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# ON THE WEYL SPECTRUM OF WEIGHT

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ABSTRACT. In this paper we study the Weyl spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator $T$ acting on an infinite dimensional Hilbert space. Main results are as follows. Firstly, we show that the Weyl spectrum of weight $\alpha$ of a polynomially $\alpha$-compact operator is finite, and that similarity preserves polynomial $\alpha$-compactness and the $\alpha$-Weyl's theorem both. Secondly, we give a sufficient condition for an operator to be the sum of an unitary and a $\alpha$-compact operators.

Throughout the paper, $H$ denotes a fixed (complex) Hilbert space of dimension $h \geq \aleph_0$, the cardinality of the set of natural numbers and we write $B(H)$ for the set of all bounded linear operators on $H$. For each cardinal $\alpha$ with $\aleph_0 \leq \alpha \leq h$, let $I_\alpha$ denote the two-sided ideal in $B(H)$ of all bounded operators of rank less than $\alpha$ and let $\mathfrak{I}_\alpha$ denote the uniform closure of $I_\alpha$. Then the $\mathfrak{I}_\alpha$ are precisely the proper closed two-sided ideals of $B(H)$. Of course, $\mathfrak{I}_{\aleph_0}$ is the ideal of compact operators and $\mathfrak{I}_h$ is the maximal closed two-sided ideal of $B(H)$. If $\aleph_0 \leq \alpha < \beta \leq h$, then $\mathfrak{I}_\alpha \subseteq \mathfrak{I}_\beta$ and $\mathfrak{I}_\alpha \neq \mathfrak{I}_\beta$. For each operator $T$, $\widehat{T}$ denotes the coset $T + \mathfrak{I}_\alpha$ in the $C^*-$algebra $B(H)/\mathfrak{I}_\alpha$. The ordinary spectrum of the canonical image $\widehat{T}$ of $T$ in the quotient $C^*-$algebra $B(H)/\mathfrak{I}_\alpha$ is called the spectrum of $T$ of weight $\alpha$ and denoted by $\sigma_\alpha(T)$. Hence $\sigma_\alpha(T)$ is nonempty and compact ([2]). $\pi_\alpha(T)$ is used to denote the approximate point spectrum of $\widehat{T}$. If $T$ is $\alpha$-compact, i.e., $T \in \mathfrak{I}_\alpha$, then $\sigma_\alpha(T) = \sigma(\widehat{T}) = \{0\}$. Since $\mathfrak{I}_\alpha$ are self-adjoint ideals, $Re\ \sigma_\alpha(T) = \{0\} = \sigma_\alpha(Re\, T)$.

In [6], Yadav and Arora defined the Weyl spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator $T$ on $H$ by

$$\omega_\alpha(T) = \cap_{K \in \mathfrak{I}_\alpha} \sigma(T + K).$$

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For each operator $T$, $\omega_\alpha(T)$ is a nonempty compact subset of $\sigma(T)$ [6, Theorem 1], and if $T$ is normal then $\sigma_\alpha(T) = \omega_\alpha(T) = \pi_\alpha(T)$ [2, Corollary 4.7.1]. Evidently, $\sigma_{\aleph_0}(T)$ and $\omega_{\aleph_0}(T)$ are the ordinary essential and Weyl spectra of $T$, respectively. In particular, $0 \notin \omega_\alpha(T)$ if and only if $T$ is of the form $S+K$, where $S$ is invertible and $K \in \mathfrak{J}_\alpha$. Again it follows from the selfadjointness of the ideal $\mathfrak{J}_\alpha$ that $\overline{\omega_\alpha(T)} = \omega_\alpha(T^*)$ for any operator $T$.

In this paper we investigate the Weyl spectrum of weight $\alpha$, $\omega_\alpha(T)$, of an operator $T$ acting on an infinite dimensional Hilbert space. Main results are as follows. Firstly, we show that the Weyl spectrum of weight $\alpha$ of a polynomially $\alpha$-compact operator is finite, and that similarity preserves polynomial $\alpha$-compactness and the $\alpha$-Weyl's theorem both. Secondly, we give a sufficient condition for an operator to be the sum of an unitary and a $\alpha$-compact operators.

LEMMA 1. ([6]) *For an arbitrary operator $T$ and a polynomial $p$,*

$$\omega_\alpha(p(T)) \subseteq p(\omega_\alpha(T)).$$

*However, if $T$ is normal then for any continuous function $f$ on $\sigma(T)$,*

$$\omega_\alpha(f(T)) = f(\omega_\alpha(T)).$$

An operator $T$ is said to be *polynomially $\alpha$-compact* if there exists a nonzero polynomial $p$ such that $p(T)$ is $\alpha$-compact [6]. Thus $T$ is polynomially $\alpha$-compact if and only if $T^*$ is polynomially $\alpha$-compact. For a normal operator $T$, the followings are equivalent([6]):

(1) $T$ is polynomially $\alpha$-compact.
(2) There exists a continuous function $f$ on $\sigma(T)$ such that $f(T)$ is $\alpha$-compact and $f$ has finitely many zeros on $\omega_\alpha(T)$.
(3) $\omega_\alpha(T)$ is finite.

From Theorem 2.4 and Theorem 4.3 in [5], we have the following structure theorem for polynomially $\alpha$-compact operators.

LEMMA 2. *Let $T \in B(H)$ with $p(T) \in \mathfrak{J}_\alpha$ for some complex polynomial $p$. Then there is an operator $C \in \mathfrak{J}_\alpha$ with $p(T+C) = 0$.*

THEOREM 3. *Let $T \in B(H)$ be a polynomially $\alpha$-compact operator. Then $\omega_\alpha(T)$ is finite.*

*Proof.* By hypothesis there exists a nonzero polynomial $p$ such that $p(T)$ is $\alpha$-compact. Then by Lemma 2, there is a $\alpha$-compact operator $C \in \mathfrak{I}_\alpha$ with $p(T+C) = 0$. Hence by the spectral mapping theorem,

$$p(\sigma(T+C)) = \sigma(p(T+C)) = \sigma(0) = \{0\},$$

which implies that $\sigma(T+C)$ is finite and therefore so is $\omega_\alpha(T)$. □

THEOREM 4. *Similarity preserves polynomial $\alpha$-compactness.*

*Proof.* Let $S, T \in B(H)$ be similar. Then there is an invertible operator $U \in B(H)$ such that $S = U^{-1}TU$. Suppose $T$ is polynomially $\alpha$-compact. Then there exists a polynomial $p$ such that $p(T)$ is $\alpha$-compact. Since $\mathfrak{I}_\alpha$ is a two-sided ideal, $p(S) = p(U^{-1}TU) = U^{-1}p(T)U$ is $\alpha$-compact. Hence $S = U^{-1}TU$ is polynomially $\alpha$-compact. □

We say ([6]) that the *$\alpha$-Weyl's theorem holds for $T$* if

$$\sigma(T) - \omega_\alpha(T) = \pi_{0\alpha}(T)$$

where $\pi_{0\alpha}(T)$ denotes the set of all isolated eigenvalues of multiplicity less than $\alpha$.

THEOREM 5. *Let $T \in B(H)$ be similar to an operator $S$. If the $\alpha$-Weyl's theorem holds for $T$, then it holds for $S$.*

*Proof.* Let $S$ be similar to $T$. Then there exists an invertible operator $P$ such that $P^{-1}TP = S$. Note that $T$ is the sum of an invertible and $\alpha$-compact operators if and only if so is $S = P^{-1}TP$. Thus

$$\omega_\alpha(S) = \omega_\alpha(P^{-1}TP) = \omega_\alpha(T). \tag{0.1}$$

By [3, Problem 75]

(0.2)

$$\sigma(S) = \sigma(P^{-1}TP) = \sigma(T) \quad \text{and} \quad \sigma_p(S) = \sigma_p(P^{-1}TP) = \sigma_p(T).$$

It suffice to show that $\ker(T-\lambda) = P(\ker(S-\lambda))$ and so $\dim\ker(T-\lambda) = \dim P(\ker(S-\lambda))$. If $x \in \ker(T-\lambda)$, then

$$\begin{aligned} S(P^{-1}x) &= (P^{-1}TP)(P^{-1}x) = P^{-1}T(PP^{-1}x) \\ &= P^{-1}Tx = P^{-1}(\lambda x) = \lambda P^{-1}x. \end{aligned}$$

Thus $P^{-1}x \in \ker(S-\lambda)$ and so $x \in P(\ker(S-\lambda))$.

Conversely if $x \in P(\ker(S-\lambda))$, then $x = Py$ for some $y \in \ker(S-\lambda)$ and so $x = Py$ and $P^{-1}TPy = \lambda y$. Hence $TPy = P(\lambda y) = \lambda Py$, i.e., $Tx = \lambda x$, and so $x \in \ker(T-\lambda)$. Therefore $\ker(T-\lambda) = P(\ker(S-\lambda))$ and so $\dim\ker(T-\lambda) = \dim P(\ker(S-\lambda)) = \dim\ker(S-\lambda)$ since $P$ is invertible.

From this it is obvious that $\pi_{0\alpha}(T) = \pi_{0\alpha}(P^{-1}TP) = \pi_{0\alpha}(S)$. Since the $\alpha$-Weyl's theorem holds for $T$, $\omega_\alpha(T) = \sigma(T) - \pi_{0\alpha}(T)$. From (0.1) and (0.2), $\omega_\alpha(S) = \omega_\alpha(P^{-1}TP) = \omega_\alpha(T) = \sigma(T) - \pi_{0\alpha}(T) = \sigma(S) - \pi_{0\alpha}(S)$. Hence the $\alpha$-Weyl's theorem holds for $S$. □

We say that $T$ in $B(H)$ is *$\alpha$-Weyl* if $T$ is of the form $S+K$, where $S$ is invertible and $K \in \mathfrak{J}_\alpha$. In this case, if $\alpha = \aleph_0$, $T$ is said to be Weyl.

THEOREM 6. *If $T$ in $B(H)$ is $\alpha$-Weyl and if $S$ in $B(H)$ is such that $\pi(S) = \pi(T)^{-1}$, then $S$ is $\alpha$-Weyl, where $\pi$ is the canonical map of $B(H)$ onto $B(H)/\mathfrak{J}_\alpha$.*

*Proof.* Since $T$ is $\alpha$-Weyl, $T = U + K$, where $U$ is invertible and $K \in \mathfrak{J}_\alpha$, and this clearly implies that $S$ is the sum of an invertible and a $\alpha$-compact operators, i.e., $S$ is $\alpha$-Weyl. □

THEOREM 7. *If $\pi(T)$ is hyponormal in $B(H)/\mathfrak{J}_\alpha$ and if $\omega_\alpha(T) \subseteq \{\lambda : |\lambda| = 1\}$, then $T$ is the sum of an unitary and a $\alpha$-compact operators.*

*Proof.* By hypothesis, 0 is not in $\omega_\alpha(T)$ and so $T = S + K$, where $S$ is invertible and $K$ is $\alpha$-compact. Hence $\pi(T) = \pi(S)$. Since $\sigma(\widehat{T}) = \sigma_\alpha(T) \subseteq \omega_\alpha(T) \subseteq \{\lambda : |\lambda| = 1\}$ and $\pi(T)$ is hyponormal, $\pi(T)$ is unitary in $B(H)/\mathfrak{J}_\alpha$ and so $\pi(S^*S) = \pi(I)$. But square roots of a positive element of a $C^*$-algebra are unique, so $\pi((S^*S)^{1/2}) = \pi(I)$.

Let the polar decomposition of $S$ be given by $S = U(S^*S)^{1/2}$, where $U$ is unitary. Then

$$\begin{aligned}\pi(T) = \pi(S) = \pi(U(S^*S)^{1/2}) &= \pi(U)\pi((S^*S)^{1/2}) \\ &= \pi(U)\pi(I) = \pi(U),\end{aligned}$$

so that $T - U$ is $\alpha$-compact. □

For an example, consider $T = U \oplus U^*$, where $U$ is the unilateral shift. In this case, $\omega(T) = \{\lambda : |\lambda| = 1\} = \sigma_e(T)$. But $T$ is not a normal operator. Since $I - UU^*$ and $UU^* - I$ are rank one operators, $\pi(T)$ is normal. By Theorem 9, $T = U \oplus U^*$ is the sum of an unitary and a compact operators; in fact $\begin{pmatrix} U & I - UU^* \\ 0 & U^* \end{pmatrix}$ is unitary-it is just the bilateral shift on $l_2(\mathbb{Z})$.

We say that $S, T \in B(H)$ are *$\alpha$-essentially similar* if there exists an invertible operator $V$ such that $VSV^{-1} - T$ is $\alpha$-compact(i.e., $S$ is similar to a perturbation of $T$ by $\alpha$-compact operator). Call $S, T \in B(H)$ *$\alpha$-essentially equivalent* if there exists an unitary operator $U$ such that $USU^{-1} - T$ is $\alpha$-compact

THEOREM 8. *If $S$ and $T$ are $\alpha$-essentially similar, then $\omega_\alpha(S) = \omega_\alpha(T)$.*

*Proof.* By hypothesis, there exists an invertible operator $U$ such that $K = USU^{-1} - T$ is $\alpha$-compact. By [6, Theorem 3], $\omega_\alpha(USU^{-1}) = \omega_\alpha(T + K) = \omega_\alpha(T)$. Also by (0.1) $\omega_\alpha(USU^{-1}) = \omega_\alpha(S)$. Thus $\omega_\alpha(S) = \omega_\alpha(T)$. □

We define a cardinal $\alpha$ to be $\alpha_0-$*irregular* if it is the sum of countably many cardinals strictly smaller than $\alpha$ ([2]). A cardinal which is not $\aleph_0-$irregular is said to be $\aleph_0-$*regular.*

THEOREM 9. *Let $H$ be a Hilbert space of dimension $h$, where $h$ is an $\aleph_0-$irregular cardinal, $h > \aleph_0$. Let $S$ and $T$ be normal elements of $B(H)/\mathfrak{I}_h$. Then the followings are equivalent:*

(1) *$S$ and $T$ are $h-$essentially equivalent, i.e., there exists an unitary operator $U$ such that $USU^* - T$ is $h-$compact.*

(2) *$S$ and $T$ are $h-$essentially similar, i.e., there exists an invertible operator $P$ such that $PSP^{-1} - T$ is $h-$compact.*
(3) $\sigma_h(S) = \sigma_h(T)$.

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3): trivial (for any $a$ and $b$).
(3) $\Rightarrow$ (1): By [2, Theorem 5.7], $S$ and $T$ are unitarily equivalent modulo an $h-$compact operator, i.e., $S$ and $T$ are $h-$essentially equivalent. □

THEOREM 10. *Let $H$ be a Hilbert space of dimension $h$, where $h$ is an $\aleph_0-$irregular cardinal, $h > \aleph_0$. If $U$ and $V$ are unitary operators such that $\omega_h(U) = \omega_h(V)$ is a proper subset of the unit circle, then $U$ and $V$ are $h-$essentially equivalent.*

*Proof.* Since $U$ and $V$ are unitary operators, $U$ and $V$ are normal operators. Thus $\sigma_h(U) = \sigma(\pi(U)) = \omega_h(U) = \omega_h(V) = \sigma_h(V)$. By [2, Theorem 5.7], $U$ and $V$ are $h-$essentially equivalent, i.e., $U$ and $V$ are unitarily equivalent modulo an $h-$compact operator. □

QUESTION. *If the complement of $\sigma_\alpha(T)$ is connected does it*

$$\omega_\alpha(T) = \sigma_\alpha(T) \ ? \tag{0.3}$$

If (0.3) is true one can get:

EXAMPLE. If $T \in B(H)$ is a polynomially $\alpha$-compact operator then

$$p(\omega_\alpha(T)) = \omega_\alpha(p(T)) \quad \text{for every polynomial } p. \tag{0.4}$$

*Proof.* If $T \in B(H)$ is polynomially $\alpha$-compact then by Theorem 3, $\omega_\alpha(T)$ is finite and hence the complement of $\sigma_\alpha(T)$ is connected. Thus by (0.3) we have that $\sigma_\alpha(T) = \omega_\alpha(T)$. We therefore have

$$\begin{aligned} p(\omega_\alpha(T)) &= p(\sigma_\alpha(T)) = p(\sigma(\widehat{T})) = \sigma(p(\widehat{T})) \\ &= \sigma(\widehat{p(T)}) = \sigma_\alpha(p(T)) \subseteq \omega_\alpha(p(T)), \end{aligned}$$

which together with Lemma 1 gives (0.4). □

If $\alpha = \aleph_0$ then the answer to the question is affirmative. Therefore if the answer to the question is negative then we will get a contrast with the ordinary case.

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