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LI-IDEALS IN LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. We define an LI-ideal of a lattice implication algebra and show that every LI-ideal is a lattice ideal. We give an example that a lattice ideal may not be an LI-ideal, and show that every lattice ideal is an LI-ideal in a lattice H implication algebra. We discuss the relationship between filters and LI-ideals, and study how to generate an LI-ideal by a set. We construct the quotient structure by using an LI-ideal, and study the properties of LI-ideals related to implication homomorphisms.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice, Y. Xu [Xu2] proposed the concept of lattice implication algebras, and discussed their some properties. Also, in [XQ1], Y. Xu and K. Y. Qin discussed the properties of lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [XQ2] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In this paper, we define an LI-ideal of a lattice implication algebra and show that every LI-ideal is a lattice ideal. We give an example that a lattice ideal may not be an LI-ideal, and show that every lattice ideal is an LI-ideal in a lattice H implication algebra. We discuss the relationship between filters and LI-ideals, and study how to generate an LI-ideal by a set. We construct the quotient structure by using an LI-ideal, and study the properties of LI-ideals related to implication homomorphisms.

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First of all, we recall a few definitions and properties.

DEFINITION 1.1 ([Xu2]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I2) $x \rightarrow x = 1$,

(I3) $x \rightarrow y = y' \rightarrow x'$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$

A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [Xu2]):

(1) $0x = 1$, $1x = x$ and $x1 = 1$.

(2) $x' = x0$.

(3) $xy \leq (yz)(xz)$.

(4) $x \vee y = (xy)y$.

(5) $x \leq y$ implies $yz \leq xz$ and $zx \leq zy$.

(6) $x \leq (xy)y$.

In a lattice H implication algebra L, the following hold (see [XQ1]):

(7) $x(xy) = xy$.

(8) $x(yz) = (xy)(xz)$.

DEFINITION 1.2 ([XQ2]). A subset F of a lattice implication algebra L is called a *filter* of L if it satisfies:

(F1) $1 \in F$,

(F2) $x \in F$ and $xy \in F$ imply $y \in F$,

for all $x, y \in L$.

DEFINITION 1.3 ([Xu1]). Let L_1 and L_2 be lattice implication algebras. A map $f : L_1 \to L_2$ is called an *implication homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in L_1$.

Moreover, if f satisfies the following conditions:

$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$
$$f(x') = (f(x))'$$

for all $x, y \in L_1$, we say that f is a *lattice implication homomorphism.*

For an implication homomorphism $f : L_1 \to L_2$, the *kernel* of f, written $Ker(f)$, is defined as follows:

$$Ker(f) := \{x \in L_1 | f(x) = 0\}.$$

PROPOSITION 1.4 ([Xu1, Theorem 1]). *If an implication homomorphism $f : L_1 \to L_2$ satisfies $f(0) = 0$, then f is a lattice implication homomorphism.*

PROPOSITION 1.5 ([Xu1, Theorem 2]). *Let $f : L_1 \to L_2$ be an implication homomorphism of lattice implication algebras. If $Ker(f) \neq \emptyset$, then $0 \in Ker(f)$.*

2. LI-ideals

We start by defining an *LI-ideal* in a lattice implication algebra.

MAIN DEFINITION. Let L be a lattice implication algebra. An *LI-ideal* A of L is a non-empty subset of L such that

(LI1) $0 \in A$,

(LI2) $(xy)' \in A$ and $y \in A$ imply $x \in A$,

for all $x, y \in L$.

Under this definition $\{0\}$ and L are the trivial examples of LI-ideals. The following example shows that there is a proper LI-ideal in a lattice implication algebra.

EXAMPLE 2.1. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 1 as a partial ordering. Define a unary operation " $'$ " and a binary operation denoted by juxtaposition on L as follows (Tables 1 and 2, respectively):

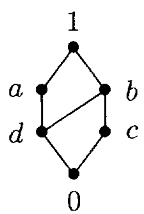

Figure 1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 1

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y,$$
$$x \wedge y := ((x'y')y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $A := \{0, c\}$ is an LI-ideal of L.

THEOREM 2.2. *Let A be an LI-ideal of a lattice implication algebra L and let $x \in A$. If $y \leq x$, then $y \in A$ for all $y \in L$.*

Proof. Since $(yx)' = 1' = 0 \in A$, it follows from (LI2) that $y \in A$. □

DEFINITION 2.3 (Burris et al. [BS, Definition 8.2]). Let L be a lattice. An *ideal* I of L is a non-empty subset of L such that

(i) $x \in I$, $y \in L$ and $y \leq x$ imply that $y \in I$,

(ii) $x, y \in I$ implies $x \vee y \in I$.

Throughout this paper we call this a *lattice ideal.*

THEOREM 2.4. *Let L be a lattice implication algebra. Every LI-ideal of L is a lattice ideal.*

Proof. Let A be an LI-ideal of L. Theorem 2.2 shows that A satisfies Definition 2.3(i). Let $x, y \in L$. Then

$$((x \vee y)y)' = (((xy)y)y)' = (xy)' \leq (x')' = x.$$

If $x, y \in A$, then $((x \vee y)y)' \in A$ and hence $x \vee y \in A$ by (LI2). Therefore A is a lattice ideal. □

REMARK 2.5. The converse of Theorem 2.4 may not be true. In fact, in Example 2.1, $A := \{0, d\}$ is a lattice ideal. But it is not an LI-ideal, since $(ad)' = d \in A$ and $a \notin A$.

THEOREM 2.6. *In a lattice H implication algebra L, every lattice ideal is an LI-ideal.*

Proof. Let A be a lattice ideal of L. Assume that $(xy)' \in A$ and $y \in A$. Note that

$$\begin{aligned} y \vee (xy)' &= (y(xy)')(xy)' \\ &= ((xy)y')(xy)' && \text{[by (I3)]} \\ &= (xy)(y')' && \text{[by (8)]} \\ &= (xy)y = x \vee y. \end{aligned}$$

It follows from Definition 2.3(ii) that $x \vee y = y \vee (xy)' \in A$. Since $x \leq x \vee y$, by Definition 2.3(i) we have $x \in A$. Clearly $0 \in A$. Hence A is an LI-ideal of L. □

For any non-empty subset A of a lattice implication algebra L, we define

$$A' := \{a' | a \in A\}.$$

It is obvious that every nonempty subset A is not a filter. Similarly the set A' for every subset A is not an LI-ideal in general. In fact the dual concept of a filter is the one of an LI-ideal in a lattice implication algebra. That is,

THEOREM 2.7. *Let A be a non-empty subset of a lattice implication algebra L. Then A is a filter of L if and only if A' is an LI-ideal of L.*

Proof. Assume that A is a filter of L. Then $1 \in A$, and so $0 = 1' \in A'$. Let $(xy)' \in A'$ and $y \in A'$ for all $x, y \in L$. Then $(xy)' = u'$ and $y = v'$ for some $u, v \in A$. Thus

$$vx' = xv' = xy = ((xy)')' = (u')' = u \in A.$$

Since A is a filter, we have $x' \in A$, and so $x = (x')' \in A'$. This proves that A' is an LI-ideal of L.

Conversely, suppose that A' is an LI-ideal of L. Then $1 \in A$ since $1' = 0 \in A'$. Let $x, y \in L$ be such that $x \in A$ and $xy \in A$. Then $x' \in A'$ and $(y'x')' = (xy)' \in A'$. As A' is an LI-ideal, it follows from (LI2) that $y' \in A'$ or $y \in A$. Hence A is a filter of L, ending the proof. □

OBSERVATION 2.8. *Suppose $\mathcal{A}$ is a non-empty family of LI-ideals of a lattice implication algebra L. Then $A = \cap\mathcal{A}$ is also an LI-ideal of L.*

Let A be a subset of a lattice implication algebra L. Then the least LI-ideal containing A is called the *LI-ideal generated* by A, written $\langle A \rangle$.

Noticing that L is clearly an LI-ideal containing A, in view of Observation 2.8, we know that the above definition $\langle A \rangle$ of A is well-defined.

The next statement gives a description of elements of $\langle A \rangle$.

THEOREM 2.9. *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L | a_n'(...(a_1'x')...) = 1 \text{ for some } a_1, ..., a_n \in A\}.$$

Proof. Denote

$$U := \{x \in L | a_n'(...(a_1'x')...) = 1 \text{ for some } a_1, ..., a_n \in A\}.$$

Since A is non-empty, there exists $a \in A$. Note that $0 \in U$ because $a'0' = a'1 = 1$. Let $(xy)' \in U$ and $y \in U$. Then there exist $a_i \in A$ $(i = 1, 2, ..., n)$ and $b_j \in A$ $(j = 1, 2, ..., m)$ such that

$$a_n'(...(a_1'((xy)')')...) = 1, \tag{*1}$$

$$b_m'(...(b_1'y')...) = 1. \tag{*2}$$

Since " $'$ " is involution, it follows from (I3) that (*1) is equivalent to

(*3) $$a'_n(...(a'_1(y'x'))...) = 1,$$

which implies from (I1) that

(*4) $$y' \leq a'_n(...(a'_1x')...).$$

Combining (*2), (*4) and (5), we get

$$\begin{aligned} 1 &= b'_m(...(b'_1y')...) \\ &\leq b'_m(...(b'_1(a'_n(...(a'_1x')...)))...), \end{aligned}$$

and hence

$$b'_m(...(b'_1(a'_n(...(a'_1x')...)))...) = 1.$$

This shows that $x \in U$. Therefore U is an LI-ideal of L containing A. Let V be any LI-ideal containing A and let $x \in U$. Then

$$a'_n(...(a'_1x')...) = 1$$

for some $a_1, ..., a_n \in A$. Thus

$$\begin{aligned} 1 &= a'_n(a'_{n-1}(...(a'_1x')...)) \\ &= a'_n((a'_{n-1}(...(a'_1x')...))')' \\ &= (a'_{n-1}(...(a'_1x')...))'a_n, \qquad \text{[by (I3)]} \end{aligned}$$

which implies that

$$((a'_{n-1}(...(a'_1x')...))'a_n)' = 1' = 0 \in V.$$

Noticing that $a_n \in A \subseteq V$ and V is an LI-ideal, we have

$$(a'_{n-1}(...(a'_1x')...))' \in V.$$

Now

$$\begin{aligned} &(a'_{n-1}(...(a'_1x')...))' \\ &= (a'_{n-1}(a'_{n-2}(...(a'_1x')...)))' \\ &= (a'_{n-1}((a'_{n-2}(...(a'_1x')...))')')' \\ &= ((a'_{n-2}(...(a'_1x')...))'a_{n-1})'. \qquad \text{[by (I3)]} \end{aligned}$$

Since $a_{n-1} \in A \subseteq V$, it follows from (LI2) that

$$(a'_{n-2}(...(a'_1x')...))' \in V.$$

Repeating the above argument we conclude that $x = (x')' \in V$. This proves that $U \subseteq V$, whence $U = \langle A \rangle$. This completes the proof. □

For any natural number n we define $n(x)y$ recursively as follows: $1(x)y = xy$ and $(n+1)(x)y = x(n(x)y)$. The following corollary is immediate from Theorem 2.9.

COROLLARY 2.10. *For any element a of a lattice implication algebra L, we have*

$$\langle a \rangle = \{x \in L | n(a')x' = 1 \text{ for some natural number } n\}.$$

Let A be an LI-ideal of a lattice implication algebra L. We define a binary relation " $\sim$ " on L as follows:

$$x \sim y \text{ if and only if } (xy)' \in A \text{ and } (yx)' \in A$$

for all $x, y \in L$.

LEMMA 2.11. *" $\sim$ " is an equivalence relation on L.*

Proof. Clearly $\sim$ is both reflexive and symmetric. To prove the transitivity, assume $x \sim y$ and $y \sim z$. Then $(xy)' \in A$, $(yx)' \in A$, $(yz)' \in A$ and $(zy)' \in A$. Since

$$((xz)'(xy)')' \leq ((xy)(xz))' \leq (yz)',$$

it follows from Theorem 2.2 that $((xz)'(xy)')' \in A$ so that $(xz)' \in A$ because $(xy)' \in A$ and A is an LI-ideal of L. Similarly, we have $(zx)' \in A$, and hence $x \sim z$. This completes the proof. □

PROPOSITION 2.12. *If $x \sim u$ and $y \sim v$, then $xy \sim uv$.*

Proof. Assume that $x \sim u$ and $y \sim v$. Then $(xu)' \in A$, $(ux)' \in A$, $(yv)' \in A$ and $(vy)' \in A$. Since

$$((xy)(xv))' \leq (yv)' \text{ and } ((xv)(xy))' \leq (vy)',$$

it follows from Theorem 2.2 that

$$((xy)(xv))' \in A \text{ and } ((xv)(xy))' \in A,$$

which means that $xy \sim xv$. Similarly we get $xv \sim uv$. By the transitivity of $\sim$, we conclude that $xy \sim uv$, ending the proof. □

We denote by A_x the equivalence class containing x and by L/A the set of all equivalence classes of L with respect to $\sim$, that is,

$$A_x := \{y \in L | x \sim y\} \text{ and } L/A := \{A_x | x \in L\}.$$

It is clear that $A_0 = A$ and $A_1 = \{y \in L | y' \in A\}$. Define binary operations "$\sqcup$", "$\sqcap$" , "$\rightrightarrows$" and unary operation "N" on L/A as follows:

$$\begin{aligned} A_x \sqcup A_y &= A_{x \vee y}, \\ A_x \sqcap A_y &= A_{x \wedge y}, \\ A_x \rightrightarrows A_y &= A_{xy}, \\ A_x^N &= A_{x'}, \end{aligned}$$

for all $A_x, A_y \in L/A$. It can be easily verified that $(L/A, \sqcup, \sqcap, A_0, A_1)$ is a bounded lattice. Moreover L/A is a lattice implication algebra, which is called the *lattice implication quotient algebra* of L by the LI-ideal A.

THEOREM 2.13. *Let $f : L_1 \rightarrow L_2$ be an implication homomorphism of lattice implication algebras. Assume that $Ker(f) \neq \emptyset$. Then $Ker(f)$ is an LI-ideal of L_1.*

Proof. Since $Ker(f) \neq \emptyset$, it follows from Proposition 1.5 that $0 \in Ker(f)$. Let $(xy)' \in Ker(f)$ and $y \in Ker(f)$. Then $f((xy)') = 0$ and $f(y) = 0$. Hence

$$0 = f((xy)') = (f(xy))' = (f(x)f(y))' = (f(x)0)' = ((f(x))')' = f(x),$$

which implies that $x \in Ker(f)$. This completes the proof. □

THEOREM 2.14. *Let $f : L \to \{0,1\}$ be an onto implication homomorphism of lattice implication algebras. Then the kernel of f is a maximal LI-ideal of L.*

Proof. Since f is onto, we know that $Ker(f) \neq \emptyset$. Hence, by Theorem 2.13, $Ker(f) = K$ is an LI-ideal of L. Suppose K is not maximal. Then there is a proper LI-ideal A containing K. Therefore there exist $x, y \in L$ such that $x \in L \setminus A$ and $y \in A \setminus K$. Thus $f(x) = f(y) = 1$, and so $f(xy) = f(x)f(y) = 1$. It follows that

$$f((xy)') = (f(xy))' = 1' = 0$$

so that $(xy)' \in K \subseteq A$. Since $y \in A$, by (LI2) we have $x \in A$, a contradiction. This completes the proof. □

THEOREM 2.15. *Let L_1 and L_2 be lattice implication algebras and let $f : L_1 \to L_2$ be an onto implication homomorphism. Then $L_1/Ker(f)$ is isomorphic to L_2.*

Proof. Let $K = Ker(f)$. Since f is onto, K is an LI-ideal of L_1 by Theorem 2.13. If $f(x) = f(y)$, then

$$f((xy)') = (f(xy))' = (f(x)f(y))' = 1' = 0.$$

Similarly $f((yx)') = 0$. Hence $(xy)' \in K$ and $(yx)' \in K$, which means that x and y belong to the same equivalent class of L_1/K. Conversely, if $x \sim y(K)$ then $(xy)' \in K$ and $(yx)' \in K$. It follows that

$$(f(x)f(y))' = f((xy)') = 0$$

and

$$(f(y)f(x))' = f((yx)') = 0.$$

Hence $f(x)f(y) = 0' = 1$ and $f(y)f(x) = 0' = 1$, whence $f(x) = f(y)$ by (I4). Therefore $\phi : L_1/K \to L_2$, $K_x \mapsto f(x)$, is a one-to-one correspondence between L_1/K and L_2. Now, for any $K_x, K_y \in L_1/K$, we have

$$\phi(K_x \rightrightarrows K_y) = \phi(K_{xy}) = f(xy) = f(x)f(y) = \phi(K_x)\phi(K_y).$$

Hence ϕ is the required isomorphism, ending the proof. □

THEOREM 2.16. *Let L_1, L_2, and L_3 be lattice implication algebras, $h : L_1 \to L_2$ an onto implication homomorphism, and $g : L_1 \to L_3$ an implication homomorphism with non-empty kernels. If $Ker(h) \subset Ker(g)$, then there is a unique implication homomorphism $f : L_2 \to L_3$ satisfying $f \circ h = g$.*

Proof. For any $y \in L_2$, there exists $x \in L_1$ such that $y = h(x)$. For the element x, put $z = g(x)$. Then we shall show that the function $f : y \mapsto z$ is well-defined and satisfies $f \circ h = g$. Let $y = h(x_1) = h(x_2)$ for $x_1, x_2 \in L_1$. Then $1 = h(x_1)h(x_2) = h(x_1x_2)$, which implies that

$$0 = 1' = (h(x_1x_2))' = h((x_1x_2)').$$

Hence $(x_1x_2)' \in Ker(h)$. Since $Ker(h) \subset Ker(g)$, we obtain

$$0 = g((x_1x_2)') = (g(x_1x_2))' = (g(x_1)g(x_2))'.$$

It follows that $g(x_1)g(x_2) = 0' = 1$ or $g(x_1) \leq g(x_2)$. Similarly, we have $g(x_2) \leq g(x_1)$. Therefore if $h(x_1) = h(x_2)$, then $g(x_1) = g(x_2)$. This shows that $f : y \mapsto z$ is well-defined, and, in above case, we have directly $g(x) = f(h(x))$, i.e., $f \circ h = g$. To show that f is an implication homomorphism, let $y_1, y_2 \in L_2$. For any $x_1, x_2 \in L_1$ such that $y_1 = h(x_1)$ and $y_2 = h(x_2)$, we have

$$\begin{aligned} f(y_1y_2) &= f(h(x_1)h(x_2)) = f(h(x_1x_2)) \\ &= g(x_1x_2) = g(x_1)g(x_2) = f(h(x_1))f(h(x_2)) \\ &= f(y_1)f(y_2). \end{aligned}$$

Hence f is an implication homomorphism. The uniqueness of f follows directly from the fact that h is an onto implication homomorphism. This completes the proof. □

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