P_ϵ) *as given by* Corollary 2.7 *when* $u_- < \alpha$, $u_+ > \beta$. *Then the sequence* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *possesses a subsequence which converges almost everywhere on* $(-\infty, \infty)$ *to functions* $(u(\xi), v(\xi))$. *On compact subsets of* $(-\infty, 0) \cup (0, \infty)$ *the convergent subsequence is bounded uniformly in* ϵ *with uniformly bounded total variation. The limit functions have bounded variation on compact subsets of* $(-\infty, 0) \cup (0, \infty)$.

LEMMA 4.6. *The functions* $u(\xi)$, $v(\xi)$ *defined by* Lemma 4.5 *satisfy the boundary conditions*

$$u(\pm\infty) = u_\pm, v(\pm\infty) = v_\pm.$$

Proof. Let $Y_\epsilon(\xi) = (u_\epsilon(\xi), v_\epsilon(\xi))^T$, $F(Y_\epsilon) = (-f(v_\epsilon), -g(u_\epsilon))^T$, T transpose. Then

$$\frac{d}{d\xi}\left(\exp\left(\frac{\xi^2}{2\epsilon}\right) Y_\epsilon'(\xi)\right) = \frac{1}{\epsilon}\left(\nabla F(Y_\epsilon) Y_\epsilon'(\xi) \exp\left(\frac{\xi^2}{2\epsilon}\right)\right)$$

and integrating from 1 to ξ, $\xi > 1$, we find

$$\exp\left(\frac{\xi^2}{2\epsilon}\right) Y_\epsilon'(\xi) - \exp\left(\frac{1}{2\epsilon}\right) Y_\epsilon'(1) = \frac{1}{\epsilon}\int_1^\xi \nabla(Y_\epsilon) Y_\epsilon'(\zeta) \exp\left(\frac{\zeta^2}{2\epsilon}\right) d\zeta$$

Since by Lemma 4.4, $|Y_\epsilon(\xi)|$ is uniformly bounded by k on $[1, \infty)$, we know $|\nabla F(Y_\epsilon)| \leq R$ for some constant $R > 0$. Thus

$$\left|\exp\left(\frac{\xi^2}{2\epsilon}\right) Y_\epsilon'(\xi)\right| \leq \left|\exp\left(\frac{1}{2\epsilon}\right) Y_\epsilon'(1)\right| + \frac{R}{\epsilon}\int_1^\xi |Y_\epsilon'(\zeta)| \exp\left(\frac{\zeta^2}{2\epsilon}\right) d\zeta$$

and using Grownwall's inequality we have

$$\left|\exp\left(\frac{\xi^2}{2\epsilon}\right)Y_\epsilon'(\xi)\right| \leq \left|\exp\left(\frac{1}{2\epsilon}\right)Y_\epsilon'(1)\right|\exp\left(\frac{R}{\epsilon}\right)(\xi-1)$$

and hence

$$|Y_\epsilon'(\xi)| \leq |Y_\epsilon'(1)|\exp\left(\frac{2R\xi-2R+1-\xi^2}{2\epsilon}\right). \tag{4.13}$$

Note that

$$\begin{aligned}
&\exp\left(\frac{\xi^2}{2\epsilon}\right)Y_\epsilon'(\xi)\\
&= z_1 + \frac{1}{\epsilon}\int_1^\xi F(Y_\epsilon(\zeta))'\exp\left(\frac{\zeta^2}{2\epsilon}\right)d\zeta\\
&= z_2 + \frac{1}{\epsilon}F(Y_\epsilon(\zeta))\exp\left(\frac{\zeta^2}{2\epsilon}\right) - \frac{1}{\epsilon^2}\int_1^\xi \zeta F(Y_\epsilon(\zeta))\exp\left(\frac{\zeta^2}{2\epsilon}\right)d\zeta
\end{aligned}$$

and hence

$$Y_\epsilon'(\xi) = z_2\exp\left(-\frac{\xi^2}{2\epsilon}\right) + \frac{1}{\epsilon}F(Y_\epsilon(\xi)) - \frac{1}{\epsilon^2}\int_1^\xi \zeta F(Y_\epsilon(\zeta))\exp\left(\frac{\zeta^2}{2\epsilon}\right)d\zeta. \tag{4.14}$$

Here

$$\begin{aligned}
&z_2\int_1^2 \exp\left(-\frac{\xi^2}{2\epsilon}\right)d\xi\\
&\quad = Y_\epsilon(2) - Y_\epsilon(1) - \frac{1}{\epsilon}\int_1^2 F(Y_\epsilon(\xi)\,d\xi + \frac{1}{\epsilon^2}\int_1^2 \zeta F(Y_\epsilon(\zeta))\exp\left(\frac{\zeta^2}{2\epsilon}\right)d\zeta.
\end{aligned} \tag{4.15}$$

Thus from (4.14) we have

$$\begin{aligned}
|Y_\epsilon'(1)| &\leq |z_2|\exp\left(-\frac{1}{2\epsilon}\right) + \frac{1}{\epsilon}|F(Y_\epsilon(1))|\\
&\leq |z_2|\exp\left(-\frac{1}{2\epsilon}\right) + \frac{\text{const}}{\epsilon}
\end{aligned} \tag{4.16}$$

From (4.15) and the inequality

$$\int_1^2 \exp\left(-\frac{\xi^2}{2\epsilon}\right) d\xi \geq \exp\left(-\frac{2}{\epsilon}\right)$$

we see that

$$|z_2| \leq \left(\text{const} + \frac{\text{const}}{\epsilon} + \frac{\text{const}}{\epsilon^2}\exp\left(\frac{2}{\epsilon}\right)\right)\exp\left(\frac{2}{\epsilon}\right)$$

and hence by (4.16) that

$$|Y_\epsilon'(1)| \leq \frac{\text{const}}{\epsilon^2}\exp\left(\frac{7}{2\epsilon}\right). \tag{4.17}$$

Now insert (4.17) into (4.13) to find that

$$|Y_\epsilon'(\xi)| \leq \frac{\text{const}}{\epsilon^2}\left(\frac{2R\xi - 2R + 8 - \xi^2}{2\epsilon}\right). \tag{4.18}$$

Thus for $\xi > R+(R^2-2R+8)^{1/2}$ (4.18) shows that $|Y_\epsilon'(\xi)| \to 0$ as $\epsilon \to 0+$. Recalling that $(u_\epsilon(\xi), v_\epsilon(\xi))$ converges pointwise to $(u(\xi), v(\xi))$, we see $(u(\xi), v(\xi))$ must be constants for $\xi > R+(R^2-2R+8)^{1/2}$. Since for any $\epsilon > 0$ $\lim_{\xi\to\infty} u_\epsilon(\xi) = u_+$, $\lim_{\xi\to\infty} v_\epsilon(\xi) = v_+$, these constants must be u_+ and v_+. A similar argument works for $\xi = -\infty$. □

COROLLARY 4.7. *The functions* $u(\xi)$, $v(\xi)$ *defined by* Lemma 4.5 *satisfy the conditions*

$$(u(\xi), v(\xi)) = \begin{cases} (u_-, v_-), & \xi < -M, \\ (u_+, v_+), & \xi > M \end{cases}$$

for some positive constant M.

LEMMA 4.8. *The functions* $(u(\xi), v(\xi))$ *defined by* Lemma 4.5 *satisfy*

$$\begin{aligned} -\xi u' - f(v)' &= 0, \\ -\xi v' - g(u)' &= 0 \end{aligned} \tag{4.19}$$

in the sense of distributions at any $\xi \neq 0$.

At any point $\xi_0 \neq 0$ *of discontinuity of* $(u(\xi), v(\xi))$ *the Rankine-Hugoniot jump conditions are satisfied:*

$$\begin{aligned} -\xi_0(u(\xi_0+) - u(\xi_0-)) - (f(v(\xi_0+)) - f(v(\xi_0-))) &= 0, \\ -\xi_0(v(\xi_0+) - v(\xi_0-)) - (g(u(\xi_0+)) - g(u(\xi_0-))) &= 0. \end{aligned} \tag{4.20}$$

Proof. By Lemma 4.5 there exists a sequence of solutions of (P_ϵ) which converges bounded almost everywhere on any compact subset of $(0,\infty)\cup(-\infty,0)$. hence if we multiply (P_ϵ) by C^∞ test functions with compact support excluding $\xi = 0$, integrate by parts, pass to the limits as the relevant sequence of ϵ's goes to zero, and use the Lebesgue dominated convergence theorem, we obtain (4.19). Equation (4.19) follows from (4.18) in the standard manner. □

DEFINITION 4.9. u, v is a distributional solution of (4.19) at $\xi = 0$ if

$$\begin{aligned} &\lim_{\xi\to 0} f(v(\xi)) = \lim_{\xi\to 0+} f(v(\xi)), \\ &\lim_{\xi\to 0} g(u(\xi)) = \lim_{\xi\to 0+} g(u(\xi)) \end{aligned} \tag{4.21}$$

LEMMA 4.10. *Assume that*

$$\frac{1}{|u|}\left|\int_\beta^u g(\xi)\,d\xi\right| \to \infty \text{ as } |u|\to\infty. \tag{4.22}$$

Then $\{u_\epsilon(\xi)\}$ *has absolutely equicontinuous integrals and the functions* $u(\xi)$, $v(\xi)$ *defined by* Lemma 4.5 *are locally integrable in* $(-\infty,\infty)$.

Proof. From (4.1), $|v_\epsilon(\xi)|$ is locally integrable. Since a subsequence of $v_\epsilon(\xi)$ converges to $v(\xi)$, Fatou's theorem implies $v(\xi)$ is locally integrable. To show locally integrability of $u(\xi)$, we will show at first $\{u_\epsilon(\xi)\}$ have absolutely equicontinuous integral. In case i(d), ii(a), iii(a) of Lemma 2.4 there is nothing to prove since $u_\epsilon(\xi)$ is monotone and hence uniformly bounded in ξ, ϵ. Theorem 3.8 implies that Case 0, i(a, b, c) were covered. We need only prove Case ii(b, c), iii(b, c, d). Consider ii(c). Given any interval (l_1, l_2) we either

(I) $(l_1,l_2) = (l_1,t_\epsilon]\cup[t_\epsilon,l_2)$ where $(l_1,t_\epsilon]$ if $u_- \le u_\epsilon(\xi)\le\beta$ and $[t_\epsilon,l_2)$ if $\beta\le u_\epsilon$, $u_\epsilon(t_\epsilon)=\beta$,

(II) $u_\epsilon\ge\beta$ on (l_1,l_2), or

(III) $u_\epsilon(\xi)\le\beta$ on (l_1,l_2).

First we consider (I). Multiply $(P_\epsilon)_1$ by $g(u)$ and $(P_\epsilon)_2$ by $f(v)$ and add. If we define $\eta(u,v) = F(v)+\int_\beta^u g(\xi)\,d\xi$, $F'(v)=f(v)$ and $\eta_\epsilon(\xi)=\eta(u_\epsilon(\xi),v_\epsilon(\xi))$ we see that

$$\epsilon\eta_\epsilon''(\xi)+\xi\eta_\epsilon'(\xi)+(f(v)g(u))'-\epsilon(u')^2g'(u)-\epsilon f'(v)(v')^2=0. \tag{4.23}$$

Let $\bar\eta = \max\{\eta(u_-,v_-),\eta(u_+,v_+)\}$. On any subinterval $(s_1,s_2) \subset [t_\epsilon, l_2)$ set

$$\zeta_\epsilon = \begin{cases} \sup\{\xi \in [t_\epsilon, s_1) | \eta_\epsilon(\xi) \le \bar\eta\} \text{ if } \eta_\epsilon(s_1) > \bar\eta, \\ \inf\{\xi \in (s_1,s_2) | \eta_\epsilon(\xi) \ge \bar\eta\} \text{ if } \eta_\epsilon(s_1) \le \bar\eta \end{cases}$$

and

$$\theta_\epsilon = \begin{cases} \inf\{\xi \in (s_2, l_2) | \eta_\epsilon(\xi) \le \bar\eta\} \text{ if } \eta_\epsilon(s_2) > \bar\eta, \\ \sup\{\xi \in (s_1,s_2) | \eta_\epsilon(\xi) \ge \bar\eta\} \text{ if } \eta_\epsilon(s_2) \le \bar\eta. \end{cases}$$

Observe that $\eta'_\epsilon(\zeta_\epsilon) \ge 0$, $\eta'_\epsilon(\theta_\epsilon) \le 0$ and

$$(4.24) \qquad \int_{s_1}^{s_2} (\eta_\epsilon(\xi) - \bar\eta)\, d\xi \le \int_{\zeta_\epsilon}^{\theta_\epsilon} (\eta_\epsilon(\xi) - \bar\eta)\, d\xi = -\int_{\zeta_\epsilon}^{\theta_\epsilon} \xi \eta'_\epsilon(\xi)\, d\xi.$$

Thus if we integrate (4.23) over $(\zeta_\epsilon, \theta_\epsilon)$ and use (4.24) we see that

$$(4.25) \qquad \begin{aligned} &\int_{s_1}^{s_2} (\eta_\epsilon(\xi) - \bar\eta)\, d\xi + \epsilon \int_{\zeta_\epsilon}^{\theta_\epsilon} ((u'_\epsilon)^2 g'(u_\epsilon) + f'(v_\epsilon)(v'_\epsilon)^2)\, d\xi \\ &\le f(v_\epsilon(\theta_\epsilon)) - f(v_\epsilon(\zeta_\epsilon))g(u_\epsilon(\zeta_\epsilon)). \end{aligned}$$

By the definitions of θ_ϵ, ζ_ϵ, $\eta(u_\epsilon(\theta_\epsilon), v_\epsilon(\theta_\epsilon))$ and $\eta(u_\epsilon(\zeta_\epsilon), v_\epsilon(\zeta_\epsilon))$ are uniformly bounded from above and since $u_\epsilon(\theta_\epsilon) \ge \beta$, η is convex at these values. This implies $u_\epsilon(\theta_\epsilon)$, $v_\epsilon(\theta_\epsilon)$, $u_\epsilon(\zeta_\epsilon)$, $v_\epsilon(\zeta_\epsilon)$ are uniformly bounded in ϵ. Hence the right-hand side of (4.25) is bounded by a constant $K = K(f,g,u_\epsilon,v_\epsilon)$ independent of ϵ. Now since $\frac{1}{u}\int_\beta^u g(s)\, ds \to \infty$ as $u \to \infty$, for any $\delta > 0$ there is $u_0 \ge \beta$ such that

$$\frac{u}{\eta(u,v)} < \frac{\delta}{2K} \text{ for all } u \ge u_0.$$

Set $l(\delta) = \frac{\delta}{(|u_-| + \beta + u_0 + \frac{\delta}{2K})}$. Fix $s_1, s_2, 0 < s_2 - s_1 < l(\delta)$. Note that for any $s_1, s_2, s_1 \in (l_1, t_\epsilon], s_2 \in (t_\epsilon, l_2)$,

$$\begin{aligned} \int_{s_1}^{s_2} u_\epsilon(\xi)\, d\xi &= \int_{s_1}^{t_\epsilon} u_\epsilon(\xi)\, d\xi + \int_{t_\epsilon}^{s_2} u_\epsilon(\xi)\, d\xi \\ &\le \beta(t_\epsilon - s_1) + \int_{t_\epsilon}^{s_2} (u_0 + \frac{\delta}{2K}\eta(u_\epsilon(\xi), v_\epsilon(\xi)))\, d\xi \\ &\le \beta(t_\epsilon - s_1) + (s_2 - t_\epsilon)u_0 + \frac{\delta}{2K}\int_{t_\epsilon}^{s_2} \eta(u_\epsilon(\xi), v_\epsilon(\xi)))\, d\xi. \end{aligned}$$

Using (4.24) with $s_2 = s_2, s_1 = t_\epsilon$,

$$\begin{aligned}\int_{s_1}^{s_2} u_\epsilon(\xi)\,d\xi &\le \beta(t_\epsilon - s_1) + (s_2 - t_\epsilon)u_0 + \frac{\delta}{2K}(K + \bar{\eta}(s_2 - s_1)) \\ &\le (s_2 - s_1)(\beta + u_0 + \frac{\bar{\eta}\delta}{2K}) + \frac{\delta}{2} \\ &\le \delta.\end{aligned}$$

If $s_1, s_2 \ge t_\epsilon$,

$$\int_{s_1}^{s_2} u_\epsilon(\xi)\,d\xi \le \int_{s_1}^{s_2} (u_0 + \frac{\delta}{2K}\eta(u_\epsilon(\xi), v_\epsilon(\xi)))\,d\xi \le \delta$$

and if $s_1, s_2 \le t_\epsilon$

$$\int_{s_1}^{s_2} u_\epsilon(\xi)\,d\xi \le \beta(s_2 - s_1) \le \delta.$$

Also since $u_\epsilon(\xi) \ge u_-$ we have

$$\int_{s_1}^{s_2} u_\epsilon(\xi)\,d\xi \ge u_-(s_2 - s_1) \ge -|u_-|(s_2 - s_1) \ge -\delta.$$

Thus we proved that

$$\left|\int_{s_1}^{s_2} u_\epsilon(\xi)\,d\xi\right| \le \delta \text{ if } 0 < s_2 - s_1 < l(\delta).$$

Now using Vitali's theorem, u is locally integrable. □

LEMMA 4.11. *The four limits which appear in (4.21) always exist and (4.21) is always satisfied. Equation (4.21) is satisfied if the sequence $\{\int_0^\xi v_\epsilon(\xi)\,d\xi\}$ is absolutely equicontinuous. Furthermore in general*

$$g(\beta) - g(\alpha) \le \lim_{\theta \to 0+} g(u(\theta)) - \lim_{\zeta \to 0-} g(u(\zeta)) \le 0.$$

Proof. Let $\{(u_\epsilon(\xi), v_\epsilon(\xi))\}$ denote the convergent subsequence of Lemma 4.5. Note that since $u_\epsilon(\xi), v_\epsilon(\xi)$ are piecewise monotone in $(-\infty, \infty)$, the limit functions $u(\xi), v(\xi)$ are also monotone and hence the set of points of continuity of u, v is dense in any finite ξ-interval. Let ζ and θ be points of continuity of $u(\xi), v(\xi), \zeta < 0 < \theta$. From the mean value theorem for every small $\epsilon > 0$ we can find $\zeta_\epsilon \in [\zeta - \epsilon^{1/2}, \zeta]$, $\theta_\epsilon \in [\theta, \theta + \epsilon^{1/2}]$ such that

$$\epsilon^{1/2} v_\epsilon'(\zeta_\epsilon) = v_\epsilon(\zeta) - v_\epsilon(\zeta - \epsilon^{1/2}), \quad \epsilon^{1/2} u_\epsilon'(\zeta_\epsilon) = u_\epsilon(\zeta) - u_\epsilon(\zeta - \epsilon^{1/2}),$$
$$\epsilon^{1/2} v_\epsilon'(\theta_\epsilon) = v_\epsilon(\theta) - v_\epsilon(\theta - \epsilon^{1/2}), \quad \epsilon^{1/2} u_\epsilon'(\theta_\epsilon) = u_\epsilon(\theta) - u_\epsilon(\theta - \epsilon^{1/2}).$$

By Lemma 2.4 there are constants K_θ, K_ζ such that

$$(4.26) \qquad \begin{aligned} |\epsilon^{1/2} v_\epsilon'(\zeta_\epsilon)| \le K_\zeta, \quad |\epsilon^{1/2} u_\epsilon'(\zeta_\epsilon)| \le K_\zeta, \\ |\epsilon^{1/2} v_\epsilon'(\theta_\epsilon)| \le K_\theta, \quad |\epsilon^{1/2} u_\epsilon'(\theta_\epsilon)| \le K_\theta. \end{aligned}$$

for ϵ sufficiently small. Now we integrate (P_ϵ) on $(\zeta_\epsilon, \theta_\epsilon)$ obtaining

$$(4.27) \qquad \begin{aligned} &\epsilon u_\epsilon'(\theta_\epsilon) - \epsilon u_\epsilon'(\zeta_\epsilon) + \theta_\epsilon u_\epsilon(\theta_\epsilon) - \zeta_\epsilon u_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi)\, d\xi \\ &\quad = f(v(\zeta_\epsilon)) - f(v(\theta_\epsilon)), \\ &\epsilon v_\epsilon'(\theta_\epsilon) - \epsilon v_\epsilon'(\zeta_\epsilon) + \theta_\epsilon v_\epsilon(\theta_\epsilon) - \zeta_\epsilon v_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi)\, d\xi \\ &\quad = g(u(\zeta_\epsilon)) - g(u(\theta_\epsilon)) \end{aligned}$$

Now let $\epsilon \to 0+$ in (4.27). Since θ, ζ are points of continuity of u, v we find by virtue of (4.26) and the Vitali's theorem that

$$(4.28) \qquad \begin{aligned} \theta u(\theta) - \zeta u(\zeta) + f(v(\theta)) - f(v(\zeta)) &= \lim_{\epsilon \to 0+} \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi)\, d\xi \\ \theta v(\theta) - \zeta v(\zeta) + g(u(\theta)) - g(u(\zeta)) &= \lim_{\epsilon \to 0+} \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi)\, d\xi \end{aligned}$$

Since the limits on the left hand side of (4.27) exists, we have from (4.1)

$$\lim_{\epsilon \to 0+} \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi)\, d\xi := S(\zeta, \theta)$$

satisfies

$$\bar{v}(\zeta - \theta) + (g(\beta) - g(\alpha)) \leq S(\zeta, \theta) \leq N_0(\zeta - \theta).$$

By Lemma 4.4 for fixed $\zeta < 0$, $S(\zeta, \theta)$ is continuous in θ, $\theta > 0$, $|\theta|$ small and for fixed $\theta > 0$, $S(\zeta, \theta)$ is continuous in ζ, $\zeta < 0$, $|\zeta|$ small. Now since $|u(\xi)|$ may be infinite only at $\xi = 0$ pointwise limits of ii(b, c), iii(b, c, d) of Lemma 2.4 shows that if $|u(0)| = \infty$, u must one of these shape shown in figure.

In all these cases (I), (II), (III) we see that

$$|\zeta u(\zeta)| \leq \int_\zeta^\theta |u(\xi)|\, d\xi,$$

$$|\theta u(\theta)| \leq \int_\zeta^\theta |u(\xi)|\, d\xi$$

But since $u(\zeta)$ is locally integrable,

$$\lim_{\zeta \to 0-} \zeta u(\zeta) = \lim_{\theta \to 0+} \theta u(\theta) = \lim_{\substack{\theta \to 0+ \\ \zeta \to 0-}} \int_\zeta^\theta u(\xi)\, d\xi = 0$$

Since $v(\xi)$ has the shape of (I) near $\xi = 0$ and v is locally integrable

$$\lim_{\zeta \to 0-} \zeta v(\zeta) = \lim_{\theta \to 0+} \theta v(\theta) = 0$$

Now let $\theta \to 0+$, $\zeta \to 0-$ along a sequence of points of continuity of u, v and possibly extract a further subsequence such that $S(\zeta, \theta)$ converges we find that

$$\lim_{\theta \to 0+} f(v(\theta)) - \lim_{\zeta \to 0-} f(v(\zeta)) = 0,$$

$$\lim_{\theta \to 0+} g(u(\theta)) - \lim_{\zeta \to 0-} g(u(\zeta)) = \lim_{\substack{\theta \to 0+ \\ \zeta \to 0-}} S(\zeta, \theta).$$

Moreover if $\int_0^\xi v_\epsilon(\xi)\, d\xi$ is absolutely equicontinuous, the Vitali's theorem implies

$$\lim_{\substack{\theta \to 0+ \\ \zeta \to 0-}} S(\zeta, \theta) = 0.$$

In general, the bounds on $S(\zeta, \theta)$ shows that

$$g(\beta) - g(\alpha) \leq \lim_{\theta \to 0+} g(u(\theta)) - \lim_{\zeta \to 0-} g(u(\zeta)) \leq 0. \qquad \square$$

THEOREM 4.12. *The functions* $u(\xi)$, $v(\xi)$ *defined by* Lemma 4.5 *is a solution of the Riemann problem provided*

$$\lim_{\xi \to 0-} g(u(\xi)) = \lim_{\xi \to 0+} g(u(\xi)).$$

Proof. Use Lemma 4.11. □

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