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IMPROVEMENTS OF THORP-ROLEWICZ THEOREMS ON OPERATOR SERIES

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ABSTRACT. In this paper, we improve a series of results for operator series by Thorp and Rolewicz.

In 1969 and 1988, B. L. D. Thorp [1] and S. Rolewicz [2] gave a series of interesting results for operator series as follows.

THEOREM 1. *Let X, Y be Banach spaces and $L(X, Y)$ the space of continuous linear operators from X into Y. The following $(A), (B), (C)$, and (D) hold.*

(A) *If X is finite-dimensional and $\{A_j\}$ is a sequence in $L(X, Y)$ such that for every $x \in X$ the series $\sum_{j=1}^{\infty} A_j(x)$ is subseries convergent, then the series $\sum_{j=1}^{\infty} A_j(x_j)$ converges for every bounded $\{x_j\} \subseteq X$ ([2], Theorem 2).*
(B) *If X is infinite-dimensional, then there exists a sequence $\{A_j\}$ in $L(X, Y)$ such that the series $\sum A_j$ is subseries convergent in the operator norm but $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| = +\infty$ for some bounded $\{x_j\} \subseteq X$ ([2], Theorem 3).*
(C) *If Y is finite-dimensional and $\{A_j\}$ is a sequence in $L(X, Y)$ such that $\sup\{\|\sum_{j=1}^{m} A_j(x_j)\| : m \in \mathbb{N}, \sup_j \|x_j\| \le 1\} < +\infty$, then $\sum_{j=1}^{\infty} \|A_j\| < +\infty$ ([2], Theorem 4).*
(D) *If Y is infinite-dimensional, then there exists $\{A_j\} \subseteq L(X, Y)$ such that the series $\sum_{j=1}^{\infty} A_j(x_j)$ converges for every bounded $\{x_j\} \subseteq X$ but $\sum_{j=1}^{\infty} \|A_j\| = +\infty$ ([2], Theorem 5).*

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Observing that for series in Banach spaces the subseries convergence, the unconditional convergence and the bounded multiplier convergence are equivalent, we can rewrite (A) to the following

(A^*) If X is finite-dimensional and $\{A_j\} \subseteq L(X,Y)$ such that the series $\sum_{j=1}^{\infty} t_j A_j(x)$ converges for every $\{t_j\} \in l^{\infty}$ and $x \in X$, then the series $\sum_{j=1}^{\infty} A_j(x_j)$ converges for every bounded $\{x_j\} \subseteq X$.

In this paper we would like to improve above results. Recall that for a Banach space X, $c_0(X)$ is the family of sequences in X tending to zero, $c(X)$ is the family of convergent sequences in X, $l^{\infty}(X)$ is the family of bounded sequences in X and for $0 < p < +\infty$, $l^p(X) = \{\{x_j\} \subseteq X : \sum_{j=1}^{\infty} \|x_j\|^p < +\infty\}$.

LEMMA 2. *Let X be a finite-dimensional Banach space with a basis $\{b_1, b_2, \ldots, b_n\}$, $\lambda = c_0$ or c or l^{∞} or l^p $(0 < p < +\infty)$. If $\{x_j\} \subseteq \lambda(X)$ where $x_j = \sum_{i=1}^{n} t_{ij} b_i$ for all j, then $\{t_{kj}\}_{j=1}^{\infty} \in \lambda$, for all $1 \leq k \leq n$.*

Proof. Define a linear operator $u : \mathbb{C}^n \to X$ by $u(t_1, t_2, \cdots, t_n) = \sum_{i=1}^{n} t_i b_i$, then u is a homeomorphism of $\mathbb{C}^n$ and X so there exists $m, M > 0$ such that

$$m\left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \leq \left\|\sum_{i=1}^{n} a_i b_i\right\| \leq M\left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2},$$

for all $(a_1, a_2, \cdots, a_n) \in \mathbb{C}^n$. Therefore, if $1 \leq k \leq n$ then

$$|t_{kj}| \leq \left(\sum_{i=1}^{n} |t_{ij}|^2\right)^{1/2} \leq \frac{1}{m}\left\|\sum_{i=1}^{n} t_{ij} b_i\right\|,$$

for all $j \in \mathbb{N}$. This shows that $\{\sum_{i=1}^{n} t_{ij} b_i\}_{j=1}^{\infty} \in \lambda(X) = c_0(X)$ or l^{∞} or $l^p(X)$ implies that $\{t_{kj}\}_{j=1}^{\infty} \in \lambda$, for all $1 \leq k \leq n$. If $\lim_j \|\sum_{i=1}^{n} t_{ij} b_i - \sum_{i=1}^{n} t_i b_i\| = 0$, then $|t_{kj} - t_k| \leq \frac{1}{m}\|\sum_{i=1}^{n}(t_{ij} - t_i) b_i\|$ for each $1 \leq k \leq n$ so $\lim_j t_{kj} = t_k$, for all $1 \leq k \leq n$ i.e. $\{t_{kj}\}_{j=1}^{\infty} \in c$, for all $1 \leq k \leq n$.□

Now we can develop (A) $(=(A^*))$ to the following

THEOREM 3. *Let X be a finite-dimensional Banach space, $\lambda(X) = c_0(X)$ or $c(X)$ or $l^\infty(X)$ or $l^p(X)$ $(0 < p < +\infty)$ and $\{A_j\}$ a sequence of linear operators from X into a topological vector space Y. If the series $\sum_{j=1}^\infty t_j A_j(x)$ converges for every $\{t_j\} \in \lambda$ and $x \in X$, then the series $\sum_{j=1}^\infty A_j(x_j)$ converges for every $\{x_j\} \in \lambda(X)$ and, in particular, if $\lambda = l^\infty$, then $\sum_{j=1}^\infty A_j(x_j)$ converges uniformly with respect to $\{x_j\} \subseteq B = \{x \in X : \|x\| \le 1\}$ and the set $\{\sum_{j=1}^\infty A_j(x_j) : \{x_j\} \subseteq B\}$ is both compact and sequentially compact in the topological vector space Y.*

Proof. Let $\{b_1, b_2, \cdots, b_n\}$ be a basis of X and $\{x_j\} \in \lambda(X)$ where $x_j = \sum_{j=1}^n t_{ij} b_i$. By Lemma 2, $\{t_{ij}\}_{j=1}^\infty \in \lambda$ and, hence, the series $\sum_{j=1}^\infty t_{ij} A_j(b_i)$ converges for each $1 \le i \le n$. Therefore,

$$\begin{aligned}
\lim_m \sum_{j=1}^m A_j(x_j) &= \lim_m \sum_{j=1}^m A_j(\sum_{j=1}^n t_{ij} b_i) \\
&= \lim_m \sum_{i=1}^n \sum_{j=1}^m t_{ij} A_j(b_i) \\
&= \sum_{i=1}^n \sum_{j=1}^\infty t_{ij} A_j(b_i)
\end{aligned}$$

i.e., $\sum_{j=1}^\infty A_j(x_j)$ converges.

Now suppose $\lambda = l^\infty$. Then the series $\sum_{j=1}^\infty A_j(x_j)$ converges for every bounded $\{x_j\} \subseteq X$ by the assumption and, hence, $\sum_{j=1}^\infty A_j(x_j)$ converges uniformly for $\{x_j\} \subseteq B = \{x \in X : \|x\| \le 1\}$ by Theorem 1 of [3]. Since X is finite-dimensional, each A_j is continuous on $(B, \|\cdot\|)$ and, letting $F(\{x_j\}) = \sum_{j=1}^\infty A_j(x_j)$, F is continuous function from the product space $B^{\mathbb{N}} = B \times B \times \cdots$ into Y ([3], Corollary 2). Observing X is finite-dimensional, B is both compact and sequentially compact. By the Tychonoff theorem and the Diagonal procedure, $B^{\mathbb{N}}$ is also both compact and sequentially compact and its continuous image $F[B^{\mathbb{N}}] = \{\sum_{j=1}^\infty A_j(x_j) : \|x_j\| \le 1, \forall j \in \mathbb{N}\}$ is both compact and sequentially compact. □

Now we improve (B) to the following

THEOREM 4. *Let X and Y be Banach spaces. If X is infinite-dimensional, then there exists a sequence $\{A_j\}$ of continuous linear operators from X into Y such that the series $\sum A_j$ is subseries convergent in the operator norm and, hence, the series $\sum_{j=1}^{\infty} t_j A_j$ is norm-convergent for every bounded sequence $\{t_j\}$ in $\mathbb{C}$ but $\sup_m \|\sum_{j=1}^{m} A_j (x_j)\| = +\infty$ for some sequence $\{x_j\}$ in X for which $x_j \to 0$ or even $\sum_{j=1}^{\infty} \|x_j\| < +\infty$.*

Proof. Let X' be the dual of X. Then $(X', \|\cdot\|)$ is an infinite-dimensional Banach space and, hence, there is a sequence $\{f_j\}$ in X' such that the series $\sum f_j$ is subseries convergent in the norm but $\sum_{j=1}^{\infty} \|f_j\| = +\infty$ by the Dvoretzky-Rogers theorem ([4], p.59). Fix a nonzero $y \in Y$ and define $A_j : X \to Y$ by $A_j(x) = f_j(x)y$, $x \in X$, $j = 1, 2, 3, \cdots$. Now let $\{j_k\}$ be an increasing sequence in $\mathbb{N}$. Then

$$\|\sum_{k=1}^{n} A_{j_k} - \sum_{k=1}^{m} A_{j_k}\| = \sup_{\|x\|\le 1} \|\sum_{k=m}^{n} A_{j_k}(x)\| = \|y\| \sup_{\|x\|\le 1} |\sum_{k=m}^{n} f_{j_k}(x)|$$
$$= \|y\|\|\sum_{k=m}^{n} f_{j_k}\| \to 0$$

as $n, m \to \infty$.

This shows that the series $\sum A_j$ is subseries convergent in the Banach space $(L(X,Y), \|\cdot\|)$ and, hence, the series $\sum_{j=1}^{\infty} t_j A_j$ converges in the operator norm for every bounded sequence $\{t_j\}$ in $\mathbb{C}$.

Now for each $j \in \mathbb{N}$ pick a $z_j \in X$ for which $\|z_j\| \le 1$ and $f_j(z_j) > \|f_j\| - 2^{-j}$. Observing $\sum_{j=1}^{\infty} \|f_j\| = +\infty$, there is a sequence $0 = n_0 < n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that

$$\|f_{n_{k-1}+1}\| + \|f_{n_{k-1}+2}\| + \cdots + \|f_{n_k}\| > k^2,\ k = 1, 2, 3, \cdots.$$

Let

$$x_j = k^{-2} z_j \text{ if } n_{k-1} < j \le n_k,\ k = 1, 2, 3, \cdots,$$

then not only $x_j \to 0$ but $\displaystyle\sum_{j=1}^{\infty} \|x_j\| < +\infty$. However,

$$
\begin{aligned}
&\|\sum_{j=1}^{n_k} A_j(x_j)\| \\
=&\|\sum_{j=1}^{n_k} f_j(x_j)y\| \\
=&\|y\||\sum_{j=1}^{n_k} f_j(x_j)| \\
\geq&\|y\|[\sum_{j=1}^{n_1} f_j(x_j) + \sum_{j=n_1+1}^{n_2} f_j(x_j) + \cdots + \sum_{j=n_{k-1}+1}^{n_k} f_j(x_j)] \\
=&\|y\|[\sum_{j=1}^{n_1} f_j(z_j) + 2^{-2} \sum_{j=n_1+1}^{n_2} f_j(z_j) + \cdots + k^{-2} \sum_{j=n_{k-1}+1}^{n_k} f_j(z_j)] \\
\geq&\|y\|[\sum_{j=1}^{n_1} \|f_j\| - \sum_{j=1}^{n_1} 2^{-j} + 2^{-2} \sum_{j=n_1+1}^{n_2} \|f_j\| - 2^{-2} \sum_{j=n_1+1}^{n_2} 2^{-j} + \\
&\quad \cdots + k^{-2} \sum_{j=n_{k-1}+1}^{n_k} \|f_j\| - k^{-2} \sum_{j=n_{k-1}+1}^{n_k} 2^{-j}] \\
\geq&\|y\|(k-1)\,.
\end{aligned}
$$

This shows that $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| = +\infty$. □

For a sequence $\{A_j\}$ in $L(X,Y)$ consider the following conditions :

1° $\sum_{j=1}^{\infty} \|A_j\| < +\infty$;

2° $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| \leq M < +\infty$, for all $\sup_j \|x_j\| \leq 1$;

3° $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| \leq P < +\infty$, $\sup_j \|x_j\| \leq 1$, $x_j \to 0$;

4° if $x_j \to 0$, then $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| < +\infty$;

5° if $\sum_{j=1}^{\infty} \|x_j\| < +\infty$, then $\sup_m \|\sum_{j=1}^{m} A_j(x_j)\| < +\infty$.

Clearly 1° $\Rightarrow$ 2° $\Rightarrow$ 3° $\Rightarrow$ 4° $\Rightarrow$ 5°. Especially, in 2° and 3° the boundedness is just the uniform boundedness on sequence families but in 4° and 5° the boundedness is sequencewise boundedness only. However, we have the following improvement of (C).

THEOREM 5. *Let X and Y be normed spaces. If Y is finite-dimensional, then for a sequence $\{A_j\} \subseteq L(X,Y)$ the conditions* 1°, 2°, 3°, 4°, *and* 5° *are equivalent.*

proof. Only need to show 5° $\Rightarrow$ 1°. Let $\{b_1, b_2, \cdots, b_n\}$ be a basis of Y. For $1 \le k \le n$ let

$$f_k(\sum_{i=1}^{n} t_i b_i) = t_k\,,$$

then f_k is a continuous linear functional and, hence, $f_k \circ A_j$ is a continuous linear functional on X for all $j \in \mathbb{N}$. Clearly, $y = \sum_{k=1}^{n} f_k(y) b_k$, for all $y \in Y$.

Suppose that $1 \le k \le n$ and $\sum_{j=1}^{\infty} \|f_k \circ A_j\| = +\infty$. Then there is an integer sequence $0 = m_0 < m_1 < m_2 < \cdots$ such that $\sum_{j=m_{p-1}+1}^{m_p} \|f_k \circ A_j\| > p^2$, for all $p \in N$. Now for each j pick a $z_j \in X$ such that $\|z_j\| \le 1$ and $(f_k \circ A_j)(z_j) > \|f_k \circ A_j\| - 2^{-j}$. Let $x_j = p^{-2} z_j$ if $m_{p-1} < j \le m_p$, $p = 1, 2, 3, \cdots$. Then not only $x_j \to 0$ but $\sum_{j=1}^{\infty} \|x_j\| < +\infty$. It is similar to the proof of Theorem 4, a calculation shows that $f_k(\sum_{j=1}^{m_p} A_j(x_j)) > p - 1$ for all $p \in \mathbb{N}$ and, hence, the set $\{\sum_{j=1}^{m} A_j(x_j) : m \in \mathbb{N}\}$ can not be bounded. This contradicts the assumption and hence, $\sum_{j=1}^{\infty} \|f_k \circ A_j\| < +\infty$ for each $1 \le k \le n$.

Observe that

$$A_j(x) = \sum_{k=1}^{n} f_k(A_j(x)) b_k = \sum_{k=1}^{n} (f_k \circ A_j)(x) b_k\,,$$

we have

$$\|A_j\| \sup_{\|x\|\leq 1} \|A_j(x)\| = \sup_{\|x\|\leq 1} \|\sum_{k=1}^{n}(f_k \circ A_j)(x)b_k\| \leq \sum_{k=1}^{n} \|f_k \circ A_j\|\|b_k\|$$

for every $j \in \mathbb{N}$ and, hence. for every $m \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{j=1}^{m} \|A_j\| &\leq \sum_{j=1}^{m}\sum_{k=1}^{n} \|f_k \circ A_j\|\|b_k\| \\ &= \sum_{k=1}^{n}(\sum_{j=1}^{m} \|f_k \circ A_j\|)\|b_k\| \\ &\leq \sum_{k=1}^{n}(\sum_{j=1}^{\infty} \|f_k \circ A_j\|)\|b_k\|,\end{aligned}$$

i.e., $\sum_{j=1}^{\infty} \|A_j\| \leq \sum_{k=1}^{n}(\sum_{j=1}^{\infty} \|f_k \circ A_j\|)\|b_k\| < +\infty.$ □

As was stated in the proof of Theorem 3, if the series $\sum_{j=1}^{\infty} A_j(x_j)$ converges for every bounded $\{x_j\} \subseteq X$, then $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to all sequences in $\{x \in X : \|x\| \leq 1\}$ (also see [5]). Thus, we can improve (D) to the following

THEOREM 6. *Let X and Y be Banach spaces. If Y is infinite-dimensional, then there exists a sequence $\{A_j\}$ of continuous linear operators from X into Y such that the series $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to all sequences $\{x_j\}$ in $\{x \in X : \|x\| \leq 1\}$ but $\sum_{j=1}^{\infty} \|A_j\| = +\infty$.*

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