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A Cross-National Study of Calculus Students' Understanding of the Function Concept

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Abstract

This paper reports results of investigating the relationship between students' performance and mathematics instructional system in understanding of the function concept. A written examination measuring calculus students' understanding of the function concept was administered to two groups of students whose educational background were different. One group consists of students who completed a pre-calculus course in Korea and the other group completed the same course in the United States.

This study investigates how students in two groups acquire an understanding of major aspects of the function concept and provided interesting insights regarding the different background and belief related to their performance.

Follow-up interviews were conducted to identify possible explanations for the different performance of the two groups in understanding the function concepts.

Results indicate that the differences came from the educational environment and individual belief.

Introduction

The function concept is an important and unifying concept in modern mathematics (Leinhardt et al., 1990) central to different branches of mathematics (Kleiner, 1989) and essential to related areas of the sciences (Seldon, 1992). Additionally, a strong understanding of the concept of function is a vital part of the background of any student hoping to understand calculus (Breidenbach et al., 1992).

In the United States, curriculum reform efforts are beginning to respond to calls for change and also in Korea, there are movements of curriculum reform calling for providing students with the possession of insights in the context of "real world" problems.

The purpose of this paper is to provide insights regarding the teaching of function topics, the creation and the development of meaningful function curriculum in two nations. Hence, it seems worthwhile to compare the conditions of students' understanding about the function concept by observing students' performance in two groups.

Since the understanding of mathematical concepts is determined by cognitive factors, we need to construct the theoretical framework within which to investigate students' understanding, in term of action, process, object, and schema. The theoretical framework for this research of understanding of function conception was developed by Dubinsky et al (1997). To give this theoretical framework more detail, an action is a transformation of objects, which is perceived by the individual as being at least somewhat external. That is, an individual whose understanding of a transformation is limited to an action conception can carry out the transformation only by reading to eternal cues that give precise details on what steps to take. Therefore action concept is the beginning of understanding a function concept.

When an action is repeated, and the individual reflects upon it, it may be interiorized into a process. That is, an internal construction. An individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps. In contrast to an action, the individual as being internal and under one's control.

When an individual reflects on operations applied to a process, becomes aware of the process as a totality, realizes that transformations (whether they are actions or process) can act on it, and is able to actually construct such transformation, then he or she is thinking of this process as an object. In this case, we say that the process has *been encapsulated to an object.*

In the course of performing of an action or process on an object, it is often necessary to de-encapsulate *object* back to the process from which it came in order to use its properties in manipulating it.

Once constructed, objects and process can be interconnected in various ways : two or more processes may be coordinated by linking them (through composition or in other ways) ; processes and objects are related by virtue of the fact that the former act on the latter. A collection of processes and objects can be organized in a structured manner to from a *schema.* Schema themselves can be treated as objects and included in the organization of "high level" schema. When this happens, we say that the schema has been *thematized* to an object.

This study was designed to guide classroom teachers and curriculum developers from two countries by providing various points of view on what factors in students' mathematics background contribute to existing differences among two groups relative to their understanding of major aspects of the function concept. More specifically, this research describes the students' abilities to :

- Interpret graphical meaning for specific point and intervals of the domain of a function;
- Recognize functions, non-function and general function types;
- Interpret and understand function notation, and
- Characterize the relationship between a function and an equation;
- Construct functions using formulas and other functions;
- Characterize "real world" function relationships using function notation;
- Operate with a particular type of function representation, such as a formula, a table, or a graph;
- Move between different representations of the same function;

- Represent and interpret covariant aspects of the function situation (i.e., recognize and characterize how change in one variable affects changes in another);
- Interpret "static" and "dynamic" functional information (i.e., interpret graphs representing position and rate of change);
- Conceptualize a function both as a process and as an object.

Methods

This study was mainly conducted by qualitative method consists of written examination, interview, and observations to get deep insights into "what do students really know?".

The subjects for this study were selected from first-semester calculus course students in a state university in the southwest area of the United States.

Among them, Group A consists of 15 Korean students completing pre-calculus courses in Korea. Group B contained 20 the United States students including 3 high school students and 17 university level students.

The 28 items written examination was administered to each group to acquire insights about different aspects of students' understanding of the function concept. The 28 item test was selected from the tests (Carlson, 98) that had categorized students' function conception as action, process, object and scheme, and appear in the Appendix A to this paper. Follow-up interviews were conducted with eight students, four from each of two groups. For administration purpose the students were numbered S1 to S4 for Korean students and S5 to S8 for the United States students.

To gain any significant insight into students' understanding of the function concepts and performances related to their educational backgrounds and beliefs, interview subjects within each group who performed at various levels on the written examination were deleted. The careful examination of some written items whose responses were various among students provided guidance for developing interview questions and for conducting individual interviews.

Results

Because of the large amount of data collected, details are presented only for selected examination items. Rubric scores for some items are in Appendix B.

For the conceptual view of the function, most of two groups were unable to give an accurate definition. Some of students (S2, S7, S8) had a pointwise view of function definition and others (S3, S4, S6) viewed function as the algebraic expression that represent a graph. When asked to state, "what is a function?" S1 student said, "kind of how x and y are related to", and S5 students responded, " it's an equation with y's and x's". These responses suggest that students mentioned only inaccurate fragments of function definition and they were unable to view a function as a covariant aspect. Hence most of students' understanding of function conception is limited to an action and not encapsulated into a process.

Most students of group A were capable of interpreting the graphical representation of the function concept for item 1 (Appendix A) and were able to manipulate the algebraic function expressions in item 2 and 5. However, each of group B students did not have the ability to distinguish the difference between "solve $g(x) = e$" and "find a root of g". In this case, we can say that all students of group B don not completely construct a *process* concept.

Analysis of interview for item 2 results reveal that although all interview subjects provided a correct justification to their correct response, the justification provided insights into how students think about the $f(x + a)$. They did not view the expression inside the parentheses (in a function statement) as the input which is processed by the function to produce out. Instead, they appear to view the evaluation of a function as nothing more than a process of algorithmically carrying out a sequence of steps. So we can say that they do not fully encapsulate *object* of a function concept, since they can not manipulate input variables.

When asked to express the capacity to move between different representations of the same function and to characterize "real world" through the function conception, the interview responses for students in group A provided that they do not construct a scheme of function conception.

How do students use the strategies (heuristic) related to the function abilities and conceptual views for solving non-routine problem? In consideration of this question, we consider some responses and ideas of students regarding test item 4, 10 and 11. On items 4 and 10, only 8 out of 35 students gave meaningful written responses.

Four (S4, S5, S7, S8) of the eight interview subjects indicated that the cars collided at $t = 1$ hr, since their "paths" are intersecting. The only one subject (S1) gave a correct answer and justification. The remaining three subject had a correct answer with no explanation for item 4.

Follow-up interviews were conducted with students S1, S2 (Korean high-performing students) and S5, S6 (the United States students with high grade A^+ in mathematics) to acquire more information regarding the students' ideas on item 4 and 10.

Interview transcripts (Group A)

Students 1

Interviewer: How did you determine that car A isahead of car B at $t = 1$ hr?

Student 1: The distance of car A moving for 1 hour is equal to the area surrounded by curve of car A, $t = 1$ and $v = 0$. Since the area under car A's graph is greater than the area under car B's graph.

Interviewer: You said that

$$F\frac{(x+y)}{2} < \frac{(F(x)+F(y)}{2}$$

is not always true when $F(x)$ is a quadratic function.

Student 1: $F(\frac{x+y}{2})$ is the function value of F at $x = \frac{x+y}{2}$.

And $\frac{F(x)+F(y)}{2}$ is the y-coordinate of intersection point given by line

$x = \frac{x+y}{2}$ and another line linked by the concavity of the curve $y = F(x)$.

Student 2

Interviewer: Explain how you sketched the graph. (item 6)

Student 2: I knew that it changed different for the bottom part because its circular and the top part has straight walls.

Interviewer: How does that affect the graph?

Student 2: Higher slope in the beginning, because the height would be changing more quickly, then in the middle the height would not change as much as it would at the bottom.

Interviewer: Good explanation. Can you say the slope of the straight line?

Student 2: Just like the curve.

Interview Transcripts (Group B)

Student 5

Interviewer: Why do you say car A and car B meet $t = 1$ hr?

Student 5: I did not pay attention. I thought that y-value is the distance.

Interviewer: Consider their speeds again. Which car is going faster for 1 hr?

Student 5: Car A. But I could not calculate it algebraically.

Interviewer: Can you attempt to rethink on the quadratic inequality? Why didn't you solve it?

Students 5: I can solve the inequality function formula by inputting numbers, but I can not solve it in general form.

Student 6

Interviewer: Can you explain your solution (item 6)?

Student 6: I tried to solve for h as a piecewise function.

Interviewer: How would you represent this graphically?

Student 6: It would be a straight line.

Written examination and individual interviews reported that though group A (Korean students) performed more or less higher than group B (United States students) on conceptual items 1, 2, 3, 5, 9 and on non-routine and challenging problems, they had a narrow view in interpreting "real world" problem by function concept. Their narrow view of functions was demonstrated by the fact that they thought any function could be defined by a single formula, all functions must be continuous and had difficulty in understanding the role of functional relationships with real world problems. However, group B had various perspectives of real world situations and they were more efficient to develop their mathematical idea in confronting problem-solving situations related to function conception.

Students' Backgrounds and Belief

There were also some high-performing students' interviews in two groups that provided interesting insights regarding the background and belief related to their successful performance in mathematics.

The interview transcription with high-performing students in group A revealed that they had many opportunities to struggle with complex tasks needed for high performing thinking skill to solve it. They also responded in a self-confident manner in solving difficult function problems because they felt to be trained for a long time in understanding function concepts and function problem solving attempt.

Learning function topics was presented earlier in Korea, Korean students would have given more opportunities to treat problems in great depth, and this might be able to learn more mathematics than their United States counterpart did.

One of major differences of the two groups regard to their function conception can be attributed to differences in the two educational systems. To pass the entrance examination of university in Korea is much more competitive than in the United States. The Korean curriculum challenged students' while completing high school. In contrast, the United States high-performing students said they studied mathematics with their own willingness. Most of them indicated that they felt very comfortable in following their school mathematics teacher who gave them appropriate questions and explanations to understand mathematical concepts rather than posing problems to solve.

Conclusion

Gaining an understanding of the many aspects of the function concept appears to be complex, as even high performing students of two groups possessed numerous misconceptions regarding many simple but essential aspects of function concepts,

They need to be engaged in activities which develop the vocabularies for constructing aspects of both algebraic and graphic function representation and interpret features of each representation, and the ability to use function to describe real world situations

This study offers insights that most of students, especially low-performing students in both groups, tended to develop superficial function understandings and replace understanding based on memorization.

Written questions and interviews reveal that among high-performing students in two groups, there is some difference of performance in understanding the function concept. When confronted with more demanding high-thinking problems, Korean students tended to be more perseverance, confident and efficient in using their heuristic. That were motivated by competitive personality, desire to do well, control of parents and educational system. In contrast, the United States students possessed a much broader view of functions in interpreting "real world" relationship. They exhibited their genuine interest in mathematics and in solving difficult problems, and tended to attribute their successful performance more to ability and less to effort.

Due to the small sample size, generalizations or strong claims can not be made. However, we hope that the present study could raise the awareness of curriculum developers and classroom teachers as to how to better design curriculum to impact the process of learning the function concept. Furthermore, a longitudinal study is necessary in order to document more carefully what happens in mathematics classrooms both in the United States and in Korea

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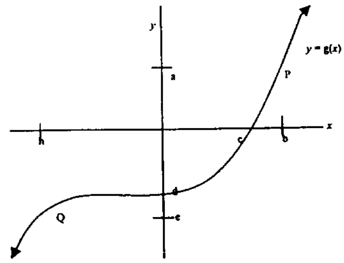
Appendix A

1. Given the following graph of the function



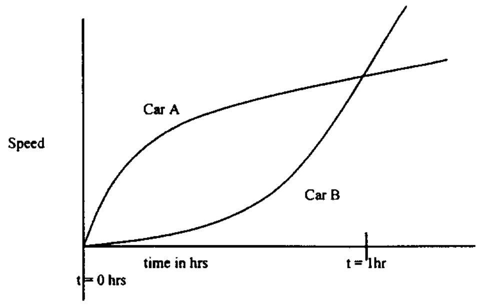
a) Evaluate g(b)
b) Evaluate g(0)
c) Solve g(0)
d) Solve g(x) = e
e) Find a root of g

2. a) Given f(x) = 2x + 3, what is the relationship between f (x + 1) and f (x) + 2 for the given function? Explain.
   b) Find k so that g (x + 1) = g (x) + k, given that g (x) = 3x + 5. Explain.
   c) Compute h (x + a) given h (x) = 2x +3.
   d) Compute f (x + a) given f (x) = 3x2 + 2x 4

3. If possible, describe the following situations using a function. If not, explain why.
   a) The string, ABCDEFG
   b) The club members dues status.

Name	Owed
Sue	$17
John	$6
Sam	$27
Bill	$0
Iris	$6
Eve	$12
Henry	$14
Louis	$6
Jane	$12

4. The given graph represents speed vs. time for two cars (Assume the cars start from the same position and are traveling in the same direction.)



a) State the relationship between the position of car A and car B at $t=1$ hr.: Explain.

b) State the relationship between the speed of car A and car B at $t=1$ hr.: Explain.

c) State the relationship between the acceleration of car A and car B at t =1 hr.: Explain.

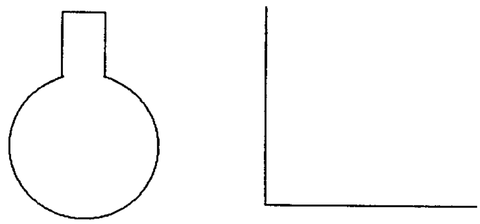
d) What is the relative position of the two cars during the time interval between $t=$ .75 hr. and $t=1$ hr.? (i.e. is one car pulling away from the other?) Explain.

5. The table on the left represents specific values of the function f(x) = x3 + 2x. Fill in the table on the right, which represents the function g(x) = x3 + 3x2 + 2x + 1.

x	y
0	0
1	0
2	0
3	6
	24

x	y
0	
1	
2	
3	
4	

6. Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water thats in the bottle.



7. a) What is a function?

b) Describe the different ways a function can be represented.

c) What is the value of studying functions?

8. Assume F (x) is any quadratic function.

a) True or False: $F(x+y/2) < F(x) + F(y)/2$

b) Justify your answer.

9. a) Find the equation of the line(s) through the point (a, a^2) that intersects the graph of $y=x^2$ exactly once.

b) Explain your solution.

10. a) Tome sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount and then again by the same amount and then again by the same amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? EXPLAIN.

b) Newt, the science nerd, then comes along and puts wheels on the bottom of the ladder. He connects them to a motor so that the bottom rolls away at a constant, but very slow, speed. Does the top of the ladder move down at a constant speed, or does it speed up or does it slow down? EXPLAIN.

c) Draw a graph which represents the relationship between the horizontal and vertical position of a ladder as it slides down a wall, starting at a vertical position and finally resting on the ground. EXPLAIN.

Appendix B

Quantitative results
Written Exam

Question No.	Group A Mean Score	GroupB Mean Score	A>B
1a)b)c)d)e)	3.29	2.11	yes
2a	2.73	2.71	no
2b	2.82	1.81	yes
3a	3.21	2.01	yes
3b	3.76	3.01	no
4a	2.21	2.54	no
6	2.96	2.01	yes
9a	2.77	2.02	no
9b	2.71	2.63	no
9c	2.68	1.67	yes
10a	3.33	2.06	yes
10b	2.85	1.78	yes
12a	2.41	2.32	no

1. Five-point rubrics were written for each exam questions.
2. The difference between the mean of group A and B is significant at $\alpha = 0.05$

함수 개념의 이해에 대한 비교 연구

윤 석 임

본 논문은 한국과 미국의 서로 다른 교육체제와 교육환경에서 고등학교 과정을 마치고 미국 남서부에 있는 한 주립대학의 첫 학기 calculus 과목을 이수하고 있는 학생들의 함수 개념에 관한 이해도를 비교 조사한 결과를 다룬다. 또한 미적분학 강의에 선행되는 함수 개념에 대하여 학생들이 실제 알고 있는 것은 무엇이고 또 두 그룹간에 함수의 이해와 성취도의 차이는 무엇에 기인하는가를 조사함으로써 두 나라의 수학교사와 교육과정 개발 담당자에게 하나의 관점을 제시하고자 한다.

함수의 개념에 관한 학생들의 다양한 인지적인 반응은 Dunbinsky(1997)의 이론을 통하여 분석하였고, 두 그룹의 우수한 학생들 간의 함수와 관련된 문제해결 능력의 성취도 차이는 지필 검사와 학생들과의 수 차례에 걸친 면담을 통하여 이루어졌다.

두 그룹의 공통점은 문제풀이 과정에서 높은 성취도를 보인 학생이라도 함수의 정의, 다양한 표현방법 및 관계 등의 개념적인 인지도에서는 정확하게 이해하지 못한다는 것이고, 서로 다른 점은 어려운 문제 풀이 과정에서 한국학생들이 미국학생보다 자신감과 지구력을 갖고 적극성을 보이고 있다는 것이다. 이는 학생 개인이 갖고있는 강력한 의지와 두 나라 사이의 다른 교육체제와 교육환경에 기인함을 지적하고자 한다.