East Asian Math. J. **14** (1998), No. 1, pp 205–217

SOME NONEXISTENCE THEOREMS OF FINITE TYPE CLOSED CURVES ON THE PSEUDO-HYPERBOLIC SPACE $H^4(-c^2)$

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ABSTRACT We obtain some theorems on nonexistence of certain finite type closed curves on the pseudo-hyperbolic space $H^4(-c^2)$ in the Minkowski spacetime E_1^5.

1. Introduction

First, we will survey briefly the fundamental concepts and properties in the pseudo-Riemannian geometry. We refer mainly to O'Neill([9]) and Chen([3],[4]). For the general concepts in the Riemannian geometry, refer to the book of Kobayashi and Nomizu([8]).

Let M be a C^∞-class differentiable manifold of dimension n and g a C^∞-class differentiable symmetric nondegenerate tensor field of type $(0,2)$ on M. The pseudo-Riemannian metric g_p at every point p of M defines the scalar product on the tangent space $T_p(M)$ of M at p. The index of g_p is not necessarily constant in general. If the index of g_p is constant $t(0 \leq t \leq n)$ on M, then we call g a *pseudo-Riemannian metric of signature* $(t, n-t)$. And a C^∞-class differentiable manifold (M, g) furnished with a pseudo-Riemannian metric g is called a *pseudo-Riemannian manifold.* A pseudo-Riemannian manifold of signature $(0, n)$ means a Riemannian manifold. Let v be a tangent vector to a pseudo-Riemannian manifold M with a pseudo-Riemannian metric g. Then v is said to be

$$\begin{aligned} &\textit{spacelike} \quad \text{if } g(v,v) > 0 \text{ or } v = 0, \\ &\textit{lightlike} \quad \text{if } g(v,v) = 0 \text{ and } v \neq 0, \\ &\textit{timelike} \quad \text{if } g(v,v) < 0 \end{aligned}$$

Received April 27, 1998.

The simplest example of pseudo-Riemannian manifold is a pseudo-Euclidean space defined as follows;

Let $(x^1, x^2, \cdots, x^m)$ be a point in the set R^m of all ordered m-tuples of real numbers. For each $t (0 \leq t \leq m)$, we define a scalar product g_0 on $T_p(R^m)$ at the point p of R^m by

$$g_0(v_p, w_p) = -\sum_{i=1}^{t} v^i w^i + \sum_{i=t+1}^{m} v^i w^i,$$

where $v_p = \sum_{i=1}^{m} v^i \partial/\partial x^i$ and $w_p = \sum_{i=1}^{m} w^i \partial/\partial x^i$. E_t^m denotes a R^m with a canonical pseudo-Riemannian metric g_0. In this case, g_0 is called a *pseudo-Euclidean metric* of signature $(t, m-t)$ and E_t^m is called a *pseudo-Euclidean space* of signature $(t, m-t)$. In particular, E_1^m is called a *Minkowski spacetime.*

Now, let $x : M \to E_t^m$ be an isometric immersion of a pseudo-Riemannian manifold M of dimension n into an m-dimensional pseudo-Euclidean space E_t^m of signature $(t, m-t)$. If x has the spectral decomposition as follows:

$$x = x_0 + \sum_{i=1}^{k} x_i : \Delta x_i = \lambda_i x_i,\ x_i \neq 0,\ \lambda_i \neq \lambda_j (i \neq j),$$

then M is called to be of *k-type submanifold* and x *k-type immersion*, where Δ is the Laplace operator on M. If one of λ_i, $i = 1, 2, \cdots, k$ is zero, then submanifold M or immersion x is said to be of *null k-type.*

The following theorem presents a necessary condition for M to be of k-type submanifold.

THEOREM A[2,3]). *Let M be a pseudo-Riemannian submanifold of E_t^m and H the mean curvature vector field of M. If M is of k-type, then there is a polynomial $P(X)$ of degree k such that $P(\Delta)H = 0$.*

If M is compact, the converse is also satisfied. That is, THEOREM B[3]. *Let M be a compact pseudo-Riemannian submanifold isometrically immersed in a pseudo-Euclidean space E_t^m. If there exists a nontrivial polynomial $P(X)$ such that $P(\Delta)H = 0$, then M is of finite type.*

REMARK. Finite type curves in a Euclidean space were investigated in [1,2,5,6] *etc.*

From now on, we will $<,>$ instead of a pseudo-Euclidean metric g_0. And we denote by $H_t^m(-c^2) = \{p \in E_{t+1}^{m+1} | <p,p> = -c^2\}$. In this case, it is called the *pseudo-hyperbolic space* of radius $c > 0$ and center o in E_{t+1}^{m+1}. For a vector $a_0 = (a_1, a_2, \cdots, a_t, \cdots, a_m)$ in E_t^m,

$$\bar{a}_0 = (-a_1, -a_2, \cdots, -a_t, a_{t+1}, a_{t+2}, \cdots, a_m)$$

is called the *conjugate vector* of a_0.

In [7] and [10], the authors proved the following

THEOREM C. *Only 1-type closed curve $\gamma(s)$ on $H_t^m(-c^2)$ is an intersection of $H_t^m(-c^2)$ and a 2-plane P lying in Π_{a_0}, where P is determined by two spacelike vectors and Π_{a_0} denotes a hyperplane through a_0 which is orthogonal to the conjugate vector $\bar{a}_0$ in the sense of Euclidean scalar product.*

Ishikawa([7]) also proved some nonexistence theorems concerning finite type closed curves on a pseudo-hyperbolic space $H^2(-c^2)$. For instance,

THEOREM D. *There exists neither 2-type closed curves nor 3-type closed curves on $H^2(-c^2)$.*

The purpose of this article is to prove some nonexistence theorems for a higher k-type ($k \geq 3$) closed curves on $H^4(-c^2)$.

2. Preliminaries

Every closed curve $\gamma : [0, 2\pi r] \to E_t^m$ of the length $2\pi r$ in E_t^m may be regarded as an isometric immersion of a circle of radius r into E_t^m. We use the arc length s as a parameter of γ. Then the Laplacian Δ on the circle is given by $\Delta = -d^2/ds^2$ and the eigenvalues are $\{(l/r)^2; l = 1, 2, \cdots\}$. The corresponding eigenspace V_l is constructed by using $cos(ls/r)$ and $sin(ls/r)$. Hence, every closed curve $\gamma : [0, 2\pi r] \to E_t^m$ has the spectral decomposition

$$\gamma(s) = a_0 + \sum_{l=1}^{\infty} \{a_l \cos(ls/r) + b_l \sin(ls/r)\},$$

where a_l, b_l are some vectors in E_t^m (see [2],[5]). In particular, if γ is a k-type closed curve of the length 2π on $H_t^m(-c^2)$, then γ can be expressed as

$$\gamma(s) = a_0 + \sum_{i=1}^{k} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}, \tag{2.1}$$

where a_i or b_i is a nonzero vector in E_t^m for each $i = 1, 2, \cdots, k$, p_i are the positive integers with $p_1 < p_2 < \cdots < p_k$ and s is the arc length parameter of γ. Because of $\gamma(s)$ being on $H_t^m(-c^2)$ and a_0 the center of mass of γ, a_0 is a timelike vector in E_{t+1}^{m+1} (see [7]). Furthermore, from $< \gamma(s),\ \gamma(s) >= -c^2$, we have the following

$$2 < a_0, a_0 > + 2c^2 + \sum_{i=1}^{k} D_{ii} = 0, \tag{2.2}$$

$$\sum_{p_i = l} M_i + \sum_{2p_i = l} A_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} A_{ij} + 2 \sum_{\substack{p_i - p_j = l \\ i > j}} D_{ij} = 0, \tag{2.3}$$

$$\sum_{p_i = l} \bar{M}_i + \sum_{2p_i = l} \bar{A}_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} \bar{A}_{ij} - 2 \sum_{\substack{p_i - p_j = l \\ i > j}} \bar{D}_{ij} = 0 \tag{2.4}$$

for each $l \in \{p_i,\ 2p_i,\ p_i + p_j,\ p_i - p_j\ ;\ 1 \leq j < i \leq k\}$, where

$$M_i = 4 < a_0, a_i >, \qquad \bar{M}_i = 4 < a_0, b_i >,$$
$$A_{ij} =< a_i, a_j > - < b_i, b_j >, \quad \bar{A}_{ij} =< a_i, b_j > + < b_i, a_j >,$$
$$D_{ij} =< a_i, a_j > + < b_i, b_j >, \quad \bar{D}_{ij} =< a_i, b_j > - < b_i, a_j > .$$

From now on, we call the real numbers M_i and $\bar{M}_i$ (resp. A_{ii} and $\bar{A}_{ii}$, A_{ij} and $\bar{A}_{ij}$, or D_{ij} and $\bar{D}_{ij}$) to be *corresponding to the integer* p_i(resp. $2p_i$, $p_i + p_j$, or $p_i - p_j$). Since s is the arc length parameter of $\gamma(s)$, we have

$$2 = \sum_{i=1}^{k} p_i^2 D_{ii}, \tag{2.5}$$

$$\sum_{2p_i=l} p_i^2 A_{ii} + 2 \sum_{\substack{p_i+p_j=l\\ i>j}} p_i p_j A_{ij} - 2 \sum_{\substack{p_i-p_j=l\\ i>j}} p_i p_j D_{ij} = 0, \tag{2.6}$$

$$\sum_{2p_i=l} p_i^2 \bar{A}_{ii} + 2 \sum_{\substack{p_i+p_j=l\\ i>j}} p_i p_j \bar{A}_{ij} + 2 \sum_{\substack{p_i-p_j=l\\ i>j}} p_i p_j \bar{D}_{ij} = 0. \tag{2.7}$$

Moreover, if $< \gamma^{(r)}(s),\ \gamma^{(r)}(s) >$ is constant $(r = 1, 2, \cdots)$, then we have

$$\sum_{2p_i=l} p_i^{2r} A_{ii} + 2 \sum_{\substack{p_i+p_j=l\\ i>j}} (p_i p_j)^r A_{ij} + (-1)^r 2 \sum_{\substack{p_i-p_j=l\\ i>j}} (p_i p_j)^r D_{ij} = 0, \tag{2.8}$$

$$\sum_{2p_i=l} p_i^{2r} \bar{A}_{ii} + 2 \sum_{\substack{p_i+p_j=l\\ i>j}} (p_i p_j)^r \bar{A}_{ij} - (-1)^r 2 \sum_{\substack{p_i-p_j=l\\ i>j}} (p_i p_j)^r \bar{D}_{ij} = 0. \tag{2.9}$$

Next, let γ be a k-type closed curve on $H_t^m(-c^2)$ given in (2.1). Divide the set $\mathfrak{A} = \{p_i,\ 2p_i,\ p_i + p_j,\ p_i - p_j\ ,\ 1 \leq j < i \leq k\}$ as the union of the subsets as follows:

$$\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_N, \tag{2.10}$$

where all elements in each subset $\mathfrak{A}_n (n = 1, 2, \cdots, N)$ are equal to each other and if $n_1 \neq n_2$, then every element in $\mathfrak{A}_{n_1}$ is not equal to any element in $\mathfrak{A}_{n_2}$.

3. Main results

Let γ be a closed k-type curve on $H_t^m(-c^2)$ in E_{t+1}^{m+1}. Then γ is expressed as $\gamma(s) = a_0 + \sum_{i=1}^{k} \{a_i \cos(p_i s/r) + b_i \sin(p_i s/r)\}$, where a_i or b_i is a nonzero vector in E_{t+1}^{m+1} for each $i = 1, 2, \cdots, k$ and p_i are the positive integers satisfying $p_1 < p_2 < \cdots < p_k$. Here s is the arc length parameter of γ and the length of γ is $2\pi r$. Therefore every k-type closed curve $\gamma(s)$ of the length 2π may be described as

$$\gamma(s) = a_0 + \sum_{i=1}^{k} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}. \tag{3.1}$$

We prove our results for $r = 1$, because the proof for case $r \neq 1$ is same as one for case of $r = 1$.

LEMMA 3.1[7]. *(1) If $< \gamma^{(r)}(s), \gamma^{(r)}(s) >$ is constant ($r = 1, 2, \cdots, l$) and the number of members in $\mathfrak{A}_n$ is less than or equal to $l+1$, then M_i and $\bar{M}_i$ (resp. A_{ii} and $\bar{A}_{ii}$, A_{ij} and $\bar{A}_{ij}$, or D_{ij} and $\bar{D}_{ij}$) of corresponding to the integer p_i (resp. $2p_i$, $p_i + p_j$, or $p_i - p_j$) in $\mathfrak{A}_n$ vanish.*

(2) In particular, for every k-type closed curve $\gamma(s)$ on $H_t^m(-c^2)$ in E_{t+1}^{m+1}, we have

$$A_{kk} = \bar{A}_{kk} = 0,$$
$$A_{k(k-1)} = \bar{A}_{k(k-1)} = 0,$$
$$A_{(k-1)(k-1)} = \bar{A}_{(k-1)(k-1)} = 0.$$

Now, let $\gamma(s)$ be a k-type closed curve on $H^4(-c^2)$ as (3.1). Then we can obtain the following three lemmas.

LEMMA 3.2. *If $\gamma(s)$ satisfies the following conditions*

$$M_k = \bar{M}_k = M_{k-1} = \bar{M}_{k-1} = 0 \text{ and } D_{k(k-1)} = \bar{D}_{k(k-1)} = 0, \tag{3.2}$$

then $a_{k-1}, b_{k-1}, a_k, b_k$ are spacelike vectors and $\{a_0, a_{k-1}, b_{k-1}, a_k, b_k\}$ forms a basis of E_1^5.

Proof. Since a_0 is a timelike vector in E_1^5 and $< a_0, a_k >=< a_0, b_k > = < a_0, a_{k-1} >=< a_0, b_{k-1} >= 0$, a_{k-1}, b_{k-1}, a_k and b_k are spacelike vectors(see [7]). Furthermore, a_{k-1}, b_{k-1}, a_k and b_k are nonzero vectors because $< a_k, a_k >=< b_k, b_k >$, $< a_{k-1}, a_{k-1} >=< b_{k-1}, b_{k-1} >$ and $\gamma(s)$ is of k-type. And, from Lemma 3.1(2) and (3.2), we have

$$< a_k, a_{k-1} >=< a_k, b_{k-1} >=< b_k, a_{k-1} >=< b_k, b_{k-1} >= 0.$$

The above equations complete the proof.

LEMMA 3.3. *Suppose that* $\{a_0, a_{k-1}, b_{k-1}, a_k, b_k\}$ *is a basis of* E_1^5 *satisfying (3.2). If a pair* $\{a_i, b_i\}(i = 1, 2, \cdots, k)$ *satisfies*

$$A_{ki} = \bar{A}_{ki} = 0 \ \textit{and} \ A_{(k-1)i} = \bar{A}_{(k-1)i} = 0,$$

then $A_{ii} = \bar{A}_{ii} = 0$ *if and only if* $M_i = \bar{M}_i = 0$.

Proof. Put $a_i = Aa_0 + Ba_{k-1} + Cb_{k-1} + Da_k + Eb_k$ and $b_i = Fa_0 + Ga_{k-1} + Hb_{k-1} + Ia_k + Jb_k$. Combining Lemma 3.1(2) and (3.2), we have $B = H$, $C = -G$, $D = J$ and $E = -I$. If $A_{ii} = \bar{A}_{ii} = 0$, then we get $A = F = 0$. Hence $M_i = 4 < a_0, a_i >= 0$ and $\bar{M}_i = 4 < a_0, b_i >= 0$.

By the same way, we can also prove the converse.

LEMMA 3.4. *Suppose that* $\{a_0, a_{k-1}, b_{k-1}, a_k, b_k\}$ *is a basis of* E_1^5 *satisfying (3.2). If a pair* $\{a_i, b_i\}(i = 1, 2, \cdots, k-2)$ *is satisfies*

$$A_{ki} = \bar{A}_{ki} = A_{(k-1)i} = \bar{A}_{(k-1)i} = 0$$

and

$$D_{ki} = \bar{D}_{ki} = D_{(k-1)i} = \bar{D}_{(k-1)i} = 0,$$

then a_i *and* b_i *are parallel to* a_0.

Proof. Put $a_i = Aa_0 + Ba_{k-1} + Cb_{k-1} + Da_k + Eb_k$ and $b_i = Fa_0 + Ga_{k-1} + Hb_{k-1} + Ia_k + Jb_k$. From Lemma 3.1(2), (3.2) and our assumptions, we have $B = C = D = E = 0$, $G = H = I = J = 0$. It follows that $a_i = Aa_0$ and $b_i = Fa_0$.

The following is an example of a 2-type closed curve in the pseudo-hyperbolic space $H^4(-c^2)$ satisfying (3.2).

EXAMPLE. The curve in E_1^5

$$\gamma(s) = \frac{1}{\sqrt{2}}\left(2,\ \cos\ s,\ \sin\ s,\ \frac{1}{2}\cos\ 2s,\ \frac{1}{2}\sin\ 2s\right)$$

is a 2-type closed curve on $H^4(-\frac{11}{8})$.

In fact, since $< \gamma(s),\ \gamma(s) >= -\frac{11}{8}$, $\gamma(s)$ is a closed curve on $H^4(-\frac{11}{8})$. And we know $< \gamma'(s),\ \gamma'(s) >= 1$. Furthermore, $\gamma(s)$ can be expressed as

$$\begin{aligned}\gamma(s) =& \frac{1}{\sqrt{2}}(2,\ 0,\ 0,\ 0,\ 0)\\ &+\left\{\frac{1}{\sqrt{2}}(0,\ 1,\ 0,\ 0,\ 0)\cos\ s + \frac{1}{\sqrt{2}}(0,\ 0,\ 1,\ 0,\ 0)\sin\ s\right\}\\ &+\left\{\frac{1}{\sqrt{2}}(0,\ 0,\ 0,\ \frac{1}{2},\ 0)\cos\ 2s + \frac{1}{\sqrt{2}}(0,\ 0,\ 0,\ 0,\ \frac{1}{2})\sin\ 2s\right\}.\end{aligned}$$

Hence $\gamma(s)$ is a 2-type closed curve satisfying the equations of (3.2).

For a higher k-type ($k \geq 3$) closed curve $\gamma(s)$ on $H^4(-c^2)$, we proved the following nonexistence theorems.

THEOREM 3.1. *There exists no 3-type closed curve $\gamma(s)$ on $H^4(-c^2)$ satisfying $M_2 = \bar{M}_2 = 0$ and $D_{32} = \bar{D}_{32} = 0$.*

Proof. We assume the existence of the 3-type closed curve

$$\gamma(s) = a_0 + a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) + b_2 \sin(p_2 s) \\ + a_3 \cos(p_3 s) + b_3 \sin(p_3 s)$$

on $H^4(-c^2)$ satisfying $M_2 = \bar{M}_2 = 0$ and $D_{32} = \bar{D}_{32} = 0$. From Lemma 3.1, we see $M_3 = \bar{M}_3 = 0$, and $A_{32} = \bar{A}_{32} = 0$. Therefore $\{a_0, a_2, b_2, a_3, b_3\}$ is a basis of E_1^5 satisfying (3.2) by Lemma 3.2 and our assumptions.

Case 1. In case of $\{p_1, p_2, p_3\} = \{p_1, 2p_1, 3p_1\}$, it follows that $\mathfrak{A} = \{p_1, p_2 - p_1, p_3 - p_2\} \cup \{2p_1, p_2, p_3 - p_1\} \cup \{p_1 + p_2, p_3\} \cup \{p_1 + p_3, 2p_2\} \cup \{p_2 + p_3\} \cup \{2p_3\}$. Applying (2.3), (2.4), (2.6) and (2.7) for the subclasses $\{p_1, p_2 - p_1, p_3 - p_2\}$ and $\{2p_1, p_2, p_3 - p_1\}$ of $\mathfrak{A}$, we obtain

$$\begin{aligned} &A_{31} = \bar{A}_{31} = 0, &&D_{31} = \bar{D}_{31} = 0, \\ &A_{21} = \bar{A}_{21} = 0, &&D_{21} = \bar{D}_{21} = 0, \\ &M_1 = \bar{M}_1 = 0. \end{aligned}$$

Substituting Lemma 3.4, we get $a_1 = b_1 = 0$. It contradicts.

Case 2. In case of $\{p_1, p_2, p_3\} = \{p_1, 2p_1, 4p_1\}$, it follows that $\mathfrak{A} = \{p_1, p_2 - p_1\} \cup \{2p_1, p_2, p_3 - p_2\} \cup \{p_1 + p_2, p_3 - p_1\} \cup \{p_3, 2p_2\} \cup \{p_1 + p_3\} \cup \{p_2 + p_3\} \cup \{2p_3\}$. From Lemma 3.1(1), we get

$$\begin{aligned} &A_{31} = \bar{A}_{31} = 0, &&D_{31} = \bar{D}_{31} = 0, \\ &A_{21} = \bar{A}_{21} = 0, &&D_{21} = \bar{D}_{21} = 0, \\ &M_1 = \bar{M}_1 = 0. \end{aligned}$$

Hence, Lemma 3.4 leads to a contradiction.

Case 3. In case of $\{p_1, p_2, p_3\} = \{p_1, 3p_1, 5p_1\}$, $\mathfrak{A} = \{p_1\} \cup \{2p_1, p_2 - p_1, p_3 - p_2\} \cup \{p_2\} \cup \{p_1 + p_2, p_3 - p_1\} \cup \{p_3\} \cup \{p_1 + p_3, 2p_2\} \cup \{p_2 + p_3\} \cup \{2p_3\}$. Applying (2.3), (2.4), (2.6) and (2.7) for the subclasses $\{2p_1, p_2 - p_1, p_3 - p_2\}$ of $\mathfrak{A}$, we obtain

$$\begin{aligned} &A_{31} = \bar{A}_{31} = 0, &&D_{31} = \bar{D}_{31} = 0, \\ &A_{21} = \bar{A}_{21} = 0, &&D_{21} = \bar{D}_{21} = 0, \\ &A_{11} = \bar{A}_{11} = 0. \end{aligned}$$

Therefore, Lemmas 3.3 and 3.4 lead to a contradiction.

Case 4. Let $\{p_1,\ p_2,\ p_3\} \neq \{p_1,\ 2p_1,\ 3p_1\}$, $\{p_1,\ 2p_1,\ 4p_1\}$ or $\{p_1,\ 3p_1,\ 5p_1\}$. In this case, each subset $\mathfrak{A}_n$ of $\mathfrak{A}$ consists of at most two elements. Hence, Lemmas 3.3 and 3.4 lead to a contradiction by Lemma 3.1(1).

Summarizing all cases, we complete the proof of this theorem.

From Theorem 3.1, we have the following corollary.

COROLLARY 3.1. *There exists no 3-type closed curve $\gamma(s)$ on $H^4(-c^2)$ satisfying $< a_0, a_2 > = < a_0, b_2 > = 0$ and $< a_3, a_2 > = < a_3, b_2 > = 0$.*

Next, we get the following

THEOREM 3.2. *There exists no 3-type closed curve with constant curvature on $H^4(-c^2)$.*

Proof. In this case, each subset $\mathfrak{A}_n$ of $\mathfrak{A}$ consists of at most three elements. Hence, by Lemmas 3.1(1) and 3.2, $\{a_0,\ a_2,\ b_2,\ a_3,\ b_3\}$ is a basis of E_1^5. This implies a contradiction by Lemma 3.4.

THEOREM 3.3. *There exists no 4-type closed curve with constant curvature on $H^4(-c^2)$ satisfying $D_{43} = \bar{D}_{43} = 0$.*

Proof. Assume the existence of the 4-type closed curve

$$\gamma(s) = a_0 + \sum_{t=1}^{4} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

satisfying the conditions. Let $\mathfrak{A}_i$ be the subclass consisting of all elements in $\mathfrak{A}$ to be equal to p_i. Then the number of elements in $\mathfrak{A}_3$ (and $\mathfrak{A}_4$) is less than or equal to three. Hence, from Lemma 3.1, we obtain $M_4 = \bar{M}_4 = 0$, $M_3 = \bar{M}_3 = 0$ and $A_{43} = \bar{A}_{43} = 0$. Thus $\{a_0, a_3, b_3, a_4, b_4\}$ is a basis of E_1^5 satisfying (3.2) by Lemma 3.2 and our assumptions.

Case 1. Let $\{p_1,\ p_2,\ p_3,\ p_4\} = \{p_1,\ 2p_1,\ 3p_1,\ 4p_1\}$, it follows that $\mathfrak{A} = \{p_1,\ p_2 - p_1,\ p_3 - p_2,\ p_4 - p_3\} \cup \{2p_1,\ p_2,\ p_3 - p_1,\ p_4 - p_2\} \cup \{p_3,\ p_1 + p_2,\ p_4 - p_1\} \cup \{p_4,\ 2p_2,\ p_1 + p_3\} \cup \{p_1 + p_4,\ p_2 + p_3\} \cup \{2p_3,\ p_2 + p_4\} \cup \{p_4 + p_3\} \cup \{2p_4\}$. Applying (2.3), (2.4), (2.8), (2.9) and Lemma 3.3 for the subclasses $\{p_2,\ 2p_1,\ p_4 - p_2,\ p_3 - p_1\}$ and $\{p_1,\ p_4 - p_3,\ p_3 - p_2,\ p_2 - p_1\}$ of $\mathfrak{A}$, we obtain

$$\begin{aligned} &A_{41} = \bar{A}_{41} = 0, \qquad D_{41} = \bar{D}_{41} = 0,\\ &A_{31} = \bar{A}_{31} = 0, \qquad D_{31} = \bar{D}_{31} = 0,\\ &M_1 = \bar{M}_1 = 0. \end{aligned}$$

Hence, Lemma 3.4 leads to a contradiction.

Case 2. In case of $\{p_1,\ p_2,\ p_3,\ p_4\} \neq \{p_1,\ 2p_1,\ 3p_1,\ 4p_1\}$, Let $\mathfrak{A}'_t$ be the subclass consisting of all elements in $\mathfrak{A}'$ to be equal to p_t. Then the number $|\mathfrak{A}'_t|$ of elements in $\mathfrak{A}'_t$ is less than or equal to three. Furthermore, $|\mathfrak{A}_{4,-1}| \leqq 3$ and $|\mathfrak{A}_{3,-1}| \leqq 3$, where $\mathfrak{A}_{4,-1}$ is the subclass of $\mathfrak{A}'$ containing $p_4 - p_1$. From Lemma 3.1(1), we get

$$\begin{aligned} &A_{41} = \bar{A}_{41} = 0, \qquad D_{41} = \bar{D}_{41} = 0,\\ &A_{31} = \bar{A}_{31} = 0, \qquad D_{31} = \bar{D}_{31} = 0,\\ &M_1 = \bar{M}_1 = 0. \end{aligned}$$

Hence, Lemmas 3.3 and 3.4 lead to a contradiction.

Summarizing above two cases, we complete the proof of this theorem.

From Theorem 3.3, we have also the following two corollaries.

COROLLARY 3.2. *There exists no 4-type closed curve with constant curvature on $H^4(-c^2)$ satisfying $< a_4, a_3 >=< a_4, b_3 >= 0$.*

COROLLARY 3.3. *There exists no 4-type closed curve $\gamma(s)$ on $H^4(-c^2)$ satisfying $< \gamma^{(l)}(s),\ \gamma^{(l)}(s) >$ is constant $(l = 2, 3)$.*

THEOREM 3.4. *There exists no 5-type closed curve* $\gamma(s)$ *on* $H^4(-c^2)$ *with* $D_{54} = \bar{D}_{54} = 0$ *satisfying* $< \gamma^{(l)}(s),\ \gamma^{(l)}(s) >$ *is constant* $(l = 2, 3)$.

Proof. Assume the existence of the 5-type closed curve

$$\gamma(s) = a_0 + \sum_{t=1}^{5} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

satisfying our conditions. In this case, $\{a_0, a_4, b_4, a_5, b_5\}$ is a basis of E_1^5 satisfying (3.2).

Case 1. Let $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5\} = \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1\}$, it follows that $\mathfrak{A} = \{p_1,\ p_2 - p_1,\ p_3 - p_2,\ p_4 - p_3,\ p_5 - p_4\} \cup \{2p_1,\ p_2,\ p_3 - p_1,\ p_4 - p_2,\ p_5 - p_3\} \cup \{p_3,\ p_1 + p_2,\ p_4 - p_1,\ p_5 - p_2\} \cup \{p_4,\ 2p_2,\ p_1 + p_3,\ p_5 - p_1\} \cup \{p_5,\ p_1 + p_4,\ p_2 + p_3\} \cup \{2p_3,\ p_1 + p_5,\ p_2 + p_4\} \cup \{p_2 + p_5,\ p_3 + p_4\} \cup \{2p_4,\ p_3 + p_5\} \cup \{p_5 + p_4\} \cup \{2p_5\}$. Applying (2.3), (2.4), (2.8), (2.9), Lemmas 3.1 and 3.3 for the subclasses $\{p_2,\ 2p_1,\ p_3 - p_1,\ p_4 - p_2,\ p_5 - p_3\}$ of $\mathfrak{A}$, we obtain

$$\begin{aligned} A_{52} &= \bar{A}_{52} = 0, \qquad & D_{52} &= \bar{D}_{52} = 0, \\ A_{42} &= \bar{A}_{42} = 0, \qquad & D_{42} &= \bar{D}_{42} = 0, \\ A_{22} &= \bar{A}_{22} = 0. \end{aligned}$$

Hence, Lemmas 3.3 and 3.4 lead to a contradiction.

Case 2. In case of $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5\} \neq \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1\}$, let $\mathfrak{A}'_t$ be the subclass consisting of all elements in $\mathfrak{A}'$ to be equal to p_t. Then the number of elements in $\mathfrak{A}'_t$ is less than or equal to four. Hence, Lemma 3.4 leads to a contradiction by Lemma 3.1.

Finally, we have also the following corollary.

COROLLARY 3 4. *There exists no 5-type closed curve* $\gamma(s)$ *on* $H^4(-c^2)$ *satisfying* $< \gamma^{(l)}(s),\ \gamma^{(l)}(s) >$ *is constant* $(l = 2, 3, 4)$.

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