Bull. Korean Math. Soc. **35** (1998), No. 1, pp. 63–74

A NEW APPROACH ON UNIQUENESS IN ELASTODYNAMICS

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ABSTRACT. Our study is dedicated to the the proof of uniqueness of solution of initiallly boundary value problem in Elastodynamics of initially stressed bodies with voids. This proof is obtained without recourse either to an energy conservation law or to any boundedness assumptions on the elastic coefficients.

1. Introduction

The theory of elastic materials with voids, or vacuous pores, is a recent generalization of the classical theory of elasticity. In fact, in this theory the authors introduce an additional degree of freedom in order to develop the mechanical behavior of porous solids in which the matrix material is elastic and the interstices are void of material. The intended applications of this theory are to geological materials, like rocks and soils and, also, to manufactured porous material. In the paper [5] of Cowin & Nunziato, the linear theory of elastic materials with voids was developed. Iesan, in [2], has established the equations of the Thermoelasticity of materials with voids. The linear theory of micropolar bodies with voids was developed in our study [3].

The present paper is concerned with the linear Elastodynamics of initially stressed materials with voids. The result presented in what follows are aimed to strengthen some theorems previously available. First, we present the basic equations and conditions of mixed initial boundary value problem in the context of linear theory of Elastodynamics of materials with voids. Next, we present a counterpart of Brun's theorem,

Received December 17, 1996.
1991 Mathematics Subject Classification: 73C35, 73C15, 73B22.
Key words and phrases: initially stressed body, Elastodynamic, voids.

[1], in the isotermal theory of elastic bodies and use the latter to prove the uniqueness. Previous studies on uniqueness have been based almost exclusively on the assumptions the elasticity tensor is positive definite or is strongly eliptic or, others, recourse to an energy conservation law. Exception include a result of the paper by Brun [1], where an assumption concerning negative deffiniteness of the initial time derivative of the relaxation tensor is used. Our objective is to obtain the uniqueness without recourse either to an energy conservation law or to any boundedness assumptions on the thermoelastic coefficients. For convenience the notations and terminology chosen are almost identical to those of [3], [4].

2. Basic equations

Let the body occupy, at time $t = 0$, a properly regular region B of the Euclidian three-dimensional space, bounded by the piece-wise smooth surface ∂B. We refer the motion of the body to a fix system of rectangular Cartesian axes $Ox_i, i = 1,2,3$. We use the summation convention over repeated indices. The subcript j after a comma indicates partial differentiation with respect to x_j. All Latin subscripts are understood to range over the integers $(1,2,3)$, while the Greek indices have the range $1,2$. A superposed dot denotes the derivative with respect to the t-time variable. The basic equations of the Elastodynamics of initially stressed bodies with voids are
- the equations of motion

$$(1)\qquad \begin{aligned} &(\tau_{ij} + \eta_{ij})_{,j} + \varrho f_i = \varrho \ddot{u}_i, \\ &\mu_{ijk,i} + \eta_{jk} + u_{j,i} M_{ik} + \varphi_{ki} M_{ji} - \varphi_{kr,i} N_{ijr} + \varrho g_{jk} = I_{kr} \ddot{\varphi}_{jr}; \end{aligned}$$

- the balance of the equilibrated forces

$$(2)\qquad h_{i,i} + \varrho L + g = \varrho \kappa \ddot{\sigma};$$

- the constitutive equations

$$(3)\qquad \begin{aligned} \tau_{ij} = {} & u_{j,k} P_{ki} + C_{ijmn} \varepsilon_{mn} + G_{ijmn} \gamma_{mn} \\ & + F_{mnrij} \chi_{mnr} + a_{ij} \sigma + d_{ijk} \sigma_{,k}, \end{aligned}$$

$$\eta_{ij} = -\varphi_{jk}M_{ik} + \varphi_{jk,r}N_{rik} + G_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn}D_{ijmnr}\chi_{mnr} + b_{ij}\sigma + e_{ijk}\sigma_{,k},$$
$$\mu_{ijk} = u_{j,r}N_{irk} + F_{ijkmn}\varepsilon_{mn}D_{mnijk}\gamma_{mn} + A_{ijkmnr}\chi_{mnr} + c_{ijk}\sigma + f_{mijk}\sigma_{,m},$$
$$h_i = d_{mni}\varepsilon_{mn} + e_{mni}\gamma_{mn} + f_{mnri}\chi_{mnr} + d_i\sigma + Q_{ij}\sigma_{,j},$$
$$g = -\tau\dot{\sigma} - a_{ij}\varepsilon_{ij} - b_{ij}\gamma_{ij} - c_{ijk}\chi_{ijk} - \xi\sigma - d_i\sigma_{,i};$$

- the geometrical equations

(4) $2\varepsilon_{ij} = u_{i,j} + u_{j,i},\ \gamma_{ij} = u_{j,i} - \varphi_{ij},\ \chi_{ijk} = \varphi_{jk,i},\ \sigma = \nu - \nu_0.$

In these equations we have used the following notations: ϱ-the constant mass density; u_i-the components of the displacement vector; φ_{jk}-the components of the dipolar displacement tensor; $\varepsilon_{ij},\ \gamma_{ij},\ \chi_{ijk}$-the kinematic characteristics of the strain; $\tau_{ij},\ \eta_{ij},\ \mu_{ijk}$-the components of the stresses; f_i-the components of the body force per unit mass; g_{jk}-the components of the dipolar body force per unit mass; L-the extrinsic equilibrated body force; h_i-components of the equilibrated stress; g-the intrinsic equilibrated force; κ-the equilibrated inertia; ν-the volume distribution function which in the reference configuration is ν_0; σ-the change in volume fraction measured from the reference state; I_{ij}-coefficients of microinertia; $C_{ijmn},\ B_{ijmn},\ G_{ijmn},\ b_{ij},\ D_{ijmnr},\ A_{ijkmnr},\ F_{ijmnr},\ e_{ijk},\ f_{mijk},\ a_{ij},\ c_{ijk},\ d_i,\ Q_{ij},\ \xi$-the characteristic constants of the material and they obey the symmetry relations

$$\begin{aligned} &C_{ijmn} = C_{mnij} = C_{jimn},\ B_{ijmn} = B_{mnij},\ G_{ijmn} = G_{ijnm},\\ &Q_{ij} = Q_{ji}, F_{ijkmn} = F_{ijknm},\ A_{ijkmnr} = A_{mnrijk},\\ &a_{ij} = a_{ji},\ I_{ij} = I_{ji},\ d_{ijk} = d_{jik}. \end{aligned} \tag{5}$$

In (??) and (??) $P_{ij} = P_{ji},\ M_{ij}$ and N_{ijk} are prescribed functions which satisfy the following equations

$$(P_{ij} + M_{ij})_{,j} = 0,\ N_{ijk,i} + M_{jk} = 0.$$

Our paper is concerned with an anisotropic and nonhomogeneous body. To avoid repeated regularity assumptions, we assume that

(1) $C_{ijmn},\ B_{ijmn},\ G_{ijmn},\ a_{ij},\ D_{ijmnr},\ A_{ijkmnr},\ F_{ijkmn},\ e_{ijk},$
$f_{mijk},\ b_{ij},\ c_{ijk}, Q_{ij},\ d_i,\ \xi$ are continuously differentiable on B;

(2) $f_i,\ g_{jk}$ and L are continuous on $B \times [0, \infty)$.

The components of the surface traction t_i, the components of the surface dipolar traction μ_{jk} and the surface equilibrated traction h at regular points of $\partial B \times [0, \infty)$ are defined by

$$t_i = (\tau_{ij} + \eta_{ij})n_j, \; \mu_{jk} = \mu_{ijk}n_i, \; h = h_i n_i$$

respectively, where we denote by n_i the components of the unit normal on ∂B, pointing towards the exterior of ∂B.
Along with (??) to (??) we shall assume that the following standard initial conditions hold

$$\begin{aligned} (6) \qquad & u_i(x,0) = u_{0i}(x), \; \dot{u}_i(x,0) = \dot{u}_{0i}(x), \; \varphi_{jk}(x,0) = \varphi_{0jk}(x), \\ & \dot{\varphi}_{jk}(x,0) = \dot{\varphi}_{0jk}(x), \; \sigma(x,0) = \sigma_0(x), \; \dot{\sigma}(x,0) = \dot{\sigma}_0(x), \; x \in B, \end{aligned}$$

where the functions u_{0i}, $\dot{u}_{0i}$, φ_{0jk}, $\dot{\varphi}_{0jk}$, σ_0, $\dot{\sigma}_0$ are prescribed.
Let ∂B_i, ∂B_i^c, $(i = 1, 2, 3)$ be subsets of the surface ∂B so that

$$\begin{aligned} \partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B, \\ \partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \emptyset. \end{aligned}$$

To the above equations we adjoin the following boundary conditions

$$\begin{aligned} & u_i = \tilde{u}_i \;\; on \;\; \partial B_1 \times [0, \infty), \; t_i = \tilde{t}_i \;\; on \;\; \partial B_1^c \times [0, \infty), \\ (7) \qquad & \varphi_{jk} = \tilde{\varphi}_{jk} \;\; on \;\; \partial B_2 \times [0, \infty), \; \mu_{jk} = \tilde{\mu}_{jk} \;\; on \;\; \partial B_2^c \times [0, \infty), \\ & \sigma = \tilde{\sigma} \;\; on \;\; \partial B_3 \times [0, \infty), \; h = \tilde{h} \;\; on \;\; \partial B_3^c \times [0, \infty), \end{aligned}$$

where $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_{jk}$, $\tilde{\mu}_{jk}$, $\tilde{\sigma}$ and $\tilde{h}$ are given functions.
In all what follows we use the assumptions

- u_{0i}, $\dot{u}_{0i}$, φ_{0jk}, $\dot{\varphi}_{0jk}$, σ_0 and $\dot{\sigma}_0$ are continuous functions on B;
- $\tilde{u}_i$, $\tilde{\varphi}_{jk}$ and $\tilde{\sigma}$ are continuous functions on $\partial B_1 \times [0, \infty)$, $\partial B_2 \times [0, \infty)$, $\partial B_3 \times [0, \infty)$, respectively;
- $\tilde{t}_i$, $\tilde{\mu}_{jk}$ and $\tilde{h}$ are continuos functions in time and are piece-wise regular on $\partial B_1^c \times [0, \infty)$, $\partial B_2^c \times [0, \infty)$ and $\partial B_3^c \times [0, \infty)$, respectively.

By a solution of the mixed initial boundary value problem of the micropolar materials with voids, in the cylinder $B \times [0, \infty)$, we mean an ordered array $(u_i, \; \varphi_{jk}, \; \sigma)$ which satisfies equations (1)-(4) and the conditions (6), (7).

3. Uniqueness result

Throughout this section it is assumed that a twice continuously differentiable solution $(u_i,\ \varphi_{jk},\ \sigma)$ exists satisfying the equations (1)-(4) and the conditions (6) and (7) on a maximal interval of existence. First, we establish some estimations and then, as a consequence, the basic theorem which proves the uniqueness of the solution.
We consider the functions K and U on $[0,\infty)$ defined by

$$(8)\qquad K(t)=\frac{1}{2}\int_B(\varrho\dot{u}_i\dot{u}_i+I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk}+\varrho\kappa\dot{\sigma}^2)dV,$$

$$(9)\qquad \begin{aligned} U(t) \;=\; & \frac{1}{2}\int_B(C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}+2G_{mnij}\varepsilon_{ij}\gamma_{mn}+2F_{mnrij}\varepsilon_{ij}\chi_{mnr}\\ &+B_{ijmn}\gamma_{ij}\gamma_{mn}+2D_{ijmnr}\gamma_{ij}\chi_{mnr}+A_{ijkmnr}\chi_{ijk}\chi_{mnr}\\ &+P_{ki}u_{j,k}u_{j,i}-2M_{ik}u_{j,i}\varphi_{jk}+2N_{rik}u_{j,i}\varphi_{jk,r}+2a_{ij}\varepsilon_{ij}\sigma\\ &+2b_{ij}\gamma_{ij}\sigma++2c_{ijk}\chi_{ijk}\sigma+2d_{ijk}\varepsilon_{ij}\sigma_{,k}+2e_{ijk}\gamma_{ij}\sigma_{,k}\\ &+2f_{mijk}\chi_{ijk}\sigma_{,m}+Q_{ij}\sigma_{,i}\sigma_{,j}+2d_i\sigma_{,i}\sigma+\xi\sigma^2)dV,\ t\in[0,\infty), \end{aligned}$$

where for convenience, we have omitted the explicit dependence of the functions on their spatial argument x and on the time t.
In all that follows where it is possible for simplicity, we shall suppress the spatial argument x or the time variable t. For any $\alpha,\ \beta\in[0,\infty)$, we also define

$$(10)\qquad \begin{aligned} G(\alpha,\ \beta) &\\ =&\int_B\varrho[f_i(x,\alpha)\dot{u}_i(x,\beta)+g_{jk}(x,\alpha)\dot{\varphi}_{jk}(x,\beta)+L(x,\alpha)\dot{\sigma}(x,\beta)]dV\\ &+\int_{\partial B}[t_i(x,\alpha)\dot{u}_i(x,\beta)+\mu_{jk}(x,\alpha)\dot{\varphi}_{jk}(x,\beta)+h(x,\alpha)\dot{\sigma}(x,\beta)]dA. \end{aligned}$$

The identity in the next Theorem is a neccesary step in the proof of the main resut.

THEOREM 1. *If the symmetry relations* (5) *are satisfied, then*

$$
\begin{aligned}
U(t)-K(t) \quad = \quad & \frac{1}{2}\int_0^t [G(t+s,t-s)-G(t-s,t+s)]ds \\
& +\frac{1}{2}\int_B \{C_{ijmn}\varepsilon_{ij}(2t)\varepsilon_{mn}(0)+G_{mnij}[\varepsilon_{ij}(2t)\gamma_{mn}(0)+\varepsilon_{ij}(0)\gamma_{mn}(2t)] \\
& +B_{ijmn}\gamma_{ij}(2t)\gamma_{mn}(0)+F_{mnrij}[\varepsilon_{ij}(2t)\chi_{mnr}(0)+\varepsilon_{ij}(0)\chi_{mnr}(2t)] \\
& +D_{ijmnr}[\gamma_{ij}(2t)\chi_{mnr}(0)+\gamma_{ij}(0)\chi_{mnr}(2t)]+A_{ijkmnr}\chi_{ijk}(2t)\chi_{mnr}(0) \\
& +P_{ki}u_{j,k}(2t)u_{j,i}(0)-M_{ik}[u_{j,i}(2t)\varphi_{jk}(0)+u_{j,i}(0)\varphi_{jk}(2t)] \\
& +N_{rik}[u_{j,i}(2t)\varphi_{jk,r}(0)+u_{j,i}(0)\varphi_{jk,r}(2t)]+a_{ij}[\varepsilon_{ij}(0)\sigma(2t) \\
& +\varepsilon_{ij}(2t)\sigma(0)]+c_{ijk}[\chi_{ijk}(0)\sigma(2t)+\chi_{ijk}(2t)\sigma(0)] \\
& +b_{ij}[\gamma_{ij}(0)\sigma(2t)+\gamma_{ij}(2t)\sigma(0)]+d_{ijk}[\varepsilon_{ij}(0)\sigma_{,k}(2t)+\varepsilon_{ij}(2t)\sigma_{,k}(0)] \\
& +e_{ijk}[\gamma_{ij}(0)\sigma_{,k}(2t)+\gamma_{ij}(2t)\sigma_{,k}(0)]+d_i[\sigma_{,i}(0)\sigma(2t)+\sigma_{,i}(2t)\sigma(0)] \\
& +f_{mijk}[\chi_{ijk}(0)\sigma_{,m}(2t)+\chi_{ijk}(2t)\sigma_{,m}(0)]+Q_{ij}\sigma_{,i}(0)\sigma_{,j}(2t)+\xi\sigma(0)\sigma(2t) \\
& -\varrho\dot{u}_i(0)\dot{u}_i(2t)-I_{kr}\dot{\varphi}_{jr}(0)\dot{\varphi}_{jk}(2t)-\varrho\kappa\dot{\sigma}(0)\dot{\sigma}(2t)\}dV, \quad t\in[0,\infty).
\end{aligned}
\tag{11}
$$

Proof. From the constitutive equations (3), we obtain

$$
\begin{aligned}
& \tau_{ij}(t-s)\dot{\varepsilon}_{ij}(t+s)+\eta_{ij}(t-s)\dot{\gamma}_{ij}(t+s) \\
& \quad +\mu_{ijk}(t-s)\dot{\chi}_{ijk}(t+s)+h_i(t-s)\dot{\sigma}_{,i}(t+s)-g(t-s)\dot{\sigma}(t+s) \\
& \quad -\tau_{ij}(t+s)\dot{\varepsilon}_{ij}(t-s)-\eta_{ij}(t+s)\dot{\gamma}_{ij}(t-s)- \\
& \quad -\mu_{ijk}(t+s)\dot{\chi}_{ijk}(t-s)-h_i(t+s)\dot{\sigma}_{,i}(t-s)+g(t+s)\dot{\sigma}(t-s) \\
& \quad =\frac{\partial}{\partial s}\{C_{ijmn}\varepsilon_{ij}(t+s)\varepsilon_{mn}(t-s)+B_{ijmn}\gamma_{ij}(t+s)\gamma_{mn}(t-s) \\
& \quad +G_{ijmn}[\varepsilon_{ij}(t+s)\gamma_{mn}(t-s)+\varepsilon_{ij}(t-s)\gamma_{mn}(t+s)] \\
& \quad +F_{mnrij}[\varepsilon_{ij}(t+s)\chi_{mnr}(t-s)+\varepsilon_{ij}(t-s)\chi_{mnr}(t+s)] \\
& \quad +D_{ijmnr}[\gamma_{ij}(t+s)\chi_{mnr}(t-s)+\gamma_{ij}(t-s)\chi_{mnr}(t+s)] \\
& \quad +A_{ijkmnr}\chi_{ijk}(t+s)\chi_{mnr}(t-s)+P_{ki}u_{j,k}(t+s)u_{j,i}(t-s) \\
& \quad -M_{ik}[u_{j,i}(t+s)\varphi_{jk}(t-s)+u_{j,i}(t-s)\varphi_{jk}(t+s)]+N_{rik}[u_{j,i}(t+s)\varphi_{jk,r}(t-s) \\
& \quad +u_{j,i}(t-s)\varphi_{jk,r}(t+s)]+a_{ij}[\sigma(t-s)\varepsilon_{ij}(t+s) \\
& \quad +\sigma(t+s)\varepsilon_{ij}(t-s)]+b_{ij}[\sigma(t-s)\gamma_{ij}(t+s)+\sigma(t+s)\gamma_{ij}(t-s)] \\
& \quad +c_{ijk}[\sigma(t-s)\chi_{ijk}(t+s)+\sigma(t+s)\chi_{ijk}(t-s)] \\
& \quad +d_{ijk}[\varepsilon_{ij}(t+s)\sigma_{,k}(t-s)+\varepsilon_{ij}(t-s)\sigma_{,k}(t+s)] \\
& \quad +e_{ijk}[\gamma_{ij}(t+s)\sigma_{,k}(t-s)+\gamma_{ij}(t-s)\sigma_{,k}(t+s)) \\
& \quad +f_{mijk}[\chi_{ijk}(t+s)\sigma_{,m}(t-s)+\chi_{ijk}(t-s)\sigma_{,m}(t+s)] \\
& \quad +d_i[\sigma_{,i}(t-s)\sigma(t+s)+\sigma_{,i}(t+s)\sigma(t-s)] \\
& \quad +Q_{ij}\sigma_{,i}(t-s)\sigma_{,j}(t+s)+\xi\sigma(t+s)\sigma(t-s)\}.
\end{aligned}
\tag{12}
$$

In view of (1), (2) and (4), it follows

$$
\begin{aligned}
&[\tau_{ij} - u_{j,k}P_{ki}](t-s)\dot{\varepsilon}_{ij}(t+s) + [\eta_{ij} + \varphi_{jk}M_{ik} - \varphi_{jk,r}N_{rik}](t-s)\dot{\gamma}_{ij}(t+s) \\
&\quad +[\mu_{ijk} - u_{j,i}N_{irk}](t-s)\dot{\chi}_{ijk}(t+s) + h_i(t-s)\dot{\sigma}_{,i}(t+s) - g(t-s)\dot{\sigma}(t+s) \\
&\quad -[\tau_{ij} - u_{j,k}P_{ki}](t+s)\dot{\varepsilon}_{ij}(t-s) - [\eta_{ij} + \varphi_{jk}M_{ik} - \varphi_{jk,r}N_{rik}](t+s)\dot{\gamma}_{ij}(t-s) \\
&\quad -[\mu_{ijk} - u_{j,i}N_{irk}](t+s)\dot{\chi}_{ijk}(t-s) - h_i(t+s)\dot{\sigma}_{,i}(t-s) + g(t+s)\dot{\sigma}(t-s) \\
&\quad = \{[\tau_{ij}(t-s) + \eta_{ij}(t-s)]\dot{u}_{,j}(t+s) + \mu_{ijk}(t-s)\dot{\varphi}_{jk}(t+s) + h_i(t-s)\dot{\sigma}(t+s)\}_{,i} \\
&\quad -\{[\tau_{ij}(t+s) + \eta_{ij}(t+s)]\dot{u}_j(t-s) + \mu_{ijk}(t+s)\dot{\varphi}_{jk}(t-s) + h_i(t+s)\dot{\sigma}(t-s)\}_{,i} \\
&\quad +\varrho[f_i(t-s)\dot{u}_i(t+s) + g_{jk}(t-s)\dot{\varphi}_{jk}(t+s) + L(t-s)\dot{\sigma}(t+s)] \\
&\quad -\varrho[f_i(t+s)\dot{u}_i(t-s) + g_{jk}(t+s)\dot{\varphi}_{jk}(t-s) + L(t+s)\dot{\sigma}(t-s)] \\
&\quad +\frac{\partial}{\partial s}[\varrho\dot{u}_i(t-s)\dot{u}_i(t+s) + I_{kr}\dot{\varphi}_{jr}(t-s)\dot{\varphi}_{jk}(t+s) + \varrho\kappa\dot{\sigma}(t-s)\dot{\sigma}(t+s)].
\end{aligned}
\tag{13}
$$

Taking into account the left sides of the equations (12) and (13), we may write

$$
\begin{aligned}
&\frac{\partial}{\partial s}\{C_{ijmn}\varepsilon_{ij}(t+s)\varepsilon_{mn}(t-s) + B_{ijmn}\gamma_{ij}(t+s)\gamma_{mn}(t-s) \\
&\quad +G_{mnij}[\varepsilon_{ij}(t+s)\gamma_{mn}(t-s) + \varepsilon_{ij}(t-s)\gamma_{mn}(t+s)] \\
&\quad +F_{mnrij}[\varepsilon_{ij}(t+s)\chi_{mnr}(t-s) + \varepsilon_{ij}(t-s)\chi_{mnr}(t+s)] + \\
&\quad +D_{ijmnr}[\gamma_{ij}(t+s)\chi_{mnr}(t-s) + \gamma_{ij}(t-s)\chi_{mnr}(t+s)] \\
&\quad +A_{ijkmnr}\chi_{ijk}(t+s)\chi_{mnr}(t-s) + P_{ki}u_{j,k}(t+s)u_{j,i}(t-s) \\
&\quad -M_{ik}[u_{j,i}(t+s)\varphi_{jk}(t-s) + u_{j,i}(t-s)\varphi_{jk}(t+s)] + N_{rik}[u_{j,i}(t+s)\varphi_{jk,r}(t-s) \\
&\quad +u_{j,i}(t-s)\varphi_{jk,r}(t+s)] + a_{ij}[\sigma(t-s)\varepsilon_{ij}(t+s) \\
&\quad +\sigma(t+s)\varepsilon_{ij}(t-s)] + b_{ij}[\sigma(t-s)\gamma_{ij}(t+s) + \sigma(t+s)\gamma_{ij}(t-s)] \\
&\quad +c_{ijk}[\sigma(t-s)\chi_{ijk}(t+s) + \sigma(t+s)\chi_{ijk}(t-s)] \\
&\quad +d_{ijk}[\varepsilon_{ij}(t+s)\sigma_{,k}(t-s) + \varepsilon_{ij}(t-s)\sigma_{,k}(t+s)] \\
&\quad +e_{ijk}[\gamma_{ij}(t+s)\sigma_{,k}(t-s) + \gamma_{ij}(t-s)\sigma_{,k}(t+s)] \\
&\quad +f_{mijk}[\chi_{ijk}(t+s)\sigma_{,m}(t-s) + \chi_{ijk}(t-s)\sigma_{,m}(t+s)] \\
&\quad +d_i[\sigma_{,i}(t-s)\sigma(t+s) + \sigma_{,i}(t+s)\sigma(t-s)] \\
&\quad +Q_{ij}\sigma_{,i}(t-s)\sigma_{,j}(t+s) + \xi\sigma(t+s)\sigma(t-s) - \varrho\dot{u}_i(t-s)\dot{u}_i(t+s) \\
&\quad -I_{kr}\dot{\varphi}_{jr}(t-s)\dot{\varphi}_{jk}(t+s) - \varrho\kappa\dot{\sigma}(t-s)\dot{\sigma}(t+s)] \\
&\quad = \{[\tau_{ij} + \eta_{ij}](t-s)\dot{u}_j(t+s) + \mu_{ijk}(t-s)\dot{\varphi}_{jk}(t+s) + h_i(t-s)\dot{\sigma}(t+s)\}_{,i} \\
&\quad -\{[\tau_{ij} + \eta_{ij}](t+s)\dot{u}_j(t-s) + \mu_{ijk}(t+s)\dot{\varphi}_{jr}(t-s) + h_i(t+s)\dot{\sigma}(t-s)\}_{,i} \\
&\quad +\varrho[f_i(t-s)\dot{u}_i(t+s) + g_{jk}(t-s)\dot{\varphi}_{jk}(t+s) + L(t-s)\dot{\sigma}(t+s) \\
&\quad -\varrho[f_i(t+s)\dot{u}_i(t-s) + g_{jk}(t+s)\dot{\varphi}_{jk}(t-s) + L(t+s)\dot{\sigma}(t-s)].
\end{aligned}
\tag{14}
$$

Now, by integrating in (14) on $B \times [0,t]$, with the aid of divergence theorem, it results

$$\begin{aligned}
&\int_0^t \int_{\partial B} [t_i(t-s)\dot{u}_i(t+s) + \mu_{jk}(t-s)\dot{\varphi}_{jk}(t+s) + h(t-s)\dot{\sigma}(t+s) \\
&-[t_i(t+s)\dot{u}_i(t-s) + \mu_{jk}(t+s)\dot{\varphi}_{jk}(t-s) - h(t+s)\dot{\sigma}(t-s)]dAds \\
&+\int_0^t \int_B \varrho[f_i(t-s)\dot{u}_i(t+s) + g_{jk}(t-s)\dot{\varphi}_{jk}(t+s) + L(t-s)\dot{\sigma}(t+s) \\
&-\varrho[f_i(t+s)\dot{u}_i(t-s) + g_{jk}(t+s)\dot{\varphi}_{jk}(t-s) + L(t+s)\dot{\sigma}(t-s)]dVds \\
&= \int_B \{C_{ijmn}\varepsilon_{ij}(2t)\varepsilon_{mn}(0) + G_{mnij}[\varepsilon_{ij}(2t)\gamma_{mn}(0) + \varepsilon_{ij}(0)\gamma_{mn}(2t)] \\
&-F_{mnrij}[\varepsilon_{ij}(2t)\chi_{mnr}(0) + \varepsilon_{ij}(0)\chi_{mnr}(2t)] + B_{ijmn}\gamma_{ij}(2t)\gamma_{mn}(0) \\
&+D_{ijmnr}[\gamma_{ij}(2t)\chi_{mnr}(0) + \gamma_{ij}(0)\chi_{mnr}(2t)] + A_{ijkmnr}\chi_{ijk}(2t)\chi_{mnr}(0) \\
&+P_{ki}u_{j,k}(2t)u_{j,i}(0) - M_{ik}[u_{j,i}(2t)\varphi_{jk}(0) + u_{j,i}(0)\varphi_{jk}(2t)] \\
&+N_{rik}[u_{j,i}(2t)\varphi_{jk,r}(0) + u_{j,i}(0)\varphi_{jk,r}(2t)] \\
&+a_{ij}[\varepsilon_{ij}(0)\sigma(2t) + \varepsilon_{ij}(2t)\sigma(0)] + b_{ij}[\gamma_{ij}(0)\sigma(2t) + \gamma_{ij}(2t)\sigma(0)] \\
&+c_{ijk}[\chi_{ijk}(0)\sigma(2t) + \chi_{ijk}(2t)\sigma(0)] + d_{ijk}[\varepsilon_{ij}(0)\sigma_{,k}(2t) + \varepsilon_{ij}(2t)\sigma_{,k}(0)] \\
&+e_{ijk}[\gamma_{ij}(0)\sigma_{,k}(2t) + \gamma_{ij}(2t)\sigma_{,k}(0)] + f_{mijk}[\chi_{ijk}(0)\sigma_{,m}(2t) + \chi_{ijk}(2t)\sigma_{,m}(0)] \\
&+Q_{ij}\sigma_{,i}(0)\sigma_{,j}(2t) + d_i[\sigma_{,i}(0)\sigma(2t) + \sigma_{,i}(2t)\sigma(0)] + \xi\sigma(0)\sigma(2t) \\
&-\varrho\dot{u}_i(0)\dot{u}_i(2t) - I_{kr}\dot{\varphi}_{jr}(0)\dot{\varphi}_{jk}(2t) - \varrho\kappa\dot{\sigma}(0)\dot{\sigma}(2t)\}dV \\
&-\int_B [C_{ijmn}\varepsilon_{ij}(t)\varepsilon_{mn}(t) + 2G_{mnij}\varepsilon_{ij}(t)\gamma_{mn}(t) + 2F_{mnrij}\varepsilon_{ij}(t)\chi_{mnr}(t) \\
&+B_{ijmn}\gamma_{ij}(t)\gamma_{mn}(t) + 2D_{ijmnr}\gamma_{ij}(t)]\chi_{mnr}(t) + A_{ijkmnr}\chi_{ijk}(t)\chi_{mnr}(t) \\
&+P_{ki}u_{j,k}(t)u_{j,i}(t) - 2M_{ik}u_{j,i}(t)\varphi_{jk}(t) + 2N_{rik}u_{j,i}(t)\varphi_{jk,r}(t) \\
&+2a_{ij}\varepsilon_{ij}(t)\sigma(t) + 2c_{ijk}\chi_{ijk}(t)\sigma(t) + 2b_{ij}\gamma_{ij}(t)\sigma(t) + 2d_{ijk}\varepsilon_{ij}(t)\sigma_{,k}(t) \\
&+2e_{ijk}\gamma_{ij}(t)\sigma_{,k}(t) + 2f_{mijk}\chi_{ijk}(t)\sigma_{,m}(t) + Q_{ij}\sigma_{,i}(t)\sigma_{,j}(t) + 2d_i\sigma_{,k}(t)s(t) \\
&+\xi\sigma^2(t) - \varrho\dot{u}_i(t)\dot{u}_i(t) - I_{kr}\dot{\varphi}_{jr}(t)\dot{\varphi}_{jk}(t) - \varrho\kappa\dot{\sigma}(t)^2]dV.
\end{aligned} \tag{15}$$

Relation (15) may be restated, equivalently

$$\begin{aligned}
&\frac{1}{2}\int_B [C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2G_{mnij}\varepsilon_{ij}\gamma_{mn} + 2F_{mnrij}\varepsilon_{ij}\chi_{mnr} + B_{ijmn}\gamma_{ij}\gamma_{mn} \\
&+2D_{ijmnr}\gamma_{ij}\chi_{mnr} + A_{ijkmnr}\chi_{ijk}\chi_{mnr} + P_{ki}u_{j,k}u_{j,i} - 2M_{ik}u_{j,i}\varphi_{jk} \\
&+2N_{rik}u_{j,i}\varphi_{jk,r} + 2a_{ij}\varepsilon_{ij}\sigma + 2b_{ij}\gamma_{ij}\sigma + 2c_{ijk}\chi_{ijk}\sigma \\
&+2d_{ijk}\varepsilon_{ij}\sigma_{,k} + 2e_{ijk}\gamma_{ij}\sigma_{,k} + 2f_{mijk}\chi_{ijk}\sigma_{,m} + 2d_i\sigma_{,i}\sigma + \xi\sigma^2]dV \\
&-\frac{1}{2}\int_B [\varrho\dot{u}_i\dot{u}_i + I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk} + \varrho\kappa\dot{\sigma}^2]dV \\
&= \int_0^t \{\frac{1}{2}\int_B \varrho[f_i(t+s)\dot{u}_i(t-s) + g_{jk}(t+s)\dot{\varphi}_{jk}(t-s) + L(t+s)\dot{\sigma}(t-s)]dV
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\partial B}[t_i(t+s)\dot{u}_i(t-s)+\mu_{jk}(t+s)\dot{\varphi}_{jk}(t-s)+h(t+s)\dot{\sigma}(t-s)]dA\\
&-\frac{1}{2}\int_{B}\varrho[f_i(t-s)\dot{u}_i(t+s)+g_{jk}(t-s)\dot{\varphi}_{jk}(t+s)+L(t-s)\dot{\sigma}(t+s)]dV\\
&-\frac{1}{2}\int_{\partial B}[t_i(t-s)\dot{u}_i(t+s)+\mu_{jk}(t-s)\dot{\varphi}_{jk}(t+s)+h(t-s)\dot{\sigma}(t+s)]dA\}ds\\
&+\frac{1}{2}\int_{B}\{C_{ijmn}\varepsilon_{ij}(2t)\varepsilon_{mn}(0)+G_{mnij}[\varepsilon_{ij}(2t)\gamma_{mn}(0)+\varepsilon_{ij}(0)\gamma_{mn}(2t)]\\
&+F_{mnrij}[\varepsilon_{ij}(2t)\chi_{mnr}(0)+\varepsilon_{ij}(0)\chi_{mnr}(2t)]+B_{ijmn}\gamma_{ij}(2t)\gamma_{mn}(0)\\
&+D_{ijmnr}[\gamma_{ij}(2t)\chi_{mnr}(0)+\gamma_{ij}(0)\chi_{mnr}(2t)]+A_{ijkmnr}\chi_{ijk}(2t)\chi_{mnr}(0)\\
&+P_{ki}u_{j,k}(2t)u_{j,i}(0)-M_{ik}[u_{j,i}(2t)\varphi_{jk}(0)+u_{j,i}(0)\varphi_{jk}(2t)]\\
&+N_{rik}[u_{j,i}(2t)\varphi_{jk,r}(0)+u_{j,i}(0)\varphi_{jk,r}(2t)]\\
&+a_{ij}[\varepsilon_{ij}(0)\sigma(2t)+\varepsilon_{ij}(2t)\sigma(0)]+b_{ij}[\gamma_{mn}(0)\sigma(2t)+\gamma_{mn}(2t)\sigma(0)]\\
&+c_{ijk}[\chi_{ijk}(0)\sigma(2t)+\chi_{ijk}(2t)\sigma(0)]+d_{ijk}[\varepsilon_{ij}(0)\sigma_{,k}(2t)+\varepsilon_{ij}(2t)\sigma_{,k}(0)]\\
&+e_{ijk}[\gamma_{ij}(0)\sigma_{,k}(2t)+\gamma_{ij}(2t)\sigma_{,k}(0)]-f_{mijk}[\chi_{ijk}(0)\sigma_{,m}(2t)+\chi_{ijk}(2t)\sigma_{,m}(0)]\\
&+Q_{ij}\sigma_{,i}(0)\sigma_{,j}(2t)+d_i[\sigma_{,i}(0)\sigma(2t)+\sigma_{,i}(2t)\sigma(0)]+\xi\sigma(0)\sigma(2t)\\
&-\varrho\dot{u}_i(0)\dot{u}_i(2t)-I_{kr}\dot{\varphi}_{jr}(0)\dot{\varphi}_{jk}(2t)-\varrho\kappa\dot{\sigma}(0)\dot{\sigma}(2t)\}dV.
\end{aligned}
$$

Taking into account the notations (8), (9), (10), we arrive at the desired result (11). □

REMARK. If we denote by $R(t)$ the last integral from (11)

$$
\begin{aligned}
R(t)=\frac{1}{2}\int_{B}&\{C_{ijmn}\varepsilon_{ij}(2t)\varepsilon_{mn}(0)+G_{ijmn}[\varepsilon_{ij}(2t)\gamma_{mn}(0)+\varepsilon_{ij}(0)\gamma_{mn}(2t)]\\
&+F_{mnrij}[\varepsilon_{ij}(2t)\chi_{mnr}(0)+\varepsilon_{ij}(0)\chi_{mnr}(2t)]+B_{ijmn}\gamma_{ij}(2t)\gamma_{mn}(0)\\
&+D_{ijmnr}[\gamma_{ij}(2t)\chi_{mnr}(0)+\gamma_{ij}(0)\chi_{mnr}(2t)]+A_{ijkmnr}\chi_{ijk}(2t)\chi_{mnr}(0)\\
&+P_{ki}u_{j,k}(2t)u_{j,i}(0)-M_{ik}[u_{j,i}(2t)\varphi_{jk}(0)+u_{j,i}(0)\varphi_{jk}(2t)]\\
&+N_{rik}[u_{j,i}(2t)\varphi_{jk,r}(0)+u_{j,i}(0)\varphi_{jk,r}(2t)]\\
&+a_{ij}[\varepsilon_{ij}(0)\sigma(2t)+\varepsilon_{ij}(2t)\sigma(0)]+b_{ij}[\gamma_{ij}(0)\sigma(2t)+\gamma_{ij}(2t)\sigma(0)]\\
&+c_{ijk}[\chi_{ijk}(0)\sigma(2t)+\chi_{ijk}(2t)\sigma(0)]+d_{ijk}[\varepsilon_{ij}(0)\sigma_{,k}(2t)+\varepsilon_{ij}(2t)\sigma_{,k}(0)]\\
&+e_{ijk}[\gamma_{ij}(0)\sigma_{,k}(2t)+\gamma_{ij}(2t)\sigma_{,k}(0)]+f_{mijk}[\chi_{ijk}(0)\sigma_{,m}(2t)+\chi_{ijk}(2t)\sigma_{,m}(0)]\\
&+Q_{ij}\sigma_{,i}(0)\sigma_{,j}(2t)+d_i[\sigma_{,i}(0)\sigma(2t)+\sigma_{,i}(2t)\sigma(0)]+\xi\sigma(0)\sigma(2t)\\
&-\varrho\dot{u}_i(0)\dot{u}_i(2t)-I_{kr}\dot{\varphi}_{jr}(0)\dot{\varphi}_{jk}(2t)-\varrho\kappa\dot{\sigma}(0)\dot{\sigma}(2t)\}dV.
\end{aligned}
\tag{16}
$$

then (11) may be restated, equivalently

$$
U(t)-K(t)=\frac{1}{2}[G(t+s,t-s)-G(t-s,t+s)]ds+R(t). \tag{17}
$$

The relations established in the next Theorem, together with the identity from Theorem 1 will be used for obtain the main result.

Theorem 2. *Let $P(t)$ be the function*

$$P(t) = \int_B \varrho[f_i\dot{u}_i + g_{jk}\dot{\varphi}_{jk} + L\dot{\sigma}]dV + \int_{\partial B}[t_i\dot{u}_i + \mu_{jk}\dot{\varphi}_{jk} + h\dot{\sigma}]dA. \tag{18}$$

Then we have the following relations

$$2U(t) = U(0) + K(0) + R(t) - \int_0^t \int_B \tau\dot{\sigma}^2 dV ds + \\ +\frac{1}{2}\int_0^t [G(t+s, t-s) - G(t-s, t+s) + 2P(s)]ds, \tag{19}$$

$$2K(t) = U(0) + K(0) - R(t) - \int_0^t \int_B \tau\dot{\sigma}^2 dV ds - \\ -\frac{1}{2}\int_0^t [G(t+s, t-s) - G(t-s, t+s) - 2P(s)]ds, \tag{20}$$

provided that the symmetry relations (5) *hold.*

Proof. With the aid of the constitutive equations (3) and the symmetry relations (5), we can write

$$\begin{aligned} &\tau_{ij}\dot{\varepsilon}_{ij} + \eta_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} + h_i\dot{\sigma}_{,i} - g\dot{\sigma} \\ &= \frac{1}{2}\frac{\partial}{\partial t}[C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2G_{mnij}\varepsilon_{ij}\gamma_{mn} + 2F_{mnrij}\varepsilon_{ij}\chi_{mnr} \\ &+B_{ijmn}\gamma_{ij}\gamma_{mn} + 2D_{ijmnr}\gamma_{ij}\chi_{mnr} + A_{ijkmnr}\chi_{ijk}\chi_{mnr} \\ &+P_{ki}u_{j,k}u_{j,i} - 2M_{ik}u_{j,i}\varphi_{jk} + 2N_{rik}u_{j,i}\varphi_{jk,r} \\ &+2a_{ij}\varepsilon_{ij}\sigma + 2b_{ij}\gamma_{ij}\sigma + 2c_{ijk}\chi_{ijk}\sigma + 2d_{ijk}\varepsilon_{ij}\sigma_{,k} \\ &+2e_{ijk}\gamma_{ij}\sigma_{,k} + 2f_{mijk}\chi_{ijk}\sigma_{,m} + Q_{ij}\sigma_{,i}\sigma_{,j} + 2d_i\sigma_{,k}\sigma] + \xi\sigma^2. \end{aligned} \tag{21}$$

On the other hand, in view of the equations of motion, (1), the balance of the equilibrated forces, (2) and the geometrical equations (4), it results

$$\begin{aligned} &\tau_{ij}\dot{\varepsilon}_{ij} + \eta_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} + h_i\dot{\sigma}_{,i} - g\dot{\sigma} \\ &= [(\tau_{ij} + \eta_{ij})\dot{u}_j + \mu_{ijk}\dot{\varphi}_{jk} + h_i\dot{\sigma}]_{,i} \\ &+\varrho(f_i\dot{u}_i + g_{jk}\dot{\varphi}_{jk} + L\dot{\sigma}) - \frac{1}{2}\frac{\partial}{\partial t}(\varrho\dot{u}_i\dot{u}_i + I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk} + \varrho\kappa\dot{\sigma}^2). \end{aligned} \tag{22}$$

From (22) and (23), by equalizing their right-hand sides, we are led to

$$
\begin{aligned}
&[(\tau_{ij} + \eta_{ij})\dot{u}_j + \mu_{ijk}\dot{\varphi}_{jk} + h_i\dot{\sigma}]_{,i} + \varrho(f_i\dot{u}_i + g_{jk}\dot{\varphi}_{jk} + L\dot{\sigma}) - \tau\dot{\sigma}^2 \\
&= \frac{1}{2}\frac{\partial}{\partial t}[C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2G_{mnij}\varepsilon_{ij}\gamma_{mn} + 2F_{mnrij}\varepsilon_{ij}\chi_{mnr} \\
&+B_{ijmn}\gamma_{ij}\gamma_{mn} + 2D_{ijmnr}\gamma_{ij}\chi_{mnr} + A_{ijkmnr}\chi_{ijk}\chi_{mnr} \\
&+P_{ki}u_{j,k}u_{j,i} - 2M_{ik}u_{j,i}\varphi_{jk} + 2N_{rik}u_{j,i}\varphi_{jk,r} \\
&+2a_{ij}\varepsilon_{ij}\sigma + 2b_{ij}\gamma_{ij}\sigma + 2c_{ijk}\chi_{ijk}\sigma + 2d_{ijk}\varepsilon_{ij}\sigma_{,k} \\
&+2e_{ijk}\gamma_{ij}\sigma_{,k} + 2f_{mijk}\chi_{ijk}\sigma_{,m} + Q_{ij}\sigma_{,i}\sigma_{,j} + 2d_i\sigma_{,k}\sigma] \\
&+\xi\sigma^2 - \varrho\dot{u}_i\dot{u}_i - I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk} - \varrho\kappa\dot{\sigma}^2).
\end{aligned}
\tag{23}
$$

By integrating in (24) over B, we conclude, with the aid of the divergence theorem and the notations (8), (9) and (19), that

$$K(t) + U(t) = P(t) - \int_B \tau\dot{\sigma}^2 dV. \tag{24}$$

If we integrate (25) from 0 to t, $t \in [0, \infty)$, we obtain

$$K(t) + U(t) = K(0) + U(0) + \int_0^t P(s)ds - \int_0^t \int_B \tau\dot{\sigma}^2 dV ds. \tag{25}$$

Now, by adding the relations (26) and (18) we establish the relation (20) and, at last, by subtracting (18) from (26), it follows the relation (21) and then the proof of Theorem 2 is complete. Theorem 1 and Theorem 2 form the basis of the following theorem which establishes the uniqueness of solution.

THEOREM 2. *Assume that*
(i) *the symmetry relations (5) are valid;*
(ii) ϱ *and* κ *are strictly positive;*
(iii) ξ *is strictly positive (or strictly negative);*
(iv) I_{ij} *is positive definite.*
Then the mixed problem of Elastodynamics of dipolar bodies with voids has at most one solution.

Proof. Assume to the contrary that there exist two solutions, say

$$(u_i^{(\alpha)},\ \varphi_{jk}^{(\alpha)},\ \sigma^{(\alpha)}), \alpha = 1, 2.$$

We denote their difference by $(v_i,\ \psi_{jk},\ S)$, such that

$$v_i = u_i^{(2)} - u_i^{(1)},\ \psi_{jk} = \varphi_{jk}^{(2)} - \varphi_{jk}^{(1)},\ S = \sigma^{(2)} - \sigma^{(1)}.$$

Because of the linearity of our problem, $(v_i,\ \psi_{jk}, S)$ is also a solution, but corresponds to null data. Thus, we conclude from (21) that

$$(26) \quad \int_B (\varrho\dot{v}_i\dot{v}_i + I_{kr}\dot{\psi}_{jr}\dot{\psi}_{jk} + \varrho\kappa\dot{S}^2)dV + \int_0^t \int_B \tau\dot{S}^2 dV ds = 0.$$

Based on the assumptions (i)-(iv), (27) implies that

$$(27) \qquad \dot{v}_i = 0, \dot{\psi}_{jk} = 0, \dot{S} = 0 \ on\ B \times [0, \infty),$$

$$(28) \qquad \int_0^t \int_B \tau\dot{S}^2 dV ds = 0, (0 \leq t < \infty).$$

Because of the fact that v_i and ψ_{jk} vanish initially, from (28) we deduce

$$(29) \qquad v_i = 0, \psi_{jk} = 0 \ \ on \ \ B \times [0, \infty).$$

Taking into account (29), (30), the relation (20) reduce to

$$\int_B \xi S^2 dV = 0,$$

so that, since $\xi > 0$ (or $\xi < 0$), we conclude that $S = 0$ on $B \times [0, \infty)$ and and the proof of Theorem 3. is complete. □

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