Bull. Korean Math. Soc. **35** (1998), No. 1, pp. 189–193

ON THE GROMOV-HAUSDORFF CONVERGENCE OF GEODESICS

YOUNG WOOK KIM

ABSTRACT. In this paper we construct a sequence of spaces which has Gromov-Hausdorff limit such that a geodesic in the limit space is not realized as a limit of geodesics in the spaces of the sequence. This contrasts with the result of Grove and Petersen in [1] where they proved otherwise for Alexandrov spaces with common curvature bounds.

1. Introduction

The Hausdorff convergence of metric spaces is a fundamental concept not only in the study of Riemannian geometry but in the study of metric differential geometry in general. Since it had been improved conceptually from the original idea of Hausdorff and introduced in 1981 by Gromov, it gave lots of stimuli in the geometry of metric spaces and much attention has been drawn to the spaces with curvature bounds. Among them the spaces with lower curvature bound share many nice properties, of which an example is the property that such spaces are closed under the Gromov-Hausdorff(or the GH) limit. This property suggests a possible natural category for doing Riemannian geometry.

In doing metric geometry in such categories, many problems of characterizing spaces — for example sphere theorems — involve careful study of metric properties and especially the study of behavior of geodesics. The reason that Hausdorff convergence fits so well with the geometry of metric spaces lies especially in the fact that, in the GH-limit, minimal

Received September 4, 1997.

1991 Mathematics Subject Classification: 53C22.

Key words and phrases: geodesic, Gromov-Hausdorff convergence.

This paper is supported (in part) by KOSEF Grant No. 951-0106-017-2 and No. 96-0701-02-01-3.

geodesics converge to minimal geodesics. This fact is so powerful that one hopes the converse to be true. That is, the geodesics in the limit space is a limit of geodesics of the sequence space. Such study was done by Grove and Petersen and was used crucially in solving problems related to the conjecture of Alexandrov.[1] In this paper they show that, for Gromov-Hausdorff limit of a sequence of Riemannian manifolds of same dimension with common curvature (upper and lower) bounds and a diameter upper bound, a geodesic in the limit space is always a limit of geodesics in the sequence manifolds. Here the essential property is the curvature bound. Now this fact poses a question if it is still true without the common curvature bound. In this paper we show that this is not true in general. In fact, we present an example of a sequence of 2-dimensional Riemannian surfaces converging to a metric space which contains a minimal geodesic which can be no limit of minimal geodesics in the sequence surfaces. This example does not have a common curvature lower bound but they are compact smooth surfaces with common diameter upper bound. (Also common curvature upper bound.)

The convergence problem of geodesics is subtle and even the GH-limit of (non-minimal) geodesics is not a geodesic in general as is seen in the following example.

EXAMPLE. Consider a closed unit disk $\Delta = \{(x, y, z) \mid x^2+y^2 \leq 1, z = 0\}$ in $\mathbb{R}^3$ and consider the boundary of the ϵ-tube of the disk.($\epsilon > 0$) As $\epsilon \to 0$, the tubular hypersurface converges to a double disk which is the identification of two closed unit disks along the boundary circles. In the tubular hypersurface, the circle of symmetry $S_\epsilon = \{(x, y, z) \mid x^2 + y^2 = (1+\epsilon)^2, z = 0\}$ is a geodesic, but the limit curve $S_0 = \{(x, y, z) \mid x^2+y^2 = 1, z = 0\}$ in the double disk is no more a geodesic.

Now the ϵ-tube is not a C^∞ manifold but we can deform it a little without breaking the symmetry and get C^∞ Riemannian manifolds which satisfies everything in the example above.

2. Contruction of the example.

For a given metric space X, we will denote by αX ($\alpha > 0$) the metric space X of which the metric is rescaled by the factor α. That is, $d_{\alpha X} = \alpha \cdot d_X$.

First, we construct a sequence X_n of metric spaces. X_1 is nothing more than an interval $[0,1]$ with standard metric. Now, let Y_{nk} be the interval $\left(1-\frac{1}{2^k}\right)\frac{1}{2^n}X_1$ with their endpoints denoted by p^0_{nk} and p^1_{nk} for $n=1,2,\dots$ and $k=0,1,2,\dots$. To define the base space X_2, we consider a disjoint union X_1 and Y_{11} and identify the points $1/2$ and 1 in X_1 with p^0_{11} and p^1_{11} in Y_{11} respectively. Then we take the inner metric induced from the metrics on X_1 and Y_{11} which defines the metric on X_2.

Now for X_n. The spaces X_n is defined from the disjoint union of the spaces $X_1, Y_{1n}, Y_{2n}, \dots, Y_{nn}$ by identifying the points as follows: (Figure 1.)

$$
\begin{aligned}
&0 \text{ in } X_1 \text{ with } p^0_{0n},\\
&\frac{1}{2} \text{ in } X_1 \text{ with } p^0_{1n}, p^0_{2n}, \dots, p^0_{nn},\\
&1=\frac{1}{2}+\frac{1}{2} \text{ in } X_1 \text{ with } p^1_{0n}, p^1_{1n},\\
&\frac{1}{2}+\frac{1}{4} \text{ in } X_1 \text{ with } p^1_{2n},\\
&\quad\vdots\\
&\frac{1}{2}+\frac{1}{2^n} \text{ in } X_1 \text{ with } p^1_{nn}.
\end{aligned}
$$

And the metric on X_n is the inner metric induced from the standard metrics on X_1 and Y_{in}'s $(i=1,\dots,n)$.

Now the space X_∞ is the disjoint union of X_1 and $Y_{n\infty}$ $(n=1,2,\dots)$ where the points are identified as follows:

$$
\begin{aligned}
&0 \text{ in } X_1 \text{ with } p^0_{0\infty},\\
&\frac{1}{2} \text{ in } X_1 \text{ with } p^0_{n\infty}\ (n=1,2,\dots),\\
&\frac{1}{2}+\frac{1}{2} \text{ in } X_1 \text{ with } p^1_{0\infty}, p^1_{1\infty},\\
&\quad\vdots\\
&\frac{1}{2}+\frac{1}{2^n} \text{ with } p^1_{n\infty},\\
&\quad\vdots
\end{aligned}
$$

And the metric on X_∞ is the induced inner metric.

Now the metric spaces X_n are compact, locally compact inner metric spaces. As $n \to \infty$, the sequence $\{X_n\}$ converges, in GH-distance, to the metric space X_∞

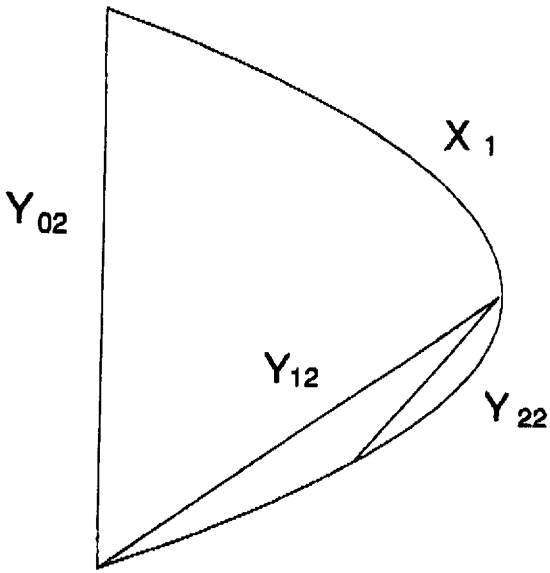

FIGURE 1.

3. Discussions

In this example, consider the curve $X_1 \subset X_\infty$. X_1 is a minimal geodesic in X_∞. Among the curves in X_n joining 0 and 1 of $X_1 \subset X_n$ (or any pair of points sufficiently near them) is a unique minimal geodesic which is the shortest one and no others are geodesics. Moreover, these geodesics do not converge as $n \to \infty$ to the geodesic $X_1 \subset X_\infty$. Therefore there exists no sequence of minimal geodesics of X_n which converges to $X_1 \subset X_\infty$.(In fact this is true for any closed interval which contains the point $1/2$ in X_1 as an interior point. Therefore there is not even a sequence of locally minimizing geodesics which converges to X_1 in X_0.)

Here X_n are simply 1-dimensional inner metric spaces. These spaces can be embedded isometrically into $\mathbb{R}^2 \subset \mathbb{R}^3$ in an obvious manner. Fix

$0 < \epsilon \ll 1$ and consider the boundary surfaces of $\epsilon/2^n$-tubes in $\mathbb{R}^3$ of X_n. Smoothing them slightly give C^∞ Riemannian surfaces which is symmetric with respect to $\mathbb{R}^2$. Denote it by Y_n. Y_n converges to X_∞ in GH-topology as $n \to \infty$. The fact that $X_1 \subset X_\infty$ is a minimal geodesic which is a limit of no minimal geodesics of Y_n is obvious from above. In fact Y_n can be made so that the surfaces have non-positive Gaussian curvatures.

The limit space not being simply connected is not essential in this phenomenon. There is such an example consisting of metric spaces where all the spaces are simply connected. (But we are not sure if we can find such a sequence of Riemannian manifolds.) The author would like to thank Prof. K. Grove for the discussions regarding the problem.

References

[1] Grove, K. and P. Petersen V, *Manifolds near the boundary of existence*, J. Diff. Geom. **33** (1991), 379–394.

DEPTARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 136–701, KOREA
E-mail: ywkim@math.korea.ac.kr