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DIRECT SUM DECOMPOSITIONS OF INDECOMPOSABLE INJECTIVE MODULES

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ABSTRACT. Matlis posed the following question in 1958: if N is a direct summand of a direct sum M of indecomposable injectives, then is N itself a direct sum of indecomposable injectives? It will be proved that the Matlis problem has an affirmative answer when M is a multiplication module, and that a weaker condition than that of M being a multiplication module can be given to the module M when M is a countable direct sum of indecomposable injectives.

0. Introduction

The universal question, raised in various categories, is whether or not a summand of unique decomposition has a unique decomposition. The uniqueness of decomposition was solved by W. Krull, O. Schmidt, R. Remak, and G. Azumaya. The general case was raised by Matlis [M58] and solved by him for injective modules over Noetherian rings. The Matlis problem was: *if an R-module M is a direct sum of indecomposable injective R-modules M_i, is every direct summand N of M also a direct sum of indecomposable injective R-modules?* It is well-known that the answer is affirmative in the following cases: (1) the number of the M_i's is finite [AF74]; (2) R is left Noetherian and M is injective [M58, P59]; (3) R is left Noetherian [M58, FW67]; (4) N is countably generated [FW67, F76, W69]; (5) M is injective [FW67]; (6) M is quasi-injective [K71]; (7) R satisfies the ascending chain condition for irreducible left ideals [Y73]; (8) M is non-singular [H83]; (9)

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end(M) is semiregular [C88]; and (10) every ideal of *end*(M) contained in $J(end(M))$ contains a non-zero idempotent [Y96].

The aim of this paper is to show that the Matlis problem has an affirmative answer when M is a multiplication R-module. Also, it will be proved that in order for the answer to be yes, a weaker condition than that of M being a multiplication R-module can be given to the module M when the index set of the i's is countable.

This adds two more special cases to the list of positive answers to the Matlis problem.

In this paper, *unless otherwise indicated, we shall not assume that our rings are commutative, but we shall always assume that every ring R has an identity element.* By *a module*, we shall always mean a unitary left R-module. For undefined terms and general information, the reader is referred to [SV72].

1. Direct Sums of Indecomposable Injective Submodules in Multiplication Modules

In this section the Matlis problem will be discussed when the given module M is a multiplication R-module. It is easy to prove the following proposition.

PROPOSITION 1.1. *Let $\{M_i\}_{i\in I}$ be a family of submodules of an R-module M. Then the following statements are equivalent:*

(*a*) $M = \sum_{i\in I} \oplus M_i$;
(*b*) *for each subset J of I,* $M = (\sum_{j\in J} M_j) \oplus (\sum_{i\in I\setminus J} M_i)$.

COROLLARY 1.2. *Let $\{M_i\}_{i\in I}$ be a family of submodules of an R-module. Then the sum $\sum_{i\in I} M_i$ is direct if and only if for each subset J of I, $\sum_{j\in J} M_j$ is direct.*

An R-module M is called a *multiplication module* provided that for each submodule N of M there exists an ideal $\mathfrak{a}$ of R such that $N = \mathfrak{a}M$.

LEMMA 1.3. *Let A, B, C be submodules of an R-module such that $A \subseteq B + C$. If $A \supseteq B$ and $A \cap C = 0$, then $A = B$.*

Proof. $A \subseteq B + [(A+B) \cap C] = B + (A \cap C) = B$. Hence $A = B$.□

A *uniform module* is a non-zero module M such that the intersection of any two non-zero submodules of M is non-zero. Compare the following result with [ST93, Theorem 5.5].

THEOREM 1.4. *Let $\{M_i\}_{i \in I}$ be a family of uniform submodules of a multiplication R-module M such that $M = \sum_{i \in I} \oplus M_i$. Then every direct summand N of M is of the form $N = \sum_{j \in J} \oplus M_j$, where J is a subset of I.*

Proof. Let N be any direct summand of M. Then there exists a submodule N' of M such that $M = N \oplus N'$.

Let

$$J = \{i \in I | M_i \cap N \neq 0\}.$$

Then clearly,

$$\sum_{j \in J} M_j \subseteq M = N + N'.$$

Since $M_i \cap N = 0$ for each $i \in I \backslash J$, it follows from [S88, Proposition 3] that

$$N = \sum_{j \in J} (M_j \cap N) + \sum_{i \in I \backslash J} (M_i \cap N) = \sum_{j \in J} (M_j \cap N) \subseteq \sum_{j \in J} M_j.$$

Let j be any element of J. Then $(M_j \cap N) \cap (M_j \cap N') = 0$ and $M_j \cap N \neq 0$. Since M_j is uniform, it follows that $M_j \cap N' = 0$. Using [S88, Proposition 3] again, we can see that $N' \subseteq \sum_{i \in I \backslash J} M_i$. Hence by Proposition 1.1

$$(\sum_{j \in J} M_j) \cap N' \subseteq (\sum_{j \in J} M_j) \cap (\sum_{i \in I \backslash J} M_i) = 0,$$

so that $(\sum_{j \in J} M_j) \cap N' = 0$.

Applying Lemma 1.3 with $A = \sum_{j \in J} M_j$, $B = N$, and $C = N'$, we get $A = B$, i.e. $N = \sum_{j \in J} M_j$. Moreover, by Corollary 1.2, the sum $\sum_{j \in J} M_j$ is direct.

It follows from Proposition 1.1 that for any subset J of I, $\sum_{j\in J} \oplus M_j$ is a direct summand of M. □

Theorem 1.4 can be restated more briefly as follows. If a multiplication R-module M is a direct sum of uniform submodules, then so is every direct summand of M.

COROLLARY 1.5. *If a multiplication R-module M is a direct sum of indecomposable injective submodules, then so is every direct summand of M.*

Proof. This follows from [H83, Lemma 4.7.15] or [GW89, Lemma 4.1]. □

2. Direct Sums of Indecomposable Injective Modules

LEMMA 2.1. *Let X, Y, Z be submodules of an R-module such that $X \oplus Y = X \oplus Z$. Then there exists an isomorphism φ from Y onto Z such that for every submodule B of Y and for every submodule C of Z, $\varphi(B) \cap C = (X \oplus B) \cap C$.*

Proof. Let $\pi : X \oplus Z \to Z$ be the canonical projection. Then for any submodule B' of $X \oplus Z$, $\pi(B') = (X \oplus B') \cap Z$. If $X \oplus Y = X \oplus Z$, then the composite map $\varphi : Y \xrightarrow{inc} X \oplus Y = X \oplus Z \xrightarrow{\pi} Z$ is an isomorphism. Now, let B be any submodule of Y and let C be any submodule of Z. Then

$$\varphi(B) \cap C = \pi(B) \cap C = (X \oplus B) \cap Z \cap C = (X \oplus B) \cap C,$$

as required. □

Compare the following theorem with Azumaya's Decomposition Theorem [AF92, Theorem 12.6].

THEOREM 2.2. *Let M be an R-module which is a direct sum of indecomposable injective submodules. Then every non-zero direct summand of M has an indecomposable injective direct summand.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of indecomposable injective submodules of an R-module M such that $M = \sum_{i\in I} \oplus M_i$. Let N be a direct summand of M. Then there is a submodule N' of M such that $M = N \oplus N'$. If N is non-zero, then there exists a finite subset J of I such that $(\sum_{j\in J} \oplus M_j) \cap N \neq 0$. Consider the family $\mathcal{F}$ of all finite subsets F of I such that $(\sum_{f\in F} \oplus M_f) \cap N \neq 0$, and consider the set $S = \{|F| | F \in \mathcal{F}\}$, where $|F|$ denotes the number of elements of F. Then S is a non-empty subset of the set $\mathbb{N}$ of natural numbers. By the well-ordering property of integers, S has the least element l. Then $l = |F_*|$ for some finite subset F_* of I with $(\sum_{f\in F_*} \oplus M_f) \cap N \neq 0$. Write $F_* = \{i_1, \cdots, i_l\}$.

Assume that $l = 1$. Then $(M_{i_1} \cap N) \cap (M_{i_1} \cap N') = 0$, $M_{i_1} \cap N \neq 0$, and M_{i_1} is indecomposable injective. So, $M_{i_1} \cap N' = 0$. Hence by [ST93, Lemma 2.10], $M_{i_1} \oplus N'$ is a direct summand of M. In fact, there exists a submodule N_1 of N such that $M = M_{i_1} \oplus N' \oplus N_1$, i.e. $N' \oplus M_{i_1} \oplus N_1 = N' \oplus N$. By Lemma 2.1, there exists an isomorphism φ from $M_{i_1} \oplus N_1$ onto N. Hence $\varphi(M_{i_1}) \oplus \varphi(N_1) = \varphi(M_{i_1} \oplus N_1) = N$ so that N has an indecomposable injective direct summand $\varphi(M_{i_1})$.

Assume now that $l \gneq 1$. Then by the minimality of l, we must have $(M_{i_1} \oplus \cdots \oplus M_{i_{l-1}}) \cap N = 0$. By [ST93, Lemma 2.10], $M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus N$ is a direct summand of M. In fact, there exists a submodule N_1' of N' such that $M = M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus N \oplus N_1'$. Write M as follows:

$$M = M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus \sum_{i\in I\setminus\{i_1,\cdots,i_{l-1}\}} \oplus M_i.$$

By Lemma 2.1, there is an isomorphism φ from $\sum_{i\in I\setminus\{i_1,\cdots,i_{l-1}\}} \oplus M_i$ onto $N \oplus N_1'$ such that $\varphi(M_{i_l}) \cap N = (M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus M_{i_l}) \cap N$. Note that

$$(\varphi(M_{i_l}) \cap N) \cap (\varphi(M_{i_l}) \cap N_1') = 0.$$

By the construction of l, $\varphi(M_{i_l}) \cap N = (M_{i_1} \oplus \cdots \oplus M_{i_l}) \cap N = (\sum_{f\in F_*} \oplus M_f) \cap N \neq 0$. $M_{i_l} \cong \varphi(M_{i_l})$ implies that $\varphi(M_{i_l})$ is indecomposable injective. Hence, as in the case of $l = 1$, we can prove that N has an indecomposable injective direct summand which is isomorphic to $\varphi(M_{i_l})$. □

COROLLARY 2.3. (*Krull and Schmidt* [AF74]) *Let M be an R-module which is a finite direct sum of indecomposable injective submodules. Then every direct summand of M is also a finite direct sum of indecomposable injective submodules.*

THEOREM 2.4. *Let M be an R-module which is a direct sum of indecomposable injective submodules. Then every finitely embedded direct summand of M is a direct sum of indecomposable injective submodules.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of indecomposable injective submodules of an R-module M such that $M = \sum_{i\in I} \oplus M_i$. Let N be any finitely embedded direct summand of M. Then there is a submodule N' of M such that $M = N \oplus N'$.

If the index set I is finite, then by Corollary 2.3, N is a finite direct sum of indecomposable injective submodules.

Assume now that the index set I is not finite and suppose on the contrary that N is not a direct sum of indecomposable injective submodules. Then N is non-zero. By Theorem 2.2, N has an indecomposable injective direct summand N_1. There is a submodule N_1' of N such that $N = N_1 \oplus N_1'$. Clearly, N_1' is neither zero nor N.

Since $M = N \oplus N' = N_1 \oplus N_1' \oplus N'$, it follows from [SV72, Lemma 3.14] that $M = M_{i_1} \oplus N_1' \oplus N'$ for some $i_1 \in I$. Since I is not finite, $I\backslash\{i_1\}$ is non-empty. Hence by Lemma 2.1, there exists an isomorphism φ_1 from $\sum_{i\in I\backslash\{i_1\}} \oplus M_i$ onto $N_1' \oplus N'$. This implies that

$$\sum_{i\in I\backslash\{i_1\}} \oplus\varphi_1(M_i) = \varphi_1(\sum_{i\in I\backslash\{i_1\}} \oplus M_i) = N_1' \oplus N'$$

Note that each $\varphi_1(M_i)$ is indecomposable injective (because $M_i \cong \varphi_1(M_i)$) and that N_1' is non-zero. We can continue to construct non-zero submodules $N_2', N_3', \cdots$ of N_1' in the same manner as N_1' was constructed such that

$$N_1' \supsetneqq N_2' \supsetneqq N_3' \supsetneqq \cdots .$$

Therefore, we get a strictly descending chain of submodules of N:

$$N \supsetneqq N_1' \supsetneqq N_2' \supsetneqq N_3' \supsetneqq \cdots .$$

For each $i (i \geqslant 1)$, let $S_i = N_i \oplus N_{i+1} \oplus \cdots$ and consider a countable family $\{S_i\}_{i=1}^{\infty}$ of indecomposable injective submodules of N. Then this family is an inverse system. In fact, let $i_1, \cdots, i_k$ be any finite number of elements of $\mathbb{N}$ and let $m = max\{i_1, \cdots, i_k\}$. Then

$$S_m = S_{i_1} \cap \cdots \cap S_{i_k}.$$

However the intersection of all the submodules of the system is zero, so it cannot be bounded below by a non-zero submodule of N. It follows from [V68, Proposition 1] or [SV72, Proposition 3.19] that N is not finitely embedded. This contradiction shows that N is a direct sum of indecomposable injective submodules. □

COROLLARY 2.5. *If an Artinian R-module M is a direct sum of indecomposable injective submodules, then every direct summand of M is a direct sum of indecomposable injective submodules.*

Proof. This follows from the proof of Theorem 2.4 or [SV72, Theorem 3.21]. □

3. Countable Direct Sums of Indecomposable Injective Modules

If an R-module M is R-isomorphic to an R-module N, then $ann_R M = ann_R N$. Hence, the following result is obtained:

PROPOSITION 3.1. *Let M be a non-zero R-module. Then the following are equivalent:*

- (a) *For every non-zero direct summand N of M $ann_R(M/N) \nsubseteq ann_R(N)$;*
- (b) *For every non-zero direct summand N of M there is an ideal $\mathfrak{a}$ in R such that $\mathfrak{a}(M/N) = 0$ but $\mathfrak{a}N \neq 0$;*
- (c) *For every proper direct summand N of M $ann_R(N) \nsubseteq ann_R(M/N)$;*
- (d) *For every proper direct summand N of M there is an ideal $\mathfrak{a}$ in R such that $\mathfrak{a}N = 0$ but $\mathfrak{a}(M/N) \neq 0$.*

DEFINITION 3.2. A non-zero R-module satisfying either (hence all) of the statements in Proposition 3.1 is called a *weak multiplication R-module.*

If a non-zero R-module M is a multiplication R-module, then M is a weak multiplication R-module. However, the converse is not true in general. For example, for any prime p in the ring $\mathbb{Z}$ of integers the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is a weak multiplication $\mathbb{Z}$-module, but $\mathbb{Z}(p^\infty)$ is not a multiplication $\mathbb{Z}$-module.

THEOREM 3.3. *Let M be an R-module which is a countable direct sum of indecomposable injective R-submodules. If M is a weak multiplication R-module, then every direct summand of M is also a direct sum of indecomposable injective R-submodules.*

Proof. Let $M = \sum_{i=1}^{\infty} \oplus M_i$ be an R-module which is a countable direct sum of indecomposable injective R-submodules. To show that every direct summand of M is a direct sum of indecomposable injective R-submodules, suppose on the contrary that there is a direct summand N of M which is not a direct sum of indecomposable injective R-submodules.

N is non-zero. By Theorem 2.2, N has an indecomposable injective direct summand N_1. Let N', N_1' be R-submodules of M, N respectively, such that $M = N \oplus N'$, $N = N_1 \oplus N_1'$. Then $M = N_1 \oplus N_1' \oplus N'$. It follows from [SV72, Lemma 3.14] that there exists a positive integer i such that the combined mapping

$$M_i \xrightarrow{inc} M \xrightarrow{proj} N_1$$

is an R-isomorphism and

$$M = M_i \oplus N_1' \oplus N'.$$

If necessary, by changing the order of $M_1, M_2, \cdots$ in the decomposition $M = \sum_{i=1}^{\infty} \oplus M_i$ we may assume that $i = 1$. Then $M_1 \cong N_1$ and $M = M_1 \oplus N_1' \oplus N'$. By Lemma 2.1, there exists an R-isomorphism φ_1 from $\sum_{i=2}^{\infty} \oplus M_i$ onto $N_1' \oplus N'$. This implies that $\sum_{i=2}^{\infty} \oplus \varphi_1(M_i) = \varphi_1(\sum_{i=2}^{\infty} \oplus M_i) = N_1' \oplus N'$. Note that each $\varphi_1(M_i)(i = 2, 3, \cdots)$ is indecomposable injective (because $M_i \overset{\varphi_1}{\cong} \varphi_1(M_i), i = 2, 3, \cdots$) and that N_1' is non-zero.

We can continue to construct indecomposable injective submodules $N_2, N_3, \cdots$ and non-zero submodules $N_2', N_3', \cdots$ of N_1' in the same manner as N_1 and N_1' were constructed such that

$$M_1 \cong N_1, M_2 \cong N_2, \cdots ; \tag{1}$$

$$N = N_1 \oplus N_1', N_1' = N_2 \oplus N_2', \cdots . \tag{2}$$

From (2), we see that the sum $\sum_{i=1}^{\infty} N_i$ is direct. Using (1), we can prove that $\sum_{i=1}^{\infty} \oplus M_i \cong \sum_{i=1}^{\infty} \oplus N_i$. Further, $\sum_{i=1}^{\infty} \oplus N_i \subsetneq N \subsetneq M$. Now taking their annihilators, we have

$$ann_R(M) \subseteq ann_R(N) \subseteq ann_R(\sum_{i=1}^{\infty} \oplus N_i) = ann_R(M),$$

so that $ann_R(N) = ann_R(M)$. This implies that

$$ann_R(N) = ann_R(N) \cap ann_R(N') \subseteq ann_R(N') = ann_R(M/N).$$

We know that N is a proper direct summand of M. Hence, M is not a weak multiplication R-module. This contradiction shows that every direct summand of M is a direct sum of indecomposable injective R-submodules. □

Note that not every direct sum of weak multiplication R-submodules of an R-module is a weak multiplication R-module. Two examples of this are given below.

EXAMPLE 3.4. Let R be a commutative local quasi-Frobenius ring, for example, $R = \mathbf{Z}/4\mathbf{Z}$. Let M be a free R-module with rank 2 and let $\{e_1, e_2\}$ be a basis for M over R. Let $M_i = \mathrm{R}e_i$ $(i = 1, 2)$. Then $M = M_1 \oplus M_2$. Since $M_1 \cong R$ and $M_2 \cong R$, it follows that M_1 and M_2 are weak multiplication R-modules. However, M_2 is a non-zero direct summand of M and $ann_R(M/M_2) = ann_R(M_1) = ann_R(M_2)$, so that M is not a weak multiplication R-module.

EXAMPLE 3.5. For any prime p in the ring $\mathbb{Z}$ of integers the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is a weak multiplication $\mathbb{Z}$-module. However, as in Example 3.4, we can show that for any prime p in the ring $\mathbb{Z}$ of integers the $\mathbb{Z}$-module $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ is not a weak multiplication $\mathbb{Z}$-module.

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References

[AF74] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, 1974.

[AF92] ———, *Rings and Categories of Modules*, Springer-Verlag, Second Edition, 1992.

[C88] Z.-Z. Chen, *On a Problem of Matlis of Krull-Schmidt's Theorem*, Kexue Tongbao (China) **7** (1988), 491-493.

[F76] C. Faith, *Algebra II Ring theory*, Springer-Verlag, 1976.

[FW67] C. Faith and E. Walker, *Direct sum representations of injective modules*, J. Algebra **5** (1967), 203-221.

[GW89] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge Univ. Press, 1989.

[H83] M. Harada, *Factor categories with applications to direct decomposition of modules*, Marcel Dekker, 1983.

[K71] U. S. Kahlon, *Problem of Krull-Schmidt-Remak-Azumaya-Matlis*, J. Indian Math. Soc. (N.S.) **35** (1971), 255-261.

[M58] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511-528.

[P59] Z. Papp, *On algebraically closed modules*, Publ. Math. Debrecen **6** (1959), 311-327.

[S88] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50** (1988), 223-235.

[ST93] P. F. Smith and A. Tercan, *Generalizations of CS-Modules*, Communications in Algebra **21** (1993), 1809–1847.

[SV72] D. W. Sharpe and P. Vámos, *Injective Modules*, Cambridge Univ. Press, 1972.

[V68] P. Vámos, *The dual of the notion of "Finitely generated"*, J. London Math. Soc. **43** (1968), 643-646.

[W69] R. B. Warfield, Jr., *Decompositions of injective modules*, Pacific J. Math. **31** (1969), 263-276.

[Y73] K. Yamagata, *A note on a problem of Matlis*, Proc. Japan Acad. **49** (1973), 145-147.
[Y96] Hua-Ping Yu, *On Krull-Schmidt and a Problem of Matlis*, Communications in Algebra **24** (1996), 2851-2858.

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