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L_1 ANALYTIC FOURIER-FEYNMAN TRANSFORM ON THE FRESNEL CLASS OF ABSTRACT WIENER SPACE

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Abstract. Let (B, H, p_1) be an abstract Wiener space and $\mathcal{F}(B)$ the Fresnel class on (B, H, p_1) which consists of functionals F of the form :

$$F(x) = \int_H \exp\{i(h, x)^\sim\}\, df(h), \quad x \in B,$$

where $(\cdot,\cdot)^\sim$ is a stochastic inner product between H and B, and f is in $\mathcal{M}(H)$, the space of complex Borel measures on H.

We introduce an L_1 analytic Fourier-Feynman transform on $\mathcal{F}(B)$ and verify the existence of the L_1 analytic Fourier-Feynman transforms for functionls in $\mathcal{F}(B)$. Furthermore, we introduce a convolution on $\mathcal{F}(B)$, and then verify the existence of the L_1 analytic Fourier-Feynman transform for the convolution product of two functionals in $\mathcal{F}(B)$, and we establish the relationships between the L_1 analytic Fourier-Feynman transform of the convolution product for two functionals in $\mathcal{F}(B)$ and the L_1 analytic Fourier-Feynman transforms for each functional. Finally, we show that most results in [7] follows from our results in Section 3.

§1. Introduction

The study of an L_1 analytic Fourier-Feynman transform on a classical Wiener space was initiated by Brue in [1]. In [2] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on a classical Wiener space. In [8] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the

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results in [2] and established several relationships between the L_1 and L_2 analytic Fourier-Feynman transform theories. In [6, 7] Huffman, Park and Skoug developed an L_p analytic Fourier-Feynman transform theory on certain classes of functionals defined on a classical Wiener space and they defined a convolution product for two functionals in the classes and then verified that the Fourier-Feynman transform of the convolution product of two functionals is the product of Fourier-Feynman transforms of each functional.

It is well known [9] that every element in the Fresnel class of the abstract Wiener space (B, H, p_1) has the analytic Wiener and Feynman integral and the Fresnel class is an analogue of the Banach algebra $\mathcal{S}$ on the classical Wiener space $C_o[0,T]$ introduced by Cameron and Storvick [3]. Moreover, the analytic Fourier-Feynman transform is based on the analytic Wiener and Feynman integral.

In this paper, we intend to make a study on an L_1 analytic Fourier-Feynman transform on the Fresnel class of the abstract Wiener space. Also we define the convolution product of certain functionals on the abstract Wiener space and establish the relationships between the L_1 analytic Fourier-Feynman transforms of each functional in $\mathcal{F}(B)$ and the L_1 analytic Fourier-Feynman transform of their convolution product. Finally, we show that most results in [7] follows from our results in Section 3.

§2. Definitions and Preliminaries

Let H be a real separable infinite dimensional Hilbert space with norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ and let $\|\cdot\|_o$ denote a fixed measurable norm on H (for definition see [11]). Let B be the completion of H with respect to the measurable norm $\|\cdot\|_o$ and $\mu_t(\, t>0\,)$ the Gauss measure on H with variance t. Then μ_t induces a cylinder set measure $\tilde{\mu}_t$ on B which in turn extends to a countably additive measure p_t on $(B, \mathcal{B}(B))$, where $\mathcal{B}(B)$ is the Borel σ-algebra of Borel sets in B. p_t is called the *Wiener measure* with variance t and it has the following properties:

$$
\begin{aligned}
&p_{st}(E) = p_t(s^{-1/2}E) \quad \text{for } s>0, \\
&p_t(-E) = p_t(E).
\end{aligned}
\tag{2.1}
$$

Let $\langle e_n \rangle$ denote a complete orthonormal system on H such that e_n's are in B^*, the topological dual space of B. For each $h \in H$ and $x \in B$, we define a *stochastic inner product* $(\cdot, \cdot)^\sim$ between H and B as follows:

$$(2.2) \qquad (h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \langle h, e_k \rangle (e_k, x), & \text{if the limit exists} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ is the natural dual pairing between B^* and B.

It is well known [9, 10] that for every $h \in H, (h,x)^\sim$ exists for p_t-a.e. $x \in B$, and is a Borel measurable functional on B having a Gaussian distribution with mean zero and variance $t|h|^2$ with respect to the Wiener measure p_t. Furthermore, it is easy to show that for each real number $\alpha, (\alpha h, x)^\sim = \alpha (h,x)^\sim = (h, \alpha x)^\sim$ holds for every $h \in H$ and $x \in B$.

Let (B, H, p_t) be an abstract Wiener space. For each $\lambda > 0$, let $\mathcal{S}_\lambda(B)$ be the completion of $\mathcal{B}(B)$ with respect to p_λ, and let $\mathcal{N}_\lambda(B) = \{A \in \mathcal{S}_\lambda(B) : p_\lambda(A) = 0\}$. Let $\mathcal{S}(B) = \bigcap\limits_{\lambda>0} \mathcal{S}_\lambda(B)$, and $\mathcal{N}(B) = \bigcap\limits_{\lambda>0} \mathcal{N}_\lambda(B)$. Every set in $\mathcal{S}(B)$ (or $\mathcal{N}(B)$) is called a *scale-invariant measurable* (or *scale-invariant null*) set. A real (or complex)-valued functional F on B is called a *scale-invariant measurable functional* if F is measurable with respect to $\mathcal{S}(B)$. A property that holds except on a scale-invariant null set is said to hold *scale-invariant almost everywhere* (briefly, s-a.e.). If two functionals F and G are equal s-a.e., then we write $F \approx G$. It is easy to show that this relation $\approx$ is an equivalence relation on the class of functionals on B. For a functional F on B, we will denote by $[F]$ the equivalence class of functionals which are equal to F s-a.e..

DEFINITION 2.1. Let (B, H, p_1) be an abstract Wiener space and let $\mathcal{M}(H)$ denote the space of all complex Borel measures on H. Consider the functional F defined for s-a.e. $x \in B$ by the formula

$$(2.3) \qquad F(x) = \int_H \exp\{i(h,x)^\sim\}\, df(h),$$

where f is in $\mathcal{M}(H)$. The class $\mathcal{F}(B)$ consists of equivalence classes $[F]$ of functionals which are equal to F s-a.e. for some f in $\mathcal{M}(H)$. We call $\mathcal{F}(B)$ the *Fresnel class* of the abstract Wiener space (B, H, p_1).

REMARKS. (1) As is customary, we will identify a functional with its equivalence class and think of $\mathcal{F}(B)$ as a class of functionals on B rather than as a class of equivalence classes.

(2) $\mathcal{M}(H)$ is a Banach algebra over the complex field under the total variation norm $\|\cdot\|$ where the convolution is taken as the multiplication (see [5]). There exists an isomorphism of Banach algebras between $\mathcal{M}(H)$ and $\mathcal{F}(B)$ [9 ; Proposition 2.1].

Throughout this paper, let $\mathbb{R}$ and $\mathbb{C}$ denote the real numbers and the complex numbers, respectively, and let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ and $\mathbb{C}_+^\sim = \{z \in \mathbb{C} : z \neq 0,\ \operatorname{Re}(z) \geq 0\}$, where $\operatorname{Re}(z)$ is the real part of the complex number z.

Let F be a complex-valued scale-invariant measurable functional on the abstract Wiener space (B, H, p_1) such that the Wiener integral

$$J[F\,;\lambda] = \int_B F(\lambda^{-1/2}x)\,dp_1(x)$$

exists as a finite number for all $\lambda > 0$. If there exists an analytic function $J^*[F\,;\lambda]$ of λ in the half-plane $\mathbb{C}_+$ such that $J^*[F\,;\lambda] = J[F\,;\lambda]$ for all $\lambda > 0$, then we define this *analytic extension* $J^*[F\,;\lambda]$ of $J[F\,;\lambda]$ to be the *analytic Wiener integral of F over B with parameter λ* and we write

$$\int_B^{anw_\lambda} F(x)\,dp_1(x) = \mathcal{I}^{anw}[F\,;\lambda] = J^*[F\,;\lambda]$$

for all $\lambda \in \mathbb{C}_+$.

Let q be a nonzero real number and F a functional on B such that the analytic Wiener integral $\mathcal{I}^{anw}[F\,;\lambda]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, then we call it the *analytic Feyman integral of F over B with parameter q* and we write

$$\int_B^{anf_q} F(x)\,dp_1(x) = \mathcal{I}^{anf}[F\,;q] = \lim_{\lambda \to -iq} \mathcal{I}^{anw}[F\,;\lambda],$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Now we are ready to define an L_1 analytic Fourier-Feynman transform on the Fresnel class $\mathcal{F}(B)$.

DEFINITION 2.2. Let F be a complex-valued functional on the abstract Wiener space (B, H, p_1) . For each $t \in \mathbb{C}_+$, we define a transform $\mathcal{F}_t F$ of F as follows :

$$(\mathcal{F}_t F)(y) = \mathcal{I}^{anw}[F(\cdot + y);t], \tag{2.4}$$

for $y \in B$, if it exists. Now we define the L_1 *analytic Fourier-Feynman transform* $\mathcal{F}_q F$ of F by the formula :

$$(\mathcal{F}_q F)(y) = \lim_{t \to -iq} (\mathcal{F}_t F)(y) \tag{2.5}$$

for s-a.e. $y \in B$, where q is nonzero real number and t approaches $-iq$ through $\mathbb{C}_+$.

We finish this section by giving the definition of the convolution product of two functionals on the abstract Wiener space (B, H, p_1).

DEFINITION 2.3. Let F and G be two complex-valued functionals on the abstract Wiener space (B, H, p_1). For each $t \in \mathbb{C}_+^\sim$, we define their *convolution product* $(F * G)_t$ as follows :

When t belongs to $\mathbb{C}_+$,

$$(F * G)_t(y) = \mathcal{I}^{anw}\Big[F\Big(\frac{1}{\sqrt{2}}(y + \cdot)\Big)\, G\Big(\frac{1}{\sqrt{2}}(y - \cdot)\Big);t\Big] \tag{2.6}$$

for $y \in B$, if it exists.

When $t = -iq$ ($q \in \mathbb{R} - \{0\}$),

$$(F * G)_q(y) = \mathcal{I}^{anf}\Big[F\Big(\frac{1}{\sqrt{2}}(y + \cdot)\Big)\, G\Big(\frac{1}{\sqrt{2}}(y - \cdot)\Big);q\Big] \tag{2.7}$$

for $y \in B$, if it exists.

§3. L_1 Analytic Fourier-Feynman Transform and Convolution

In this section, we first show that the L_1 analytic Fourier-Feynman transform exists for functionals in the Fresnel class $\mathcal{F}(B)$ and it belongs to the Fresnel class $\mathcal{F}(B)$. And we establish the relationships between the L_1 analytic Fourier-Feynman transform of the convolution product for two functionals and the L_1 analytic Fourier-Feynman transforms for each functional.

THEOREM 3.1. *Let $F \in \mathcal{F}(B)$ be given by the formula (2.3). Then the transform $\mathcal{F}_t F$ exists for all $t \in \mathbb{C}_+$, and is expressed by the formula*

$$(\mathcal{F}_t F)(y) = \int_H \exp\Big\{-\frac{1}{2t}|h|^2 + i(h,y)^{\sim}\Big\}\, df(h) \tag{3.1}$$

for s-a.e. $y \in B$, where f is in $\mathcal{M}(H)$.

Moreover, the L_1 analytic Fourier-Feynman transform $\mathcal{F}_q F$ ($q \in \mathbb{R} - \{0\}$) belongs to the Fresnel class $\mathcal{F}(B)$, and it is expressed by the formula

$$(\mathcal{F}_q F)(y) = \int_H \exp\Big\{-\frac{i}{2q}|h|^2 + i(h,y)^{\sim}\Big\}\, df(h) \tag{3.2}$$

for s-a.e. $y \in B$, where f is in $\mathcal{M}(H)$.

Proof. We first show that the transform $\mathcal{F}_t F$ exists for $t > 0$. Using Fubini's Theorem and the well-known integration formula :

$$\int_B \exp\{it(h,x)^{\sim}\}\, dp_1(x) = \exp\Big\{-\frac{t^2}{2}|h|^2\Big\}, \quad h \in H,\ t \in \mathbb{R}, \tag{3.3}$$

we obtain, for all $t > 0$,

$$\begin{aligned}(\mathcal{F}_t F)(y) &= \int_B \int_H \exp\Big\{i\Big(h, \frac{x}{\sqrt{t}} + y\Big)^{\sim}\Big\}\, df(h)\, dp_1(x)\\ &= \int_H \exp\{i(h,y)^{\sim}\}\Big\{\int_B \exp\Big\{i\Big(h, \frac{x}{\sqrt{t}}\Big)^{\sim}\Big\}\, dp_1(x)\Big\}\, df(h)\\ &= \int_H \exp\Big\{-\frac{1}{2t}|h|^2 + i(h,y)^{\sim}\Big\}\, df(h).\end{aligned}$$

for s-a.e. $y \in B$.

Now we can verify with the help of Morera's Theorem that the last expression is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\mathbb{C}_+^{\sim}$ for all $y \in B$, because f is in $\mathcal{M}(H)$. Therefore the transform $\mathcal{F}_t F$ exists for all $t \in \mathbb{C}_+$, and we can show that the formulas (3.1) and (3.2) hold.

Finally we shall show that $\mathcal{F}_q F\,(q \in \mathbb{R} - \{0\})$ belongs to $\mathcal{F}(B)$. Define a set function $\eta : \mathcal{B}(H) \to \mathbb{C}$ as follows :

$$\eta(E) = \int_E \exp\Big\{-\frac{i}{2q}|h|^2\Big\}\, df(h), \quad E \in \mathcal{B}(H),$$

where $\mathcal{B}(H)$ is the Borel σ-algebra of H. Then it is easy to show that η belongs to the Banach algebra $\mathcal{M}(H)$ of complex Borel measures on $\mathcal{B}(H)$. And the formula (3.2) is expressed as follows :

$$(\mathcal{F}_q F)(y) = \int_H \exp\{i(h,y)^\sim\}\, d\eta(h).$$

Thus $\mathcal{F}_q F$ belongs to $\mathcal{F}(B)$. □

REMARKS. (1) We define the transform $\mathcal{F}_t^n F$ of the functional F on the abstract Wiener space (B, H, p_1) as follows :

$$\mathcal{F}_t^n F = \underbrace{(\mathcal{F}_t \circ \cdots \circ \mathcal{F}_t)}_{n}(F),$$

that is, $\mathcal{F}_t^n$ means the n-times composition of $\mathcal{F}_t$, where $n = 0, 1, 2, \cdots$ and $t > 0$. When $n = 0$, $\mathcal{F}_t^0 F$ is equal to F ; When $n = 1$, $\mathcal{F}_t^1 F$ is equal to $\mathcal{F}_t F$.

When t belongs to $\mathbb{C}_+^\sim$, the transform $\mathcal{F}_t^n F$ means the analytic extension of $\mathcal{F}_t^n F\ (t > 0)$ as the function of $t \in \mathbb{C}_+^\sim$.

We have already shown that for every $F \in \mathcal{F}(B)$, the transforms $\mathcal{F}_t F\,(t \in \mathbb{C}_+)$ and $\mathcal{F}_q F\,(q \neq 0)$ belong to the Fresnel class $\mathcal{F}(B)$ again. Hence, using the mathematical induction, we can obtain the following result :

For every $F \in \mathcal{F}(B)$ and $t \in \mathbb{C}_+$, the formula

$$(\mathcal{F}_t^n F)(y) = \int_H \exp\Big\{-\frac{n}{2t}|h|^2 + i(h,y)^\sim\Big\}\, df(h) \tag{3.4}$$

holds for s-a.e. $y \in B$, where $n = 0, 1, 2, \cdots$. In particular, when $t = -iq\,(q \in \mathbb{R} - \{0\})$,

(3.5)

$$(\mathcal{F}_q^n F)(y) = \lim_{t \to -iq} (F_t^n F)(y) = \int_H \exp\Big\{-\frac{in}{2q}|h|^2 + i(h,y)^\sim\Big\}\, df(h).$$

When $n = 1$ in the formulas (3.4) and (3.5), the formulas (3.4) and (3.5) are reduced to the formulas (3.1) and (3.2), respectively.

(2) From the formulas (3.1) and (3.2) in Theorem 3.1, we deduce the expression for the analytic Wiener integral and the analytic Feynman integral of a functional F in $\mathcal{F}(B)$ as follows :

Taking $t = z$ ($z \in \mathbb{C}_+$) and $y = 0$ in the formula (3.1), we obtain the formula :

$$\mathcal{F}_z F(0) = \int_H \exp\Big\{-\frac{1}{2z}|h|^2\Big\}\, df(h) = \mathcal{I}^{anw}[\,F\,;z\,].$$

This coincides with the analytic Wiener integral of $F \in \mathcal{F}(B)$ which is obtained in [9; Proposition 2.2].

Next taking $y = 0$ in the formula (3.2), we obtain the formula :

$$\mathcal{F}_q F(0) = \int_H \exp\Big\{-\frac{i}{2q}|h|^2\Big\}\, df(h) = \mathcal{I}^{anf}[\,F\,;q\,].$$

This coincides with the analytic Feynman integral of $F \in \mathcal{F}(B)$ which is obtained in [9; Proposition 2.2].

THEOREM 3.2. *Let F and G be in $\mathcal{F}(B)$ which are given by the formula (2.3). Then the convolution product $(F * G)_t$ exists for each $t \in \mathbb{C}_+^{\sim}$ and is expressed by the formula*

$$(3.6)\quad (F*G)_t(y) = \int_{H^2} \exp\Big\{-\frac{1}{4t}|u-v|^2 + \frac{i}{\sqrt{2}}(u+v,y)^{\sim}\Big\}\, df(u)\, dg(v)$$

for s-a.e. $y \in B$, where f and g are in $\mathcal{M}(H)$.

Furthermore, for each $t \in \mathbb{C}_+^{\sim}$, the convolution product $(F * G)_t$ belongs to the Fresnel class $\mathcal{F}(B)$. In particular, when $t = -iq$ ($q \in \mathbb{R} - \{0\}$), the convolution product $(F * G)_q$ is expressed by the formula

$$(3.7)\quad (F*G)_q(y) = \int_{H^2} \exp\Big\{-\frac{i}{4q}|u-v|^2 + \frac{i}{\sqrt{2}}(u+v,y)^{\sim}\Big\}\, df(u)\, dg(v)$$

for s-a.e. $y \in B$, where f and g are in $\mathcal{M}(H)$.

Proof. Proceeding as in the proof of Theorem 3.1, for all $t > 0$ and s-a.e. $y \in B$, we have

$$\begin{aligned}
(F * G)_t(y) &= \int_B F\Big(\frac{1}{\sqrt{2}}\Big(y + \frac{x}{\sqrt{t}}\Big)\Big) G\Big(\frac{1}{\sqrt{2}}\Big(y - \frac{x}{\sqrt{t}}\Big)\Big)\, dp_1(x) \\
&= \int_B \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(u, y + \frac{x}{\sqrt{t}}\Big)^{\sim}\Big\}\, df(u)\Big] \\
&\cdot \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(v, y - \frac{x}{\sqrt{t}}\Big)^{\sim}\Big\}\, dg(v)\Big]\, dp_1(x) \\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v, y)^{\sim}\Big\} \\
&\cdot \Big[\int_B \exp\Big\{\frac{i}{\sqrt{2t}}(u-v, x)^{\sim}\Big\}\, dp_1(x)\Big]\, df(u)\, dg(v) \\
&= \int_{H^2} \exp\Big\{-\frac{1}{4t}|u-v|^2 + \frac{i}{\sqrt{2}}(u+v, y)^{\sim}\Big\}\, df(u)\, dg(v).
\end{aligned}$$

Now we can verify with the help of Morera's Theorem that the last expression is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t over $\mathbb{C}_+^{\sim}$ for all y in B, beacuse f and g are in $\mathcal{M}(H)$. By Definition 2.3, we conclude that the formulas (3.6) and (3.7) hold.

Next we shall show that $(F * G)_t$ belongs to $\mathcal{F}(B)$ for every $t \in \mathbb{C}_+^{\sim}$. Let t be in $\mathbb{C}_+^{\sim}$ and define a set function $\nu : \mathcal{B}(H^2) \to \mathbb{C}$ by

$$\nu(E) = \int_E \exp\Big\{-\frac{1}{4t}|u-v|^2\Big\}\, df(u)\, dg(v), \quad E \in \mathcal{B}(H^2).$$

Then ν is a complex Borel measure on $\mathcal{B}(H^2)$. Now define a function $\varphi : H^2 \to H$ as follows :

$$\varphi(u, v) = \frac{1}{\sqrt{2}}(u+v), \quad (u, v) \in H^2.$$

Then φ is continuous , and so it is a Borel measurable function. Hence $\mu = \nu \cdot \varphi^{-1}$ is a complex Borel measure on $\mathcal{B}(H)$. Using the Change of

Variable Formula, we have

$$\begin{aligned}(F*G)_t(y) &= \int_{H^2} \exp\Big\{-\frac{1}{4t}|u-v|^2 + \frac{i}{\sqrt{2}}(u+v,y)^\sim\Big\}\,df(u)\,dg(v)\\ &= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v,y)^\sim\Big\}\,d\nu(u,v)\\ &= \int_{H} \exp\big\{i(w,y)^\sim\big\}\,d\mu(w).\end{aligned}$$

Thus $(F*G)_t$ belongs to $\mathcal{F}(B)$. □

Our next theorem shows that the L_1 analytic Fourier-Feynman transform of the convolution product for two functionals in the Fresnel class $\mathcal{F}(B)$ is a product of transforms for each functional.

THEOREM 3.3. *Let F and G be as in* Theorem 3.2. *Then the transform $\mathcal{F}_t(F*G)_t$ exists for all $t \in \mathbb{C}_+$, and is given by the formula*

$$(3.8)\qquad (\mathcal{F}_t(F*G)_t)(y) = (\mathcal{F}_tF)\Big(\frac{y}{\sqrt{2}}\Big)\cdot(\mathcal{F}_tG)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in B$.

Moreover, the L_1 analytic Fourier-Feynman transform $\mathcal{F}_q(F*G)_q$ ($q \in \mathbb{R}-\{0\}$) is given by the formula

$$(3.9)\qquad (\mathcal{F}_q(F*G)_q)(y) = (\mathcal{F}_qF)\Big(\frac{y}{\sqrt{2}}\Big)\cdot(\mathcal{F}_qG)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in B$.

Proof. We first show that the formula (3.8) holds for all $t > 0$. Using Fubini's Theorem and the formulas (3.1), (3.3) and (3.6), we have, for s-a.e. $y \in B$,

$$(\mathcal{F}_t(F*G)_t)(y) = \int_B (F*G)_t\Big(\frac{x}{\sqrt{t}}+y\Big)\,dp_1(x)$$

$$
\begin{aligned}
&= \int_B \int_{H^2} \exp\Big\{-\frac{1}{4t}|u-v|^2 + \frac{i}{\sqrt{2}}\Big(u+v, \frac{x}{\sqrt{t}}+y\Big)^{\sim}\Big\} \\
&\quad \cdot df(u)\,dg(v)\,dp_1(x) \\
&= \int_{H^2} \exp\Big\{-\frac{1}{4t}|u-v|^2 + \frac{i}{\sqrt{2}}(u+v,y)^{\sim}\Big\} \\
&\quad \cdot \Big[\int_B \exp\Big\{\frac{i}{\sqrt{2t}}(u+v,x)^{\sim}\Big\}\,dp_1(x)\Big]\,df(u)\,dg(v) \\
&= \int_{H^2} \exp\Big\{-\frac{1}{2t}(|u|^2+|v|^2) + \frac{i}{\sqrt{2}}(u+v,y)^{\sim}\Big\} \\
&\quad \cdot df(u)\,dg(v) \\
&= (\mathcal{F}_t F)\Big(\frac{y}{\sqrt{2}}\Big)\cdot(\mathcal{F}_t G)\Big(\frac{y}{\sqrt{2}}\Big).
\end{aligned}
$$

Since $\mathcal{F}_t F$ and $\mathcal{F}_t G$ are analytic functions of t over $\mathbb{C}_+$, we conclude that $\mathcal{F}_t(F*G)_t$ is an analytic function of t over $\mathbb{C}_+$. Moreover $\mathcal{F}_t F$ and $\mathcal{F}_t G$ are bounded continuous functions of t over $\mathbb{C}_+^{\sim}$ for all $y \in B$ and so is $\mathcal{F}_t(F*G)_t$. Therefore $\mathcal{F}_t(F*G)_t$ exists for all $t \in \mathbb{C}_+^{\sim}$, and we obtain the formulas (3.8) and (3.9). □

In our next theorem we establish an interesting Parseval's identity for two functionals in the Fresnel class $\mathcal{F}(B)$.

THEOREM 3.4. *Let F and G be as in* Theorem 3.2. *Then the following Parseval's identity*

$$
\mathcal{F}_{-q}\big(\mathcal{F}_q(F*G)_q\big)(0) = \mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\cdot G\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0). \tag{3.10}
$$

holds for each $q \in \mathbb{R}-\{0\}$.

Proof. First of all, we show that the transform $\mathcal{F}_t\big(\mathcal{F}_q(F*G)_q\big)(0)$ exists for all $t>0$ and $q \in \mathbb{R}-\{0\}$. Using Fubini's Theorem and the formulas (3.2), (3.3) and (3.9), we have, for all $t>0$,

$$
\begin{aligned}
&\mathcal{F}_t\big(\mathcal{F}_q(F*G)_q\big)(0) \\
&= \mathcal{F}_t\Big((\mathcal{F}_q F)\Big(\frac{\cdot}{\sqrt{2}}\Big)\cdot(\mathcal{F}_q G)\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(0) \\
&= \int_B (\mathcal{F}_q F)\Big(\frac{x}{\sqrt{2t}}\Big)\cdot(\mathcal{F}_q G)\Big(\frac{x}{\sqrt{2t}}\Big)\,dp_1(x)
\end{aligned}
$$

$$= \int_B \Big[\int_H \exp\Big\{ -\frac{i}{2q}|u|^2 + i\Big(u, \frac{x}{\sqrt{2t}}\Big)^\sim \Big\} \, df(u) \Big]$$
$$\cdot \Big[\int_H \exp\Big\{ -\frac{i}{2q}|v|^2 + i\Big(v, \frac{x}{\sqrt{2t}}\Big)^\sim \Big\} \, dg(v) \Big] \, dp_1(x)$$
$$= \int_{H^2} \exp\Big\{ -\frac{i}{2q}(|u|^2 + |v|^2) \Big\} \Big[\int_B \exp\Big\{ \frac{i}{\sqrt{2t}}(u+v, x)^\sim \Big\} \, dp_1(x) \Big]$$
$$\cdot \, df(u) \, dg(v)$$
$$= \int_{H^2} \exp\Big\{ -\frac{i}{2q}(|u|^2 + |v|^2) - \frac{1}{4t}|u+v|^2 \Big\} \, df(u) \, dg(v).$$

Since the last expression has an analytic extension for t over $\mathbb{C}_+$, and is a bounded continuous function of t over $\mathbb{C}_+^\sim$, by letting $t \to iq$ ($q \in \mathbb{R} - \{0\}$) through $\mathbb{C}_+$, we have

$$\begin{aligned} \mathcal{F}_{-q}\big(\mathcal{F}_q(F*G)_q\big)(0) &= \lim_{t \to iq} \mathcal{F}_t\big(\mathcal{F}_q(F*G)_q\big)(0) \\ &= \int_{H^2} \exp\Big\{ -\frac{i}{4q}|u-v|^2 \Big\} \, df(u) \, dg(v). \end{aligned} \tag{3.11}$$

Next we show that the transform $\mathcal{F}_t\big(F(\frac{\cdot}{\sqrt{2}})G(-\frac{\cdot}{\sqrt{2}})\big)(0)$ exists for all $t > 0$. Using the formula (3.3) and Fubini's Theorem,

$$\mathcal{F}_t\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big) \, G\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$
$$= \int_B F\Big(\frac{x}{\sqrt{2t}}\Big) \, G\Big(-\frac{x}{\sqrt{2t}}\Big) \, dp_1(x)$$
$$= \int_B \Big[\int_H \exp\Big\{ i\Big(u, \frac{x}{\sqrt{2t}}\Big)^\sim \Big\} \, df(u) \Big] \Big[\int_H \exp\Big\{ i\Big(v, -\frac{x}{\sqrt{2t}}\Big)^\sim \Big\} \, dg(v) \Big] \, dp_1(x)$$
$$= \int_{H^2} \Big[\int_B \exp\Big\{ \frac{i}{\sqrt{2t}}(u-v, x)^\sim \Big\} \, dp_1(x) \Big] \, df(u) \, dg(v)$$
$$= \int_{H^2} \exp\Big\{ -\frac{1}{4t}|u-v|^2 \Big\} \, df(u) \, dg(v).$$

Since the last expression has an analytic extension for t over $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\mathbb{C}_+^\sim$, by letting

$t \to -iq$ ($q \in \mathbb{R} - \{0\}$) through $\mathbb{C}_+$, we have

(3.12)
$$\mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0) = \lim_{t\to -iq} \mathcal{F}_t\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$
$$= \int_{H^2} \exp\Big\{-\frac{i}{4q}|u-v|^2\Big\}\, df(u)\, dg(v).$$

From the formulas (3.11) and (3.12), we have the desired result. □

COROLLARY 3.5. *Let $F \in \mathcal{F}(B)$ be given by the formula* (2.3). *Then the following formulas hold for each $q \in \mathbb{R} - \{0\}$:*

$$\mathcal{F}_{-q}(\mathcal{F}_q(F * F)_q)(0) = \mathcal{F}_{-q}\Big((\mathcal{F}_q F)^2\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$
(3.13)
$$= \mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,F\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0),$$

and

$$\mathcal{F}_{-q}(\mathcal{F}_q(F * 1)_q)(0) = \mathcal{F}_{-q}\Big((\mathcal{F}_q F)\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$
(3.14(a))
$$= \mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(0),$$

$$\mathcal{F}_{-q}(\mathcal{F}_q(1 * G)_q)(0) = \mathcal{F}_{-q}\Big((\mathcal{F}_q G)\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$
(3.14(b))
$$= \mathcal{F}_q\Big(G\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0).$$

Proof. By taking $G \equiv F$ in the formulas (3.9) and (3.10) we obtain the formula (3.13). By taking $G \equiv 1$ in the formulas (3.9) and (3.10) we obtain the formula(3.14(a)) . By taking $F \equiv 1$ in the formulas (3.9) and (3.10) we obtain the formula (3.14(b)). □

Because we have already proved that the L_1 analytic Fourier-Feynman transform $\mathcal{F}_q F$ of F in the Fresnel class $\mathcal{F}(B)$ belongs to $\mathcal{F}(B)$ again, where q is in $\mathbb{R} - \{0\}$, we can deduce the following theorem.

THEOREM 3.6. *Let F and G be in $\mathcal{F}(B)$ which are given by the formula (2.3). Then for each $q \in \mathbb{R} - \{0\}$, the formula*

$$\Big((\mathcal{F}_q F) * (\mathcal{F}_q G)\Big)_{-q}(y) = \mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\, G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y) \tag{3.15}$$

holds for s-a.e. $y \in B$.

Proof. With the help of the formulas (3.2), (3.3) and Fubini's Theorem, we first calculate the convolution product $\big((\mathcal{F}_q F) * (\mathcal{F}_q G)\big)_t(y)$ for each $t > 0$ and s-a.e. $y \in B$ as follows :

$$\begin{aligned}
&\big((\mathcal{F}_q F) * (\mathcal{F}_q G)\big)_t(y) \\
&= \int_B (\mathcal{F}_q F)\Big(\frac{1}{\sqrt{2}}\Big(y + \frac{x}{\sqrt{t}}\Big)\Big)\, (\mathcal{F}_q G)\Big(\frac{1}{\sqrt{2}}\Big(y - \frac{x}{\sqrt{t}}\Big)\Big)\, dp_1(x) \\
&= \int_B \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(u, y + \frac{x}{\sqrt{t}}\Big)^{\sim} - \frac{i}{2q}|u|^2\Big\}\, df(u)\Big] \\
&\cdot \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(v, y - \frac{x}{\sqrt{t}}\Big)^{\sim} - \frac{i}{2q}|v|^2\Big\}\, dg(v)\Big]\, dp_1(x) \\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v, y)^{\sim} - \frac{i}{2q}\Big(|u|^2 + |v|^2\Big)\Big\} \\
&\cdot \Big[\int_B \exp\Big\{\frac{i}{\sqrt{2t}}(u - v, x)^{\sim}\Big\}\, dp_1(x)\Big]\, df(u)\, dg(v) \\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v, y)^{\sim} - \frac{i}{2q}\Big(|u|^2 + |v|^2\Big) - \frac{1}{4t}|u - v|^2\Big\}\, df(u)\, dg(v).
\end{aligned}$$

But we can verify with the help of Morera's Theorem that the last expression is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\mathbb{C}_+^{\sim}$. Therefore, by letting $t \to iq$ ($q \in \mathbb{R} - \{0\}$) through $\mathbb{C}_+$,we have, s-a.e $y \in B$,

$$\begin{aligned}
\big((\mathcal{F}_q F) * (\mathcal{F}_q G)\big)_{-q}(y) &= \lim_{t \to iq} \big((\mathcal{F}_q F) * (\mathcal{F}_q G)\big)_t(y) \\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v, y)^{\sim} - \frac{i}{4q}|u+v|^2\Big\}\, df(u) dg(v).
\end{aligned} \tag{3.16}$$

Next let us calculate the transform $\mathcal{F}_t\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y)$ for each $t > 0$ and s-a.e $y \in B$. Then we have , for each $t > 0$ and s-a.e. $y \in B$,

$$\begin{aligned}
&\mathcal{F}_t\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y)\\
&= \int_B F\Big(\frac{1}{\sqrt{2}}\Big(\frac{x}{\sqrt{t}}+y\Big)\Big)\,G\Big(\frac{1}{\sqrt{2}}\Big(\frac{x}{\sqrt{t}}+y\Big)\Big)\,dp_1(x)\\
&= \int_B \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(u, \frac{x}{\sqrt{t}}+y\Big)^{\sim}\Big\}\,df(u)\Big]\\
&\cdot \Big[\int_H \exp\Big\{\frac{i}{\sqrt{2}}\Big(v, \frac{x}{\sqrt{t}}+y\Big)^{\sim}\Big\}\,dg(v)\Big]dp_1(x)\\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v,y)^{\sim}\Big\}\Big[\int_B \exp\Big\{\frac{i}{\sqrt{2t}}(u+v,x)^{\sim}\Big\}\,dp_1(x)\Big]\\
&\cdot\, df(u)\,dg(v)\\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v,y)^{\sim} - \frac{1}{4t}|u+v|^2\Big\}\,df(u)\,dg(v).
\end{aligned}$$

But we can verify with the help of Morera's Theorem that the last expression is an analytic function of t throughout $\mathbb{C}_+$, and is a bounded continuous function of t throughout $\mathbb{C}_+^{\sim}$. Therefore, by letting $t \to -iq$ ($q \in \mathbb{R} - \{0\}$) through $\mathbb{C}_+$, we obtain, s-a.e. $y \in B$,

$$\begin{aligned}
&\mathcal{F}_q\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y) = \lim_{t\to -iq} \mathcal{F}_t\Big(F\Big(\frac{\cdot}{\sqrt{2}}\Big)\,G\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y)\\
&= \int_{H^2} \exp\Big\{\frac{i}{\sqrt{2}}(u+v,y)^{\sim} - \frac{i}{4q}|u+v|^2\Big\}\,df(u)dg(v).
\end{aligned} \tag{3.17}$$

From the formulas (3.16) and (3.17), we have the desired result. □

§4. Corollaries

In this section we apply our results in the preceding section to the classical Wiener space to obtain some results in [7] as corollaries.

Fix $T>0$ and let $B_o \equiv C_o[0,T]$ be the real separable Banach space of all real-valued continuous functions f on the closed interval $[0,T]$ which vanish at 0 and equip B_o with the uniform norm. Let $(B_o, \mathcal{W}(B_o), m_w)$ be the classical Wiener space , where m_w is the Wiener measure on the σ-algebra $\mathcal{W}(B_o)$ which is the completion of Borel σ-algebra $\mathcal{B}(B_o)$.

Put

$$H_o = \Big\{ f \in C_o[0,T] : f(t) = \int_o^t v(s)ds, \;\; v \text{ is in } L_2[0,T], \; t \in [0,T] \Big\},$$

and define an inner product $\langle \cdot , \cdot \rangle$ on H_o as follows :

$$\langle f, g \rangle = \int_0^T (Df)(s)\,(Dg)(s)\,ds, \quad f, g \in H_o,$$

where $Df \equiv \frac{df}{ds}$, the derivative of f. Then H_o is a real separable infinite dimensional Hilbert space , and $(B_o,\, H_o, m_w)$ is a typical example of an abstract Wiener space (see [11]). It is well known [9] that for each $h \in H_o$,

$$(h,x)^{\sim} = \int_0^T (Dh)(s)\,\widetilde{dx}(s)$$

holds for s-a.e. $x \in H_o$, where $\int_0^T (Dh)(s)\,\widetilde{dx}(s)$ is the Paley-Wiener-Zygmund stochastic integral of Dh (see [3]).

In [3] Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of functionals on B_o given by

$$\mathcal{S} = \Big\{ F : F(x) = \int_{L_2[0,T]} \exp\Big\{ i \int_0^T v(s)\widetilde{dx}(s) \Big\}\, df(v), \; f \in \mathcal{M}(L_2[0,T]) \Big\}.$$

Let I be the unitary operator from $L_2[0,T]$ onto H_o given by

$$Iv(t) = \int_0^t v(s)\,ds, \qquad \text{for } v \in L_2[0,T] \text{ and } t \in [0,T].$$

If

$$F(x) = \int_{L_2[0,T]} \exp\Big\{ i \int_0^T v(s)\,\widetilde{dx}(s) \Big\}\, df(v) \tag{4.1}$$

for some $f \in \mathcal{M}(L_2[0,T])$, then we have

$$F(x) = \int_{H_o} \exp\{i(h,x)^\sim\}\, d(f \circ I^{-1})(h).$$

Conversely, if

$$F(x) = \int_{H_o} \exp\{i(h,x)^\sim\}\, df(h)$$

for some $f \in \mathcal{M}(H_o)$, then we have

$$F(x) = \int_{L_2[0,T]} \exp\Big\{i \int_0^T v(s)\, \tilde{d}x(s)\Big\}\, d(f \circ I)(v).$$

Thus we show that $F \in \mathcal{S}$ if and only if $F \in \mathcal{F}(B_o)$ (see [9]).

COROLLARY 4.1. (Theorem 3.1 in [7]) *Let $F \in \mathcal{S}$ be given by the formula* (4.1). *Then the L_1 analytic Fourier-Feynman transform $T_q^{(1)}F$ exists for all $q \in \mathbb{R} - \{0\}$, and the following formula*
(4.2)

$$(T_q^{(1)}F)(y) = \int_{L_2[0,T]} \exp\Big\{i \int_0^T v(t)\, \tilde{d}y(t) - \frac{i}{2q} \int_0^T v^2(t) dt\Big\}\, df(v)$$

holds for s-a.e. $y \in B_o$.

Proof. Taking $H = H_o$, $\mathcal{F}_q F = T_q^{(1)}F$, and $h = Iv$ for some $v \in L_2[0,T]$ in the formula (3.2) of Theorem 3.1, we have the desired result. □

COROLLARY 4.2. (Theorem 3.2 in [7]) *Let F and G be elements of $\mathcal{S}$ with corresponding complex Borel measures f and g in $\mathcal{M}(L_2[0,T])$. Then the convolution product $(F * G)_q$ exists for all $q \in \mathbb{R} - \{0\}$, and the following formula*

$$(F * G)_q(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}} \int_0^T (v(t) + w(t))\, \tilde{d}y(t)\Big\} \cdot \exp\Big\{-\frac{i}{4q} \int_0^T (v(t) - w(t))^2\, dt\Big\}\, df(v)\, dg(w) \tag{4.3}$$

holds for s-a.e. $y \in B_o$.

Proof. Taking $H = H_o$ and $u = Iv$ and $v = Iw$ for some v and w in $L_2[0,T]$ in the formula (3.7) of Theorem 3.2, we have the desired result. □

COROLLARY 4.3. (Theorem 3.3 in [7]) *Let F and G be as in Corollary 4.2. Then, for all $q \in \mathbb{R} - \{0\}$, the following formula*

$$(4.4) \qquad \big(T_q^{(1)}(F * G)_q\big)(y) = \big(T_q^{(1)}F\big)\Big(\frac{y}{\sqrt{2}}\Big)\big(T_q^{(1)}G\big)\Big(\frac{y}{\sqrt{2}}\Big)$$

holds for s-a.e. $y \in B_o$.

Proof. Taking $\mathcal{F}_q F = T_q^{(1)} F$ for every $F \in \mathcal{S}$ in the formula (3.9) of Theorem 3.3, we have the desired result. □

COROLLARY 4.4. (Theorem 3.4 in [7]) *Let F and G be as in Corollary 4.2. Then, for all $q \in \mathbb{R} - \{0\}$, the Parseval's identity*

$$\begin{aligned} &\int_{C_o[0,T]}^{anf_{-q}} \big(T_q^{(1)}(F * G)_q\big)(x)\, m_w(dx) \\ (4.5) \qquad &\equiv \int_{C_o[0,T]}^{anf_{-q}} \big(T_q^{(1)}F\big)(x/\sqrt{2})\big(T_q^{(1)}G\big)(x/\sqrt{2})\, m_w(dx) \\ &= \int_{C_o[0,T]}^{anf_q} F(x/\sqrt{2})G(-x/\sqrt{2})\, m_w(dx) \end{aligned}$$

holds .

Proof. Taking $\mathcal{F}_q F(0) = \int_{C_o[0,T]}^{anf_q} F(x)\, m_w(dx)$ and $\mathcal{F}_q F = T_q^{(1)} F$ for every $F \in \mathcal{S}$ in the formula (3.10) of Theorem 3.4, we have the desired result. □

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