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EXOTIC SYMPLECTIC STRUCTURES ON $S^3 \times \mathbb{R}$

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ABSTRACT. We construct exotic symplectic structures on $S^3 \times \mathbb{R}$ which is obtained by the symplectic sum of two smooth symplectic four-manifolds with exotic symplectic structures, each of which is diffeomorphic to $\mathbb{R}^4$.

1. Introduction

Let ω_0 be the standard symplectic structure on $\mathbb{R}^{2n}$ and $L \subset \mathbb{R}^{2n}$ be a closed Lagrangian submanifold. In [3], Gromov have shown the following theorem:

THEOREM (GROMOV). *As a cohomology class $[\omega_0]$ is non-zero in $H^2(\mathbb{R}^{2n}, L; \mathbb{R})$. The form ω_0 has a potential ψ on $\mathbb{R}^{2n}$, i.e., $\omega_0 = d\psi$. Furthermore, $[\psi|_L] \neq 0$ in $H^1(L;\mathbb{R})$.*

The Lagrangian submanifold L in a $2n$-dimensional symplectic manifold M is called exact(non-exact) if the restriction to the Lagrangian L of the potential is exact(non-exact). Thus, in the above Theorem, L is a non-exact Lagrangian in $\mathbb{R}^{2n}$.

Gromov have also proved that there are no exact Lagrangian subvarieties in $\mathbb{R}^{2n}$, for the standard symplectic structure. Recently, Bates and Peschke [1] have explicitly endowed a manifold M diffeomorphic to $\mathbb{R}^4$ with a symplectic form ω admitting a Lagrangian torus T such that $[\omega] = 0$ in $H^2(M, T; \mathbb{R})$. Hence T is an exact Lagrangian. By Gromov's theorem, (M, ω) does not symplectically embed in $(\mathbb{R}^4, \omega_0)$, such a structure ω is called an exotic symplectic structure on M.

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Let M_i $(i = 1, 2)$ be smooth symplectic four-manifolds diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting Lagrangian tori $(T_1^{'})^i$ $(i = 1, 2)$.

In section 2, we introduce the symplectic sum of these two manifolds and construct symplectic forms ω_M on the sum $M = M_1 \sharp_\psi M_2$ from symplectic forms on the M_i $(i = 1, 2)$. We first show that

LEMMA 2.3. *$M = M_1 \sharp_\psi M_2 \cong ((M_1 - \overset{\circ}{S}_1) - K) \cup_\varphi ((M_2 - \overset{\circ}{S}_2) - j_2(D^2)) \cong S^3 \times \mathbb{R}$, where $\overset{\circ}{S}_i$ are the interior surfaces of S_i on $(T_1^{'})^i$ with the boundaries $S_i^1 = j_i(\partial D^2)$ $(i = 1, 2)$. Hence $H^1(T_2^{'}\ ;\ \mathbb{R}) \cong H^2(M, T_2^{'}\ ;\ \mathbb{R})$ is an isomorphism, where $T_2^{'}$ is a Lagrangian surface of genus 2 in M.*

In section 3, we show the process of constructing symplectic forms ω'_M on $M = M_1 \sharp_\psi M_2 \cong S^3 \times \mathbb{R}$ from exotic symplectic forms on two smooth symplectic four-manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$.

In section 4, we get the following two Lemmas 4.1 and 4.2 from each case of manifolds (M, ω_M) and (M, ω'_M):

LEMMA 4.1. *The symplectic forms ω_M admit a non-exact Lagrangian surface $T_2^{'}$ of genus 2 in M and hence $[\omega_M] \neq 0$ in $H^2(M, T_2^{'}; \mathbb{R})$.*

LEMMA 4.2. *The symplectic forms ω'_M admit an exact Lagrangian surface T_2 of genus 2 in M and hence $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.*

By the Lemmas 4.1 and 4.2, we can get the following Theorem 4.3.

THEOREM 4.3. *The symplectic forms ω_M on the symplectic sum M of two smooth symplectic four-manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting non-exact Lagrangian tori $(T_1^{'})^i$ $(i = 1, 2)$ admit a non-exact Lagrangian surface $T_2^{'}$ of genus 2 and $[\omega_M] \neq 0$ in $H^2(M, T_2^{'}; \mathbb{R})$.*

On the other hand, the symplectic forms ω'_M on the symplectic sum M of two smooth symplectic four-manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting exact Lagrangian tori T_1^i $(i = 1, 2)$ admit an exact Lagrangian surface T_2 of genus 2 and $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$. Therefore, $(M,\ \omega'_M)$ does not symplectically diffeomorphic to $(M,\ \omega_M)$.

2. Symplectic sums

Let M_i $(i = 1,2)$ be smooth symplectic four- manifolds which are diffeomorphic to $\mathbb{R}^4$. Let $\mathbb{R}^4$ be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$ and let (r,θ), (s,ϕ) be polar coordinates on each factor. That is, if (x_1, x_2) and (y_1, y_2) are rectangular coordinates on each factor of $\mathbb{R}^2 \times \mathbb{R}^2$, then $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $y_1 = s\cos\phi$, $y_2 = s\sin\phi$. Suppose that $\mathbb{R}^4$ has a standard symplectic structure $\omega_{\mathbb{R}^4} = \sum_{i=1}^2 dx_i \wedge dy_i$.

Let $T_1 = \{ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 | \ x_1^2 + x_2^2 = \frac{\pi}{2}; \ y_1^2 + y_2^2 = \frac{\pi}{2} \} = \{ (\sqrt{\frac{\pi}{2}}\cos\theta, \sqrt{\frac{\pi}{2}}\sin\theta, \sqrt{\frac{\pi}{2}}\cos\phi, \sqrt{\frac{\pi}{2}}\sin\phi) \in \mathbb{R}^4 | \ 0 \le \theta < 2\pi, \ 0 \le \phi < 2\pi \}$. Let $j : T_1 \longrightarrow \mathbb{R}^4$ be an embedding defined by $j(r\cos\theta, \ r\sin\theta, \ s\cos\phi, \ s\sin\phi) = (r\cos\theta, \ s\cos\phi, \ r\sin\theta, \ s\sin\phi)$. Then $T_1' = j(T_1)$ is a torus defined by $x_1^2 + y_1^2 = \frac{\pi}{2}$ and $x_2^2 + y_2^2 = \frac{\pi}{2}$, and a closed Lagrangian in $\mathbb{R}^4$ with respect to $\omega_{\mathbb{R}^4}$ since $\omega_{\mathbb{R}^4}|_{T_1'} = j^*\omega_{\mathbb{R}^4}$ and

$$
\begin{aligned}
& j^*\omega_{\mathbb{R}^4}(m)(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}) \\
&= \omega_{\mathbb{R}^4}(j(m))(dj(\frac{\partial}{\partial\theta}|_m), \ dj(\frac{\partial}{\partial\phi}|_m)) \\
&= (dx_1 \wedge dy_1 \ + \ dx_2 \wedge dy_2)(j(m)) \\
&\qquad (-r\sin\alpha \cdot \frac{\partial}{\partial x_1}|_{j(m)} - r\cos\alpha \cdot \frac{\partial}{\partial y_1}|_{j(m)}, \\
&\qquad - s\sin\beta \cdot \frac{\partial}{\partial x_2}|_{j(m)} + s\cos\beta \cdot \frac{\partial}{\partial y_2}|_{j(m)}) \\
&= \ -r\sin\alpha \cdot 0 \ - \ 0 \cdot r\cos\alpha \ + \ 0 \cdot s\cos\beta \ + \ s\sin\beta \cdot 0 \\
&= 0
\end{aligned}
$$

for all $m = (r\cos\theta, r\sin\theta, s\cos\phi, s\sin\phi) \in T_1$.

By Gromov's theorem in section 1, $[\omega_{\mathbb{R}^4}] \neq 0$ in $H^2(\mathbb{R}^4, T_1'; \mathbb{R})$ and $[\sum_{i=1}^2 x_i dy_i|_{T_1'}] \neq 0$ in $H^1(T_1'; \mathbb{R})$. If we take φ_i as diffeomorphism from M_i to $\mathbb{R}^4$ such that $\varphi_i^{-1}(T_1') = (T_1')^i$ and if we set $\omega_{M_i} = \varphi_i^*\omega_{\mathbb{R}^4}$ as symplectic structures on M_i $(i = 1,2)$, then $(T_1')^i$ are closed Lagrangian tori in M_i since $\omega_{M_i}|_{(T_1')^i} = \varphi_i^*\omega_{\mathbb{R}^4}|_{(T_1')^i} = \omega_{\mathbb{R}^4}|_{T_1'} = 0$. Moreover, $(T_1')^i$ are non-exact Lagrangian tori in M_i since $[\varphi_i^*(\sum_{i=1}^2 x_i dy_i)|_{(T_1')^i}]$

$= [\sum_{i=1}^{2} x_i dy_i|_{T_1'}] \neq 0$ in $H^1((T_1')^i;\mathbb{R})$. By isomorphisms $H^1((T_1')^i;\mathbb{R}) \cong H^2(M_i,(T_1')^i;\mathbb{R})$, $[\omega_{M_i}] \neq 0$ in $H^2(M_i,(T_1')^i;\mathbb{R})$.

Let D^2 be the standard closed 2-dimensional disk of radius $\sqrt{\pi}$ with symplectic structure $\omega_{D^2} = dx_1 \wedge dy_1$. Let $h : (D^2, \partial D^2) \to (\mathbb{R}^4, T_1')$ be defined by $h(x_1, y_1) = (\frac{x_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, -\frac{x_1}{\sqrt{2}})$, and let $j_i = \varphi_i^{-1} \circ h : (D^2, \partial D^2) \to (M_i, (T_1')^i)$. Then j_i are symplectic embeddings satisfying $j_i(\partial D^2) \subset (T_1')^i$ and $(j_i(D^2) - j_i(\partial D^2)) \cap (T_1')^i = \emptyset$ $(i = 1, 2)$ since $j_i^* \omega_{M_i} = j_i^* \varphi_i^* \omega_{\mathbb{R}^4} = (\varphi_i \circ j_i)^* \omega_{\mathbb{R}^4} = h^* \omega_{\mathbb{R}^4}$ and

$$\begin{aligned} h^*\omega_{\mathbb{R}^4} &= h^*(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\ &= \frac{1}{\sqrt{2}}\, dx_1 \wedge \frac{1}{\sqrt{2}} dy_1 + \frac{1}{\sqrt{2}} dy_1 \wedge (-\frac{1}{\sqrt{2}}) dx_1 \\ &= \frac{1}{2} dx_1 \wedge dy_1 - \frac{1}{2} dy_1 \wedge dx_1 \\ &= dx_1 \wedge dy_1 \\ &= \omega_{D^2}. \end{aligned}$$

We can choose a fiber-orientation reversing bundle isomorphism $\psi : \nu_1 \to \nu_2$. We choose fiber metrics on ν_i such that ψ is isometric. Let ν_i^0 be disk bundles in ν_i $(i = 1,\ 2)$. Then there is an orientation-preserving diffeomorphism $\varphi = \iota \circ \psi : \nu_1^0 - j_1(D^2) \to \nu_2^0 - j_2(D^2)$, where the map $\iota : \nu_2^0 - \{0 - section\} \to \nu_2^0 - \{0 - section\}$ is defined by $\iota(x) = (\frac{1}{\pi||x||^2} - 1)^{1/2} x$.

Now we construct suitable models for tubular neighborhoods of the submanifolds $j_i(D^2)$ in M_i $(i = 1,\ 2)$. Let ν_i denote the $SO(2)$-vector bundles over D^2 and let ν_i^0 denote the sub-disk bundles of radius $\pi^{-1/2}$ $(i = 1,\ 2)$. Let $\pi : S \to D^2$ be the 2-sphere bundle obtained by gluing together ν_1^0 and ν_2^0 using ι defined in the above statement. We may take the sphere bundle S over D^2 as $D^2 \times S^2$. Let $i_0,\ i_\infty : D^2 \to S$ be 0-sections of ν_1^0 and ν_2^0 with images D_0 and D_∞, respectively. Thus, $\nu_1^0 = S - D_\infty$.

Considering cylindrical polar coordinates $(\theta,\ x_3)$ on $S^2 - \{(0,\ 0,\ \pm 1)\}$ where $0 \leq \theta < 2\pi$ and $-1 \leq x_3 \leq 1$, we can take a symplectic form ω_{S^2} on S^2 as the area form $\omega_{S^2} = d\theta \wedge dx_3$ induced by the Euclidean metric. Hence we may choose a closed 2-form η on the sphere bundle

$S \cong D^2 \times S^2$ over D^2 as ω_{S^2}. Then η has the following properties: $i_0^*\eta = \eta|_{i_0(D^2)} = \eta|_{D_0} = 0$ and $\eta|_{S^2} = d\theta \wedge dx_3$ is the symplectic form. By the method of Thurston[8], we can thus construct the set of symplectic forms on S as $\{\omega_t = \pi^*\omega_{D^2} + t \cdot \eta \mid 0 < t \leq t_1\}$ for some sufficiently small constant $t_1 > 0$.

On the other hand, there is a smooth orientation-preserving embedding $f : \nu_1^0 \to M_1$ (into any preassigned neighborhood of $j_1(D^2)$) with $f \circ i_0 = j_1$. And $f|_{D_0} : (D_0, \omega_{t_1}) \to (M_1, \omega_{M_1})$ is symplectic, since $i_0^*\omega_t = i_0^*\pi^*\omega_{D^2} + t \cdot i_0^*\eta = (\pi \circ i_0)^*\omega_{D^2} = \omega_{D^2}$, $f \circ i_0 = j_1$ and j_1 is symplectic. Thus we get the following Theorem 2.1 which is the same result as Gompf's.

THEOREM 2.1. *Let (ν_1^0, ω_t) , (M_1, ω_{M_1}), D_0 and $f : \nu_1^0 \to M_1$ be the same as above. Then there is a compactly supported isotopy rel D_0 from f to an embedding $\tilde{f} : \nu_1^0 \to M_1$ that is symplectic in a neighborhood of D_0.*

Proof. It can be proved by the same way as the proof of Lemma 2.1 in [2]. □

Weinstein's integral operator $I : \Omega^2(\nu_1^0) \to \Omega^1(\nu_1^0)$ is defined by $I(\eta) = \int_0^1 \pi_s^*(X_s \lrcorner \eta) ds$, where $\pi_s : \nu_1^0 \to \nu_1^0$ $(0 \leq s \leq 1)$ is a multiplication by s in this bundle structure, $X_s = \frac{d}{ds}\pi_s$ the corresponding vector field, and $\lrcorner$ denotes contraction. The key property of I is that if η satisfies $d\eta = 0$ and $i_0^*\eta = 0$, then $dI(\eta) = \eta$. Set $\varphi = I(\eta)$, and define Y_t by $Y_t \lrcorner \omega_t = -\varphi$, $0 < t \leq t_1$. Then Y_t $(0 < t \leq t_1)$ is a time-dependent vector field on ν_1^0 that vanishes on D_0 and $SO(2)$-invariant. For any $SO(2)$-invariant compact subset $K \subset \nu_1^0$ and fixed $t_0 \in (0, t_1]$, Y_t integrates to an $SO(2)$-equivariant flow $F : K \times J \to \nu_1^0$,where J is some neighborhood of t_0 in $(0, t_1]$ and $F_{t_0} = id_K$. Since $\frac{d}{dt}(F_t^*\omega_t) = dF_t^*(Y_t \lrcorner \omega_t) + F_t^*(\frac{d}{dt}\omega_t) = -F_t^* d\varphi + F_t^*\eta = -F_t^*\eta + F_t^*\eta = 0$, $F_t^*\omega_t$ is independent of t.

For $x \in \nu_1^0$, let $D(x)$ be the closed disk in the fiber $\pi^{-1}(\pi(x))$ that is bounded by the $SO(2)$-orbit of x. Let $A(x) = \int_{D(x)} \eta$ be the η-area of $D(x)$. Then $A : \nu_1^0 \to [0, 1)$ is a smooth, $SO(2)$-invariant, proper surjection that increases radially. The ω_t-area of $D(x)$ is given by $\int_{D(x)} \omega_t = \int_{D(x)}(\pi^*\omega_{D^2} + t \cdot \eta) = t \int_{D(x)} \eta = t \cdot A(x)$. Fix $x \in \nu_1^0$ and

$t_0 \in (0, t_1]$, and integrate Y_t as above to obtain a flow of $D(x)$ with $F_{t_0} = id_{D(x)}$. Let $x(t) = F_t(x)$ be the trajectory of x, with $x(t_0) = x$. Since F is $SO(2)$-equivariant, $\partial F_t D(x) = \partial D(F_t(x)) = \partial D(x(t))$. Thus the ω_t-area of $D(x(t))$ is $t \cdot A(x(t)) = \int_{D(x(t))} \omega_t = \int_{F_t D(x)} \omega_t = \int_{D(x)} F_t^* \omega_t = \int_{D(x)} F_{t_0}^* \omega_{t_0} = t_0 \cdot A(x)$, and hence $A(x(t)) = \frac{t_0}{t} \cdot A(x)$, which tells us that all flow lines of Y_t are decreasing in A. Since $A : \nu_1^0 \to [0,1)$ is proper, flow lines cannot escape from ν_1^0 as t increases, and the flow is globally defined as a map $F : \nu_1^0 \times [t_0, t_1] \to \nu_1^0$.

For any $x \in \nu_1^0$, $A(x) < 1$, so $A(F_{t_1}(x)) = A(x(t_1)) < \frac{t_0}{t_1}$. Thus, we may arrange for $F_{t_1}(\nu_1^0)$ to lie in any preassigned neighborhood V of D_0 by choosing t_0 sufficiently small. Since $F_{t_1} : (\nu_1^0, \omega_{t_0}) \to (\nu_1^0, \omega_{t_1})$ is symplectic, we get the following result with the neighborhood $V = \nu_1^0$ of D_0: For the neighborhood ν_1^0 of D_0 in (ν_1^0, ω_{t_1}), there is a t_0 with $0 < t_0 \leq t_1$ such that, for all positive $t \leq t_0$, (ν_1^0, ω_t) embeds symplectically in ν_1^0 rel D_0 . From the above fact and Theorem 2.1, we can get a symplectic embedding $\hat{f} : (\nu_1^0, \omega_t) \to (M_1, \omega_{M_1})$ with $\hat{f} \circ i_0 = j_1$, for any fixed $t \in (0, t_0]$ with t_0 suitably small, and $\hat{f}$ is isotopic rel D_0 to f.

We would like to find a similar map from a neighborhood of D_∞ in (S, ω_t) into a neighborhood of $j_2(D^2)$ in M_2. By construction, $\nu_2^0 = S - D_0$ canonically identifies the normal bundles ν_∞ and ν_0 of D_∞ and D_0 (reversing fiber-orientation). We also have isomorphisms $f_* : \nu_0 \to \nu_1$ and $\psi : \nu_1 \to \nu_2$ (the latter reversing orientation). Let $\psi'' : \nu_\infty \to \nu_2$ denote the composite of these (which preserves orientation). Then there is a smooth embedding $g : S - D_0 \to M_2$ (independent of t) with $g \circ i_\infty = j_2$ and $g_* = \psi''$ on ν_∞. Clearly, $M = M_1 \sharp_\psi M_2$ could be constructed as a smooth manifold by composing f^{-1} and g. However, we cannot perturb g to be symplectic, since we have $i_\infty^* \omega_t = \omega_{D^2} + t \cdot i_\infty^* \eta$. To remedy this, we choose a smooth map $\mu : S \to S$ that radially rescales ν_1^0, fixing a neighborhood of D_∞ and collapsing a neighborhood of D_0 onto D_0. By composing g^{-1} and μ, we may assume that g^{-1} extends to a smooth map $\lambda : N \to S$ with $\lambda(N - g(S - D_0)) \subset D_0$, where N is a neighborhood of $\overline{g(S - D_0)}$. Let $\zeta = \lambda^* \eta$. Then ζ is a closed 2-form that vanishes on $N - g(S - D_0)$, since $i_0^* \eta = 0$. And ζ can be extended over M_2 as follows:

$$\zeta = \begin{cases} \lambda^* \eta & \text{over } g(S - D_0) \\ 0 & \text{over } M_2 - g(S - D_0). \end{cases}$$

ζ is determined by g and η (so it is independent of λ and t) and $j_2^* \zeta = i_\infty^* \eta$. Let's replace ω_{M_2} by $\tilde{\omega}_{M_2} = \omega_{M_2} + t \cdot \zeta$. Since non-degeneracy is an open condition, $\tilde{\omega}_{M_2}$ will be symplectic on M_2 provided that $0 < t \leq t_0$ for t_0 sufficiently small. Furthermore, $g|_{D_\infty} : (D_\infty, \omega_t) \to (M_2, \tilde{\omega}_{M_2})$ is a symplectic embedding. Hence we can get the same result as Theorem 2.1 for the smooth embedding g, and by this result, there is a compactly supported isotopy rel D_∞ from g to $\tilde{g} : (S - D_0, \omega_t) \to (M_2, \tilde{\omega}_{M_2})$ which is symplectic on a neighborhood U_∞ of D_∞.

Now we perform the symplectic summation. Let $W = \tilde{g}(U_\infty - D_\infty)$ be a neighborhood of one end of the open manifold $(M_2 - \mathring{S}_2) - j_2(D^2)$, where $\mathring{S}_i$ are the interior surfaces of S_i on $(T_1')^i$ with the boundaries $S_i^1 = j_i(\partial D^2)$ $(i = 1, 2)$. The map $\tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \to (\nu_1^0, \omega_t)$ symplectically identifies the ends of $((M_2 - \mathring{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$ and (ν_1^0, ω_t). Let $K = \hat{f}(\nu_1^0 - U_\infty)$ and let φ be the inverse of the symplectic embedding $\hat{f} \circ \tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \to (M_1, \omega_{M_1})$. We use φ to glue together the two ends of $((M_1 - \mathring{S}_1) - K, \omega_{M_1})$ and $((M_2 - \mathring{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$. The resulting symplectic manifold is diffeomorphic to M. As in [2], we can get a unique isotopy class of symplectic forms on M as follows:

$$\omega_M = \begin{bmatrix} \omega_{M_1} \quad \text{on } M_1 - \nu_1^0 \\ \{(1-s)\omega_{M_1} + s \cdot \pi^* \omega_{D^2} \mid 0 \leq s < 1\} \quad \text{on } cl(\nu_1^0) \\ \{\tilde{\omega}_{M_2} = \omega_{M_2} + t \cdot \zeta \mid 0 < t \leq t_0\} \quad \text{on } M_2 - j_2(D^2). \end{bmatrix}$$

THEOREM 2.2. *In the above notation, we have the following results:*

(1) *The symplectic sum (M, ω_M) is a smooth symplectic four-manifold with symplectic structures ω_M,*

(2) $T_2' = (T_1')^1 \sharp (T_1')^2$ *is a non-exact Lagrangian surface of genus 2 in* M *with respect to* ω_M,

(3) $[\omega_M] \neq 0$ *in* $H^2(M, T_2'; \mathbb{R})$.

((2) *and* (3) *will be shown in* Lemma 4.1.)

LEMMA 2.3. $M = M_1 \sharp_\psi M_2 \cong ((M_1 - \overset{\circ}{S}_1) - K) \cup_\varphi ((M_2 - \overset{\circ}{S}_2) - j_2(D^2)) \cong S^3 \times \mathbb{R}$, *where* $\overset{\circ}{S}_i$ *are the interior surfaces of* S_i *on* $(T_1')^i$ *with the boundaries* $S_i^1 = j_i(\partial D^2)$ $(i = 1, 2)$. *Hence* $H^1(T_2' ; \mathbb{R}) \cong H^2(M, T_2' ; \mathbb{R})$ *is an isomorphism, where* T_2' *is a Lagrangian surface of genus 2 in* M.

Proof. We know that $M \cong ((M_1 - \overset{\circ}{S}_1) - K) \cup_\varphi ((M_2 - \overset{\circ}{S}_2) - j_2(D^2)) \cong S^3 \times (-\infty, 0) \cup_\varphi S^3 \times (0, \infty) \cong S^3 \times (-\infty, 0] \cup_\varphi S^3 \times [0, \infty)$. Since $\varphi = (\hat{f} \circ \tilde{g}^{-1})^{-1} = \tilde{g} \circ \hat{f}^{-1}$ glues together the two ends of $((M_1 - \overset{\circ}{S}_1) - K, \omega_{M_1})$ and $((M_2 - \overset{\circ}{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$, $M \cong S^3 \times \mathbb{R}$. □

3. The construction of an exotic symplectic form

In this section we would like to construct symplectic forms on $S^3 \times \mathbb{R}$ from exotic symplectic forms on two smooth symplectic manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$. In section 4 we will prove that the symplectic forms are exotic.

Let $\psi \in \Omega^1(\mathbb{R}^3)$ be such that the pull-back of ψ to the torus vanishes and $d\psi \neq 0$, and let $\chi \in \Omega^1(\mathbb{R}^3)$ be such that $\chi \wedge d\psi$ is a volume on $\mathbb{R}^3$. Let $\rho = \psi + x^4 \cdot \chi \in \Omega^1(\mathbb{R}^4)$. We define τ to be the smooth one-form on $\mathbb{R}^4$ given by

$$\tau = r^2 \cos r^2 d\theta + s^2 \cos s^2 d\phi,$$

where $\mathbb{R}^4$ may be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$ and (r, θ), (s, ϕ) are polar coordinates on each factor.

For details, we take $\psi = (p^{-1})^* i^* \tau$, $\chi = (p^{-1})^* i^* \xi$, and $\xi = *(d\tau \wedge d\phi^2)$, where S^3 is a three sphere defined by $r^2 + s^2 = \phi^2$, $i : S^3 \rightarrow \mathbb{R}^4$ the standard embedding, and $p : S^3 - \{x\} \rightarrow \mathbb{R}^3$ the stereographic projection, where x is a point in $S^3 - T_1$. Then there is an open ball B in

$\mathbb{R}^3$ containing $p(T_1)$ and an interval I about $x^4 = 0$ so that $\omega'_{M'} (= d\rho)$ is a symplectic form on a smooth symplectic four-manifold $M' (\cong B \times I)$ diffeomorphic to $\mathbb{R}^4$. We see that τ vanishes only on the torus T_1 defined by $x_1^2 + x_2^2 = r^2 = \pi/2$ and $y_1^2 + y_2^2 = s^2 = \pi/2$. T_1 is an exact Lagrangian torus in M', since $\rho \mid_{T_1} = 0$ and $\omega'_{M'} \mid_{T_1} = d\rho \mid_{T_1} = 0$. By an isomorphism $H^1(T_1 ; \mathbb{R}) \cong H^2(B \times I,\ T_1 ; \mathbb{R})$, the relative class $[\omega'_{M'}]$ vanishes in $H^2(B \times I,\ T_1; \mathbb{R})$. We call this structure $\omega'_{M'}$ an exotic symplectic structure on M'. By the same procedure as in the section 2 with $h : (D^2, \partial D^2) \to (\mathbb{R}^4, T_1)$ defined by $h(x_1, y_1) = (\frac{x_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, -\frac{x_1}{\sqrt{2}})$, we can get a unique isotopy class of symplectic forms on $M = M_1 \sharp_\psi M_2$, where $\omega'_{M_i} = d\rho_i$ are exotic symplectic forms on M_i as follows:

$$\omega'_M = \begin{cases} \omega'_{M_1} = d\rho_1 & \text{on } M_1 - \nu_1^0 \\ \{(1-s)\omega'_{M_1} + s \cdot \pi^* \omega_{D^2} \mid 0 \le s < 1\} & \text{on } cl(\nu_1^0) \\ \{\tilde{\omega}'_{M_2} = \omega'_{M_2} + t \cdot \zeta \mid 0 < t \le t_0\} & \text{on } M_2 - j_2(D^2). \end{cases}$$

THEOREM 3.1. *In the above notations, we have the following results:*

(1) *The symplectic sum $(M,\ \omega'_M)$ is a smooth symplectic four-manifold with symplectic structures ω'_M,*
(2) *$T_2 = T_1^1 \sharp T_1^2$ is an exact Lagrangian surface of genus 2 in M with respect to ω'_M,*
(3) *$[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.*

((2) *and* (3) *will be shown in* Lemma 4.2.)

We also have the following Lemma 3.2 which is similar to Lemma 2.3.

LEMMA 3.2. *$H^1(T_2; \mathbb{R}) \cong H^2(M, T_2; \mathbb{R})$ is an isomorphism, where T_2 is a Lagrangian surface of genus 2 in M.*

4. Exotic symplectic structures

Let (M, ω_M) be the smooth symplectic four-manifold in Theorem 2.2. (1).

LEMMA 4.1. *The symplectic forms ω_M admit a non-exact Lagrangian surface T_2' of genus 2 in M and hence $[\omega_M] \neq 0$ in $H^2(M, T_2'; \mathbb{R})$.*

Proof. Let $S_i^1 = j_i(\partial D^2) \cap (T_1')^i$ $(i = 1, 2)$. Let's divide the surface T_2' into 3 parts $[T_2' \cap (M_1 - \nu_1^0)] \cup [T_2' \cap cl(\nu_1^0)] \cup [T_2' \cap (M_2 - j_2(D^2))]$. In the first part, $\omega_{M_1}|_{T_2' \cap (M_1 - \nu_1^0)} = 0$, since $T_2' \cap (M_1 - \nu_1^0) \subset (T_1')^1$ and $\omega_{M_1}|_{(T_1')^1} = 0$. In the second part, $\omega_{M_1}|_{T_2' \cap cl(\nu_1^0)} = 0$, since $T_2' \cap cl(\nu_1^0)$ $= S_1^1 \subset (T_1')^1$. And $\pi^*\omega_{D^2}|_{T_2' \cap cl(\nu_1^0)} = \omega_{D^2}|_{S_1^1} = 0$. Thus $(1 - s)\omega_{M_1} + s \cdot \pi^*\omega_{D^2}|_{T_2' \cap cl(\nu_1^0)} = 0$ $(0 \leq s < 1)$. In the third part, $\omega_{M_2}|_{T_2' \cap (M_2 - j_2(D^2))}$ $= 0$, since $T_2' \cap (M_2 - j_2(D^2)) \subset (T_1')^2$ and $\omega_{M_2}|_{(T_1')^2} = 0$. Also ζ is zero on $T_2' \cap (M_2 - j_2(D^2))$, since ζ is zero on $M_2 - g(S - D_0)$ and T_2' $\cap (M_2 - j_2(D^2)) \subset M_2 - g(S - D_0)$. Thus $\tilde{\omega}_{M_2}|_{T_2' \cap (M_2 - j_2(D^2))} = 0$ and hence, T_2' is a Lagrangian surface of genus 2 in $(M = M_1 \sharp_\psi M_2, \omega_M)$.

Let's examine the exactness of the Lagrangian surface T_2' in M. $\varphi_i^*(\sum_{i=1}^2 x_i dy_i)|_{(T_1')^i} = \sum_{i=1}^2 x_i dy_i|_{T_1'} = j^*(\sum_{i=1}^2 x_i dy_i)$ can be locally written by $\frac{\pi}{2}(\sin\theta\cos\phi$ - $\cos\theta\sin\phi)d\phi$. Let S_0 be a meridian in the torus T_1' with $\theta = 0$. Then we have

$$\begin{aligned}\int_{S_0} j^*(\sum_{i=1}^{2} x_i dy_i) &= -\frac{\pi}{2}\int_0^{2\pi} \sin\phi d\phi \\ &= \frac{\pi}{2} \cdot 4[\cos\phi]_0^{\frac{\pi}{2}} \\ &\neq 0.\end{aligned}$$

Since $\int_{\varphi_1^{-1}(j(S_0))} \varphi_1^*(\sum_{i=1}^2 x_i dy_i)|_{T_2' \cap (M_1 - \nu_1^0)} = \int_{S_0} j^*(\sum_{i=1}^2 x_i dy_i) \neq 0$, $\varphi_1^*(\sum_{i=1}^2 x_i dy_i)|_{T_2' \cap (M_1 - \nu_1^0)}$ is not exact. Thus T_2' is a non-exact Lagrangian in M. By the isomorphism in Lemma 2.3, $[\omega_M] \neq 0$ in $H^2(M, T_2'; \mathbb{R})$. □

Let (M, ω_M') be the smooth symplectic four-manifold in Theorem 3.1.(1).

LEMMA 4.2. *The symplectic forms ω_M' admit an exact Lagrangian surface T_2 of genus 2 in M and hence $[\omega_M'] = 0$ in $H^2(M, T_2; \mathbb{R})$.*

Proof. By the same method shown in the first part of the proof of Lemma 4.1, we can easily see that $\omega'_M|_{T_2} = 0$ and hence T_2 is also a Lagrangian surface of genus 2 in $(M = M_1 \sharp_\psi M_2,\ \omega'_M)$.

Let's examine the exactness of the Lagrangian surface T_2 in M. $\rho_1|_{T_2 \cap (M_1 - \nu_1^0)} = 0$, since $T_2 \cap (M_1 - \nu_1^0) \subset T_1^1$ and $\rho_1|_{T_1^1} = 0$. Moreover $\pi^*(x_1 dy_1)|_{T_2 \cap cl(\nu_1^0)} = x_1 dy_1|_{S_1^1}$ is an exact form. Therefore $(1-s)\rho_1 + s \cdot \pi^*(x_1 dy_1)|_{T_2 \cap cl(\nu_1^0)}$ is exact. We know that $\tilde{\omega}_{M_2}\ |_{T_2 \cap (M_2 - j_2(D^2))} = d\rho_2|_{T_2 \cap (M_2 - j_2(D^2))}$, since ζ is zero on $T_2 \cap (M_2 - j_2(D^2)) \subset M_2 - g(S - D_0)$ and that $\rho_2|_{T_2 \cap (M_2 - j_2(D^2))} = 0$, since $T_2 \cap (M_2 - j_2(D^2)) \subset T_1^2$ and $\rho_2|_{T_1^2} = 0$. Thus T_2 is an exact Lagrangian in M and we conclude Lemma 4.2 by the use of Lemma 3.2. □

By the Lemmas 4.1, 4.2, we can get the following Theorem 4.3.

THEOREM 4.3. *The symplectic forms ω_M on the symplectic sum M of two smooth symplectic four-manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting non-exact Lagrangian tori $(T_1')^i$ $(i = 1, 2)$ admit a non-exact Lagrangian surface T_2' of genus 2 and $[\omega_M] \neq 0$ in $H^2(M, T_2'; \mathbb{R})$.*

On the other hand, the symplectic forms ω'_M on the symplectic sum M of two smooth symplectic four-manifolds M_i $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting exact Lagrangian tori T_1^i $(i = 1, 2)$ admit an exact Lagrangian surface T_2 of genus 2 and $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$. Therefore, $(M,\ \omega'_M)$ does not symplectically diffeomorphic to $(M,\ \omega_M)$.

In addition, we can show the exoticities of ω_M and ω'_M for any closed 2-form (not necessarily exact) η on the sphere bundle $S \cong D^2 \times S^2$ over D^2 with $i_0^* \eta = 0$ and η restricting to a symplectic form on each fiber, since $T_2' \cap ((M_2 - \mathring{S}_i) - j_2(D^2)) \subset M_2 - g(S - D_0)$.

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