East Asian Math J. 14 (1998), No 1, pp. 149–155

GK-DIMENSIONS OF EXTENSION ALGEBRAS

CHAN HUH

1. Introduction

Throughout K is a field and R is a K-algebra with identity. If R is finitely generated as an algebra and V is a finite dimensional K-subspace of R containing identity, then it is well known that the extended real number $\overline{\lim} \log_n(\dim_k V^n)$ is independent of the generating subspace V. This number, denoted GKdim(R), is called the Gelfand-Kirillov dimension (or simply, GK-dimension) of R. If R is not finitely generated as an algebra, the GK-dimension of R is defined by

$$\begin{aligned}&\text{GKdim}(R)\\ &= \sup\{\text{GKdim}(S) | S \text{ is a finitely generated subalgebra of } R\}.\end{aligned}$$

If $\theta : R \longrightarrow R$ is a K-monomorphism, then $\text{GKdim}(R[x;\theta]) \geq \text{GKdim}(R) + 1$ in general. In section 2, we investigate conditions on θ for which the equality holds, that is, $\text{GKdim}(R[x;\theta]) = \text{GKdim}(R) + 1$.

Note that if Ω is a left Ore set of R, then the left quotient algebra $\Omega^{-1}R$ contains R as a subalgebra. So $\text{GKdim}(\Omega^{-1}R) \geq \text{GKdim}(R)$. In section 3, we give some conditions on Ω for which the two GK-dimensions coincide, and also generalize a theorem[2, Proposition 4.2.] of G.Krause and T.Lenagan.

2. GK-dimensions of skew polynomial rings

Let R be a K-algebra and $\theta : R \longrightarrow R$ be a K-monomorphism of R. The left skew polynomial ring $R[x;\theta]$ of R by θ is the set of all

---

Received March 17, 1998

This work was done while the author was partly supported by the Nulwon Cultral Foundation, 1997.

polynomials,

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \ (a_i \in R)$$

with usual addition and multiplication induced by the commutation rule: $xa = \theta(a)x$ for all $a \in R$. Since R is a subalgebra of $R[x;\theta]$, $\mathrm{GKdim}(R[x;\theta]) \geq \mathrm{GKdim}(R)$.

More precisely, we have the following:

LEMMA 1. *If θ is a K-monomorphism of R, then $GKdim(R[x;\theta]) \geq GKdim(R) + 1$.*

Proof. Let V be any finite dimensional subspace of R containing 1. Then $W = V \oplus Kx$ is a finite dimensional subspace of $R[x;\theta]$ with 1. Since

$$V^n \oplus V^nx \oplus \cdots \oplus V^nx^n \subset W^{2n},$$

we have $(n+1)\dim_K V^n \leq \dim_K W^{2n}$ for all $n \geq 1$, and hence

$$\begin{aligned}\mathrm{GKdim}(R[x;\theta]) &\geq \overline{\lim} \log_n(\dim_K W^{2n}) \\ &\geq \overline{\lim} \log_n((n+1)\dim_K V^n) \\ &= \overline{\lim} \log_n(\dim_K V^n) + 1.\end{aligned}$$

But V is arbitrary, $\mathrm{GKdim}(R[x;\theta]) \geq \mathrm{GKdim}(R) + 1$.

In general, GK-dimension can increase by more than 1, in fact, it increases by any positive integer and can become infinite when passing to suitable skew polynomial rings.

THEOREM 2. *Let n be any nonnegative integer and let p be a positive integer or $p = \infty$. Then there exist an algebra R and a K-automorphism θ of R such that $GKdim(R) = n$ and $GKdim(R[x;\theta]) = GKdim(R) + p$.*

Proof. See[1, Theorem 3.4 and Corollary 3.5].

However in the proof of Theorem 2, the algebra R is neither finitely generated nor left Noetherian. The following example shows that there is a finitely generated Noetherian algebra R with a K-monomorphism θ such that $\mathrm{GKdim}(R) = 1$ and $\mathrm{GKdim}(R[x;\theta]) = \infty$.

EXAMPLE 3. Let $R = K[t]$ be the polynomial ring in an indeterminate t. Then clearly R is a finitely generated Noetherian K-algebra with Gk-dimension one. Let $\theta : R \longrightarrow R$ be the K-monomorphism defined by $\theta(f(t)) = f(t^2)$ for all $f(t) \in R$. Thus in the skew polynomial ring $R[x;\theta]$, $xf(t) = f(t^2)x$ for all $f(t)$. Now note that $V = K \oplus Kt \oplus Kx$ is a generating subspace of $R[x;\theta]$, hence $\text{GKdim}(R[x;\theta]) = \overline{\lim} \log_n(\dim_K V^n)$.

For each integer n, define a subset X_n of V^{2n+1} by

$$X_n = \{t^{i_0}xt^{i_1}\cdots t^{i_{n-1}}xt^{i_n} \mid i_j = 0 \text{ or } 1 \text{ for all } j = 0,1,\ldots,n\}.$$

Since $i_j = 0$ or 1, and $t^{i_0}xt^{i_1}\cdots t^{i_{n-1}}xt^{i_n} = t^{(i_0+2i_1+2^2i_2+\cdots+2^ni_n)}x^n$, the set

$$\begin{aligned} X_n &= \{t^{i_0}xt^{i_1}\cdots t^{i_{n-1}}xt^{i_n} \mid i_j = 0 \text{ or } 1\} \\ &= \{t^{(i_0+2i_1+\cdots+2^ni_n)}x^n \mid i_j = 0 \text{ or } 1\} \end{aligned}$$

is linearly independent over K with 2^{n+1} elements. Therefore, $\dim_K V^{2n+1} \geq 2^{n+1}$, and hence $\dim_K V^{2(n+1)} \geq \dim_K V^{2n+1} \geq 2^{n+1}$, or $\dim_K V^{2n} \geq 2^n$ for all $n \geq 1$. Thus

$$\begin{aligned} \text{GKdim}(R[x;\theta]) &= \overline{\lim} \log_n(\dim_K V^n) \\ &= \overline{\lim} \log_n(\dim_k V^{2n}) \geq \overline{\lim} \log_n(2^n) = \infty. \end{aligned}$$

Consequently, $\text{GKdim}(R[x;\theta]) = \infty$, while R is a finitely generated Noetherian algebra with $\text{GKdim}(R) = 1$.

The equaltity, $\text{GKdim}(R[x;\theta]) = \text{GKdim}(R) + 1$, holds for a wide class of algebras and K-monomorphisms by the following results.

LEMMA 4. *Let θ be a K-monomorphism of R. Suppose that for each finite dimensional subspace U, there exists a finite dimensional subspace V of R such that $U \subset V$ and $\theta(V) \subset V$. Then $GKdim(R[x;\theta]) = GKdim(R) + 1$.*

Proof. The proof is essentially the same as for [1, Corollary 2.4].

COROLLARY 5. *Let R be a finite dimensional algebra over K. Then for any automorphism θ of R, GKdim$(R[x;\theta]) = 1$.*

THEOREM 6. *If θ is a K-automorphism of R of finite order (that is, $\theta^p = id$, the identity map of R for some positive integer p), then GKdim$(R[x;\theta]) =$ GKdim$(R) + 1$.*

Proof. Suppose θ^p is the identity map ($p \geq 1$), and U is a finite dimensional subspace of R. Let $V = U + \theta(U) + \cdots + \theta^{p-1}(U) = \sum_{i=0}^{p-1} \theta^i(U)$; then V is a finite dimensional subspace such that $U \subset V$ and $\theta(V) \subset V$. Therefore, GKdim$(R[x;\theta]) =$ GKdim$(R) + 1$ by Lemma 5.

3. GK-dimensions of quotient rings

Recall that a left Ore set Ω is a nonempty multiplicatively closed subset consisting of regular elements in R satisfying $Ra \cap \Omega r \neq \phi$ for all $a \in \Omega$ and $r \in R$. Thus if Ω is a multiplicatively closed set of regular elements in the center of R, Ω is automatically a left Ore set of R. It is a well known fact that if Ω is a left Ore set of R, then there exists an extension ring $\Omega^{-1}R$, which is called a left quotient ring, and whose elements are of the form $a^{-1}r$ where $a \in \Omega, r \in R$.

PROPOSITION 7. *Let Ω be a multiplicatively closed subset of regular esements in the center of R. Then GKdim$(\Omega^{-1}R) =$ GKdim(R).*

Proof. See [2, Proposition 4.2].

COROLLARY 8. *Let R be a commutative algebra and Ω be the set of all regular elements of R. Then GKdim$(\Omega^{-1}R) =$ GKdim(R). In particular, if R is an integral domain containing K, then GKdim$(Q(R)) =$ GKdim(R) where $Q(R)$ is the field of fractions of R.*

The result in Proposition 7 may not be true if Ω is not contained in the center by the following example [2, Example 8.18]:

EXAMPLE 9. Let K be a field of characteristic zero, and let $R = K[x, y| \, xy = yx + 1]$ be the first Weyl algebra over K. Then GKdim(R) = 2. Since R is a Noetherian domain, $\Omega = R - \{0\}$ is a left Ore set of R. L.Makar-Limanov [3] showed that GKdim($\Omega^{-1}R$) = ∞, hence GKdim($\Omega^{-1}R$) $\neq$ GKdim(R).

We now give a generalization of Proposition 7 as follows:

LEMMA 10. *Let Ω be a left Ore set of R. Suppose that for each $a \in \Omega$ and each finite dimensional subspace U there exists a finite dimensional subspace V of R such that $U \subset V$ and $aV \subset Va$, then GKdim($\Omega^{-1}R$) = GKdim(R).*

Proof. Let W be a finite dimensional subspace of $\Omega^{-1}R$. Then $aW \subset R$ for some $a \in \Omega$. If we set $U = K + aW$, then U is a finite dimensional subspace of R and $W \subset a^{-1}U$. By hypothesis there is a finite dimensional subspace V of R such that $U \subset V$ and $aV \subset Va$ (or equivalently, $Va^{-1} \subset a^{-1}V$). Thus we have $W \subset a^{-1}U \subset a^{-1}V$, and hence $W^n \subset (a^{-1}V)^n \subset a^{-n}V^n$ for all integer $n \geq 1$. So $\dim_K W^n \leq \dim_K(a^{-n}V^n) = \dim_K V^n$, and therefore,

$$\begin{aligned}
&\mathrm{GKdim}(\Omega^{-1}R) \\
&= \sup\{\overline{\lim} \log_n(\dim_K W^n) | \; W \text{ is a finite dimensional subspace of } \Omega^{-1}R\} \\
&\leq \sup\{\overline{\lim} \log_n(\dim_K V^n) | \; V \text{ is a finite dimensional subspace of } R\} \\
&= \mathrm{GKdim}(R).
\end{aligned}$$

But clearly GKdim($\Omega^{-1}R$) $\geq$ GKdim(R), cosequently GKdim($\Omega^{-1}R$) = GKdim(R).

For later use we mention some remarks:

REMARKS.

(1) Let Ω be a multiplicatively closed subset of regular elements of R. Suppose that $aR \subset Ra$ for $a \in \Omega$, then Ω is a left Ore set of R In order to see this, if $a \in \Omega$ and $r \in R$ then since $aR \subset Ra$, $ar = r'a$ for some r'. Hence $ar = r'a \in \Omega r \cap Ra$, and so $\Omega r \cap Ra$ is nonempty.

(2) Let θ be a K-monomorphism and let $R[x;\theta]$ be the skew polynomial ring of R by θ. Then the set $X = \{x^i \mid i = 0, 1, 2, \cdots\}$ is a left Ore suset of $R[x;\theta]$, because for each $i(\geq 1)$ $x^i R[x;\theta] \subset R[x;\theta]x^i$, and by (1).

COROLLARY 11. *Let R be a finite dimensional algebra over K, and θ be a K-automorphism of R. Then $X = \{x^i \mid i \geq 0\}$ is a left Ore set of $R[x;\theta]$ and $GKdim(X^{-1}R[x;\theta]) = GKdim(R[x;\theta]) = 1$.*

Proof. It follows from Remark(2) that X is a left Ore set of $R[x;\theta]$. Since R is finite dimensional, R itself is a finite dimensional generating subspace such that $\theta(R) = R$. Therefore by Lemma 4, $\text{GKdim}(R[x;\theta]) = \text{GKdim}(R) + 1 = 1$.

Now if U is a finite dimensional subspace of $R[x;\theta]$, then $U \subset (R + \sum_{j=1}^{m} Kx^j)$ for some $m \geq 1$. Since $x^i R = \theta^i(R)x^i = Rx^i$, we have $x^i(R + \sum_{j=1}^{m} Kx^j) = (R + \sum_{j=1}^{m} Kx^j)x^i$ for all $i \geq 1$. Therefore $\text{GKdim}(X^{-1}R[x;\theta]) = \text{GKdim}(R[x;\theta]) = 1$.

THEOREM 12. *Let Ω be a multiplicatively closed set of regular elements of R. If for each element $a \in \Omega$, $aR \subset Ra$ and a^p is central ($p \geq 1$ an integer depending on a), then Ω is a left Ore set of R with $GKdim(\Omega^{-1}R) = GKdim(R)$.*

Proof. By Remark(1) above, Ω is a left ore set of R. Now let U be a finite dimensional subspace of R and $a \in \Omega$. Since $aR \subset Ra$ and a^p is an element of the center for some integer $p \geq 1$, we can find finite dimensional subspaces, $U = U_{(0)}, U_{(1)}, \ldots, U_{(p-1)}$, such that

$$aU = U_{(1)}a,\ aU_{(1)} = U_{(2)}a, \ldots, aU_{(p-2)} = U_{(p-1)}a,\ aU_{(p-1)} = Ua.$$

If we take $V = \sum_{i=0}^{p-1} U_{(i)}$, then V is a finite dimensional subspace of R satisfying $U \subset V$ and $aV \subset Va$. Therefore, $\text{GKdim}(\Omega^{-1}R) = \text{GKdim}(R)$ by Lemma 10.

EXAMPLE 13. Let θ be a K-automorphism of R of order p. Then by Remark(2), the set $X = \{x^i \mid i \geq 0\}$ is a left Ore set of $R[x;\theta]$. Since $x^p r = \theta^p(r)x^p = rx^p$ for all $r \in R$, x^p (and hence, $(x^i)^p$) is central. So by Theorems 6 and 12, $\text{GKdim}(X^{-1}R[x;\theta]) = \text{GKdim}(R[x;\theta]) = \text{GKdim}(R) + 1$.

References

[1] C. Huh and C Kim, *Gelfand-Kirillof dimension of skew polynomial rins of automorphism type*, Comm in Algebra **24(7)** (1996), 2317-2323.

[2] G. Krause and T Lenagan, *Growth of Algebras and Gelfand-Kirillov Dimension, Research Note in Math., 116*, Pitman, London, 1985

[3] L Makar-Limanov, *The skew field of fractions of the Weyl algebra contains a free subalgebra*, Comm in Algebra **11** (1983), 2003-2006.

Department of Mathematics
Pusan National University
Pusan 609-735, Korea