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CONVERGENCE THEOREMS FOR NONLINEAR MAPPINGS IN CONVEX METRIC SPACES

JEONG SHEOK UME* AND HEE SOO PARK

1. Introduction

In 1970, Takahashi [4] introduced a notion of convex metric spaces and proved a fixed point theorem for nonlinear mappings on convex metric spaces.

Ding[1] also studied Ishikawa's iteration scheme[2] to construct fixed points of quasi-contractive, generalized quasi-constractive, and quasi-nonexpansive mappings in convex metric spaces.

Recently, Kada-Suzuki-Takahashi [4] introduced the concept of w-distance on a metric space and obtained a nonconvex minimization theorem.

In this paper, using an Ishikawa type iterative process and the concept of a p-convex metric space, we obtain a unique common fixed point for nonlinear mappings which generalize quasi-contractive mappings [1].

2. Preliminaries

DEFINITION 2.1 [4]. Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

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X together with a convex structure W is called a convex metric space. A nonempty subset K is said to be convex if $W(x,y,\lambda) \in K$ for all $(x,y,\lambda) \in K \times K \times I$.

In what follows, if (X,d) is a metric space and nonempty subset K of X is convex, then K is said to be metric d-convex.

DEFINITION 2.2 [3]. Let X be a metric space with metric d. Then a function $p : X \times X \to [0,\infty)$ is called a w-distance on X if the following are satisfied

(1) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in X$;
(2) for any $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

In particular, if d is a metric for any set X, then clearly d is a w-distance on X.

DEFINITION 2.3. Let (X,d) be a metric space with a w-distance p and $I = [0,1]$. A mapping $W : X \times X \times X \times I \times I \to X$ is said to be a p-convex structure on X if for each $(x,y,z,\lambda,\omega) \in X \times X \times X \times I \times I$ with $0 \le \lambda + \omega \le 1$ and $u \in X$,

$$\begin{aligned} p(u, W(x,y,z,\lambda,\omega)) &\le \lambda p(u,x) + \omega p(u,y) \\ &\quad + (1-\lambda-\omega)p(u,z), \\ p(W(x,y,z,\lambda,\omega),u) &\le \lambda p(x,u) + \omega p(y,u) \\ &\quad + (1-\lambda-\omega)p(z,u). \end{aligned} \tag{2.1}$$

X together with a p-convex structure W is called a p-convex metric space. A nonempty subset K of a p-convex metric space X is said to be a p-convex if $W(x,y,z,\lambda,\omega) \in K$ for all $(x,y,z,\lambda,\omega) \in K \times K \times K \times I \times I$ with $0 \le \lambda + \omega \le 1$.

EXAMPLE 2.4. Let X be the set of all real numbers with usual metric and let K be the set of all nonnegative real numbers. Let I be closed interval $[0,1]$. Define a mapping $W : X \times X \times X \times I \times I \to X$

by $W(x,y,z,\lambda,\omega) = \lambda x + \omega y + (1-\lambda-\omega)z$ for every $x,y,z \in K$ and for every $\lambda,\omega \in I$ with $0 \le \lambda + \omega \le 1$. Let $p : X \times X \to [0,\infty)$ be a mapping such that

$$p(x,y) = \max\left\{ |\frac{1}{5}x - y|, \frac{1}{5}|x-y| \right\}$$

for all $x,y \in K$. Then it is clear that K together with a p-convex structure W is a p-convex on X but not metric p-convex on X.

LEMMA 2.5 [3]. *Let X be a metric space with metric d, and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converges to 0, and let $x,y,z \in X$. Then the following hold:*

(1) *If $p(x_n,y) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*
(2) *if $p(x_n,y_n) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;*
(3) *if $p(x_n,x_m) \le \alpha_n$ for any $n,m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *if $p(y,x_n) \le \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

3. Main results

THEOREM 3.1. *Let K be a nonempty closed p-convex subset of a complete metric space X with a w-distance p, and let $S,T : K \to K$ are two mappings such that*

$$\begin{aligned}(3.1)\qquad &\max\{p(Sx,Sy),p(Tx,Ty),p(Sx,Ty),p(Tx,Sy)\}\\ &\le q\cdot\max\{p(x,y),p(y,x),p(x,Sx),p(x,Sy),\\ &\quad p(x,Tx),p(x,Ty),p(y,Sx),p(y,Sy),\\ &\quad p(y,Tx),p(y,Ty),p(Sx,x),p(Sy,x),\\ &\quad p(Tx,x),p(Ty,x),p(Sx,y),p(Sy,y),\\ &\quad p(Tx,y),p(Ty,y),p(Sx,Sx),p(Tx,Tx),\\ &\quad p(Sy,Sy),p(Ty,Ty),p(Sx,Tx),\\ &\quad p(Tx,Sx),p(Sy,Ty),p(Ty,Sy)\}\end{aligned}$$

for all $x, y \in K$ and some $q \in [0,1)$ and such that

$$\inf\{p(x,y) + p(x,Sx) + p(x,Tx) : x \in K\} > 0$$

for every $y \in K$ with $y \neq Sy$ or $y \neq Ty$. Suppose that $\{x_n\}$ is an Ishikawa type iterative scheme defined by

$$\begin{aligned} (3.2) \qquad & x_0 \in K, \\ & x_{n+1} = W(Sy_n, Ty_n, x_n, \alpha_n, \beta_n), \\ & y_n = W(Sx_n, Tx_n, x_n, \gamma_n, \rho_n), \ n = 0,1,2,\cdots, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\rho_n\}$ satisfy that $0 \leq \alpha_n, \beta_n, \gamma_n, \rho_n \leq 1$, $0 \leq \alpha_n + \beta_n < 1$, $0 \leq \gamma_n + \rho_n < 1$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n)^2$ diverges. Then $\{x_n\}$ converges to a unique common fixed point of S and T in K.

Proof. Let M be the set of all nonnegative integers. For $n, m \in M$ with $0 \leq n < m$, we denote

$$(3.3) \qquad D_{n,m} = \bigcup_{j=n}^{m} \{x_j, y_j, Sx_j, Tx_j, Sy_j, Ty_j\}$$

and $\delta(D_{n,m}) = \sup\{p(x,y) : x, y \in D_{n,m}\}$ denotes a p-diameter of $D_{n,m}$. By (3.1), we easily show that for all $k, l \in M$ with $n \leq k, l \leq m$, $\max\{p(Ua, Vb) : U, V = S$ or T and $a, b = x_k, x_l, y_k$ or $y_l\} \leq q.\delta(D_{n,m})$.

Now we show that

$$\begin{aligned} (3.4) \qquad \delta(D_{n,m}) = \max\{ & p(x_n, x_n), p(x_n, Sx_j), \\ & p(x_n, Sy_j), p(x_n, Tx_j), p(x_n, Ty_j), \\ & p(Sx_j, x_n), p(Sy_j, x_n), p(Tx_j, x_n), \\ & p(Ty_j, x_n) : n \leq j \leq m\}. \end{aligned}$$

We consider the following several cases:

Case 1. Let $\delta(D_{n,m}) = \max\{p(x_i, Sx_j) : n \leq i, j \leq m\}$. In the case, if $\delta(D_{n,m}) = p(x_{k+1}, Sx_l)$ for $n \leq k < m$ and $n \leq l \leq m$, then, by (2.1),(3.2) and (3.3), we have

$$\begin{aligned}
\delta(D_{n,m}) &= p(W(Sy_k, Ty_k, x_k, \alpha_k, \beta_k), Sx_l) \\
&\leq \alpha_k p(Sy_k, Sx_l) + \beta_k p(Ty_k, Sx_l) \\
&\quad + (1 - \alpha_k - \beta_k) p(x_k, Sx_l) \\
&\leq \alpha_k \cdot q \cdot \delta(D_{n,m}) + \beta_k \cdot q \cdot \delta(D_{n,m}) \\
&\quad + (1 - \alpha_k - \beta_k) p(x_k, Sx_l) \\
&\leq \alpha_k \delta(D_{n,m}) + \beta_k \delta(D_{n,m}) + (1 - \alpha_k - \beta_k) p(x_k, Sx_l)
\end{aligned}$$

It follows that $\delta(D_{n,m}) = p(x_k, Sx_l)$. By induction, we obtain $\delta(D_{n,m}) = p(x_n, Sx_l)$ and so (3.4) holds.

Case 2. Let $\delta(D_{n,m}) = \max\{p(x_i, Sy_j) : n \leq i, j \leq m\}, \delta(D_{n,m}) = \max\{p(x_i, Tx_j) : n \leq i, j \leq m\}$ or $\delta(D_{n,m}) = \max\{p(x_i, Ty_j) . n \leq i, j \leq m, \}$. In these cases, by the method similar to Case 1, (3.4) holds.

Case 3. Let $\delta(D_{n,m}) = \max\{p(Sx_j, x_i) : n \leq i, j \leq m\}$. If $\delta(D_{n,m}) = p(Sx_l, x_{k+1})$ for $n \leq k < m$ and $n \leq l \leq m$, then by (2.1),(3.2) and (3.3), we have

$$\begin{aligned}
\delta(D_{n,m}) &= p(Sx_l, W(Sy_k, Ty_k, x_k, \alpha_k, \beta_k)) \\
&\leq \alpha_k p(Sx_l, Sy_k) + \beta_k p(Sx_l, Ty_k) + (1 - \alpha_k - \beta_k) p(Sx_l, x_k) \\
&\leq \alpha_k \delta(D_{n,m}) + \beta_k \cdot q \cdot \delta(D_{n,m}) + (1 - \alpha_k - \beta_k) p(Sx_l, x_k) \\
&\leq \alpha_k \delta(D_{n,m}) + \beta_k \delta(D_{n,m}) + (1 - \alpha_k - \beta_k) p(Sx_l, x_k).
\end{aligned}$$

It follows that $\delta(D_{n,m}) = p(Sx_l, x_k)$. By induction, we obtain $\delta(D_{n,m}) = p(Sx_l, x_n)$ and so (3.4) holds.

Case 4. Let $\delta(D_{n,m}) = \max\{p(Sy_j, x_i) : n \leq i, j \leq m\}, \delta(D_{n,m}) = \max\{p(Tx_j, x_i) : n \leq i, j \leq m\}$ or $\delta(D_{n,m}) = \max\{p(Ty_j, x_i) : n \leq i, j \leq m, \}$. In these cases, by the method similar to Case 3, and so (3.4) holds.

Case 5. Let $\delta(D_{n,m}) = p(y_k, Sx_l)$ or $\delta(D_{n,m}) = p(Sx_l, y_k)$ for some

k, l with $n \le k, l \le m$. In these cases, by (2.1),(3.2) and (3.3) we have

$$\begin{aligned}\delta(D_{n,m}) &= p(W(Sx_k, Tx_k, x_k, \gamma_k, \rho_k), Sx_l) \\ &\le \gamma_k p(Sx_k, Sx_l) + \rho_k p(Tx_k, Sx_l) + (1-\gamma_k-\rho_k)p(x_k, Sx_l) \\ &\le \gamma_k \cdot q \cdot \delta(D_{n,m}) + \rho_k \cdot q \cdot \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(x_k, Sx_l) \\ &\le \gamma_k \delta(D_{n,m}) + \rho_k \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(x_k, Sx_l)\end{aligned}$$

or

$$\begin{aligned}\delta(D_{n,m}) &= p(Sx_l, W(Sx_k, Tx_k, x_k, \gamma_k, \rho_k)) \\ &\le \gamma_k p(Sx_l, Sx_k) + \rho_k p(Sx_l, Tx_k) + (1-\gamma_k-\rho_k)p(Sx_l, x_k) \\ &\le \gamma_k \cdot q \cdot \delta(D_{n,m}) + \rho_k \cdot q \cdot \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(Sx_l, x_k) \\ &\le \gamma_k \delta(D_{n,m}) + \rho_k \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(Sx_l, x_k)\end{aligned}$$

and so $\delta(D_{n,m}) = p(x_k, Sx_l)$ or $\delta(D_{n,m}) = p(Sx_l, x_k)$. It follows from Case 3 that (3.4) holds.

Case 6. Let $\delta(D_{n,m}) = p(y_k, Tx_l)$ or $\delta(D_{n,m}) = p(Tx_l, y_k)$ for some k, l with $n \le k, l \le m$. Using the same argument as in Case 5, (3.4) holds.

Case 7. Let $\delta(D_{n,m}) = p(y_k, Sy_l)$ or $\delta(D_{n,m}) = p(Sy_l, y_k)$ for some k, l with $n \le k, l \le m$. In these cases, by (2.1),(3.2) and (3.3), we have

$$\begin{aligned}\delta(D_{n,m}) &= p(W(Sx_k, Tx_k, x_k, \gamma_k, \rho_k), Sy_l) \\ &\le \gamma_k p(Sx_k, Sy_l) + \rho_k p(Tx_k, Sy_l) + (1-\gamma_k-\rho_k)p(x_k, Sy_l) \\ &\le \gamma_k \cdot q \cdot \delta(D_{n,m}) + \rho_k \cdot q \cdot \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(x_k, Sy_l) \\ &\le \gamma_k \delta(D_{n,m}) + \rho_k \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(x_k, Sx_l)\end{aligned}$$

or

$$\begin{aligned}\delta(D_{n,m}) &= p(Sy_l, W(Sx_k, Tx_k, x_k, \gamma_k, \rho_k)) \\ &\le \gamma_k p(Sy_l, Sx_k) + \rho_k p(Sy_l, Tx_k) + (1-\gamma_k-\rho_k)p(Sy_l, x_k) \\ &\le \gamma_k \cdot q \cdot \delta(D_{n,m}) + \rho_k \cdot q \cdot \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(Sy_l, x_k) \\ &\le \gamma_k \delta(D_{n,m}) + \rho_k \delta(D_{n,m}) + (1-\gamma_k-\rho_k)p(Sy_l, x_k)\end{aligned}$$

and so $\delta(D_{n,m}) = p(x_k, Sy_l)$ or $\delta(Sy_l, x_k)$. It follows from Case 1 and Case 3 that (3.4) holds.

Case 8. Let $\delta(D_{n,m}) = p(y_k, Ty_l)$ or $\delta(D_{n,m}) = p(Ty_l, y_k)$ for some k, l with $n \le k, l \le m$. Using the same argument as in Case 7, (3.4) holds.

Case 9. Let $\delta(D_{n,m}) = \max\{p(x_i, x_j) : n \le i, j \le m\}$. If $\delta(D_{n,m}) = p(x_{k+1}, x_l)$ or $\delta(D_{n,m}) = p(x_l, x_{k+1})$ for some k, l with $n \le k < m$ and $n \le l \le m$, then by (2.1),(3.2) and (3.3), we have

$$\begin{aligned}\delta(D_{n,m}) &= p(W(Sy_k, Ty_k, x_k, \alpha_k, \beta_k), x_l)\\ &\le \alpha_k p(Sy_k, x_l) + \beta_k p(Ty_k, x_l) + (1 - \alpha_k - \beta_k p(x_k, x_l)\end{aligned}$$

or

$$\begin{aligned}\delta(D_{n,m}) &= p(x_l, W(Sy_k, Ty_k, x_k, \alpha_k, \beta_k))\\ &\le \alpha_k p(x_l, Sy_k) + \beta_k p(x_l, Ty_k) + (1 - \alpha_k - \beta_k)p(x_l, x_k).\end{aligned}$$

It follows that $\delta(D_{n,m}) = p(Sy_k, x_l), \delta(D_{n,m}) = p(x_l, Sy_k), \delta(D_{n,m}) = p(Ty_k, x_l), \delta(D_{n,m}) = p(x_l, Ty_k)$, $\delta(D_{n,m}) = p(x_k, x_l)$ or $\delta(D_{n,m}) = p(x_l, x_k)$. If $\delta(D_{n,m}) = p(Sy_k, x_l), \delta(D_{n,m}) = p(x_l, Sy_k), \delta(D_{n,m}) = p(Ty_k, x_l)$ and $\delta(D_{n,m}) = p(x_l, Ty_k)$, then, by Case 2 and Case 4, we have $\delta(D_{n,m}) = p(Sy_k, x_n)$, $\delta(D_{n,m}) = p(x_n, Sy_k)$, $\delta(D_{n,m}) = p(Ty_k, x_n)$ or $\delta(D_{n,m}) = p(x_n, Ty_k)$. If $\delta(D_{n,m}) = p(x_k, x_l)$ or $\delta(D_{n,m}) = p(x_l, x_k)$, then, by induction, we obtain $\delta(D_{n,m}) = p(x_n, x_l)$ and $\delta(D_{n,m}) = p(x_l, x_n)$. Again by (2 1), (3.2) and (3 3), and we have $\delta(D_{n,m}) = p(x_n, x_n)$, $\delta(D_{n,m}) = p(x_n, Sy_l)$, $\delta(D_{n,m}) = p(x_n, Ty_l)$, $\delta(D_{n,m}) = p(Sy_l, x_n)$ or $\delta(D_{n,m}) = p(Ty_l, x_n)$. Thus (3.4) holds.

Case 10. Let $\delta(D_{n,m}) = p(y_k, x_l)$ or $\delta(D_{n,m}) = p(x_l, y_k)$ for some k, l with $n \le k, l \le m$ In these cases, it follows from Case 2, Case 3 and Case 4 that (3.4) holds.

Case 11. Let $\delta(D_{n,m}) = p(y_k, y_l)$ for some k, l with $n \le k, l \le m$. In this case, using the same arguments as in Case 9, (3.4) holds. Hence (3.4) is true. Since the remaining cases are impossible, (3.4) is proved.

By (3.4) , for $m \in M$ we have

$$(3.5) \qquad \delta(D_{0,m}) = \max\{p(x_0, x_0), p(x_0, Sx_j), \\ p(x_0, Sy_j), p(x_0, Tx_j), p(x_0, Ty_j), \\ p(Sx_j, x_0), p(Sy_j, x_0), p(Tx_j, x_0), \\ p(Ty_j, x_0) : 0 \leq j \leq m\}.$$

By simple calculus, we have

$$\delta(D_{0,m}) \leq \frac{1}{1-q}\{p(x_0, Sx_0) + p(Sx_0, x_0)\}.$$

By (3.2), (3.4) and simple calculus, we have for any $n, m \in M$ with $0 \leq n < m$

$$(3.6) \qquad \delta(D_{n,m}) = \max\{p(x_n, x_n), p(x_n, Sx_j), \\ p(x_n, Sy_j), p(x_n, Tx_j), p(x_n, Ty_j), \\ p(Sx_j, x_n), p(Sy_j, x_n), p(Tx_j, x_n), \\ p(Ty_j, x_n) : n \leq j \leq m\} \\ \leq [1 - (1-q)(\alpha_{n-1} + \beta_{n-1})^2]\delta(D_{n-1,m}).$$

By induction, we have

$$(3.7) \qquad \delta(D_{n,m}) \leq \prod_{j=0}^{n-1}[1 - (1-q)(\alpha_j + \beta_j)^2] \cdot \delta(D_{0,m}) \\ \leq \frac{1}{1-q}\{p(x_0, Sx_0) + p(Sx_0, x_0)\} \\ \cdot \prod_{j=0}^{n-1}[1 - (1-q)(\alpha_j + \beta_j)^2].$$

Since $1 - q > 0$ and $\sum_{j=0}^{\infty}(\alpha_j + \beta_j)^2$ diverges, we have $\prod_{j=0}^{\infty}[1 - (1 - q) \cdot (\alpha_j + \beta_j)^2] = 0$. Thus $\lim_{n,m\to\infty} \delta(D_{n,m}) = 0$. By Lemma 2.5, this implies that $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, Sx_n) = \lim_{n\to\infty} p(x_n, Sy_n) = \lim_{n\to\infty} p(x_n, Tx_n) =$

$\lim_{n\to\infty} p(x_n, Ty_n) = \lim_{n\to\infty} p(Sx_n, x_n) = \lim_{n\to\infty} p(Sy_n, x_n) = \lim_{n\to\infty} p(Tx_n, x_n) = \lim_{n\to\infty} p(Ty_n, x_n) = 0$ Let $\lim_{n\to\infty} x_n = z \in K$. Then, since $\{x_n\}$ converges to z and $p(x_n, \cdot)$ is lower semicontinuous, by (3.7),

$$\begin{aligned} p(x_n, z) &\le \lim_{m\to\infty} p(x_n, x_m) \\ &\le \lim_{m\to\infty} \inf \delta(D_{n,m}) \\ &\le \frac{1}{1-q}\{p(x_0, Sx_0) + p(Sx_0, x_0)\} \prod_{j=o}^{n-1} [1 - (1-q)\cdot(\alpha_j + \beta_j)^2]. \end{aligned}$$

Thus we obtain $\lim_{n\to\infty} p(x_n, z) = 0$. Assume that $z \neq Sz$ or $z \neq Tz$. Then by hypothesis we have

$$\begin{aligned} 0 &< \inf\{p(x, z) + p(x, Sx) + p(x, Tx) : x \in K\} \\ &\le \inf\{p(x_n, z) + p(x_n, Sx_n) + p(x_n, Tx_n) : n \in N\} = 0, \end{aligned}$$

where N is the set of all positive integers. This is a contradiction. Therefore, we have $z = Sz = Tz$. We claim that z is the unique common fixed point of S and T in K. Let $z = Sz = Tz$ and $v = Sv = Tv$. Then, by (3.1) we have

$$\begin{aligned} p(v, z) &= p(Sv, Tz) \\ &\le q \cdot \max\{p(z, v), p(v, z), p(z, z), p(v, v)\}, \end{aligned}$$

and similarly,

$$\begin{aligned} p(z, v) &= p(Sz, Tv) \\ &\le q \cdot \max\{p(z, v), p(v, z), p(z, z), p(v, v)\} \\ p(z, z) &= p(Sz, Tz) \le q \cdot p(z, z) \end{aligned}$$

and

$$p(v, v) = p(Sv, Tv) \le q \cdot p(v, v).$$

Therefore we have

$$\begin{aligned} &\max\{p(z, v), p(v, z), p(z, z), p(v, v)\} \\ &\quad\le q \cdot \max\{p(z, v), p(v, z), p(z, z), p(v, v)\}. \end{aligned}$$

Since $0 < q < 1$, we obtain $\max\{p(z,v), p(v,z), p(z,z), p(v,v)\} = 0$. This implies that $p(z,z) = p(v,v) = p(z,v) = p(v,z) = 0$. Then, by Lemma 2.5, $z = v$. Therefore $\{x_n\}$ converges to the unique common fixed point of S and T in K. □

THEOREM 3.2. *Let K be a nonempty closed p-convex subset of a complete metric space X with a w-distance p, and let $S, T : K \to K$ be two mappings satisfying (3.1),*

$$\inf\{p(x,y) + p(x,Sx) + p(x,Tx) : x \in K\} > 0$$

for all $y \in K$ with $y \neq Sy$ or $y \neq Ty$, and

$$\max\{p(x,x), p(y,y)\} \leq \max\{p(x,y), p(y,x)\} \tag{3.8}$$

for all $x, y \in K$. Suppose that $\{x_n\}$ is an Ishikawa type iterative scheme defined by

$$\begin{aligned} x_0 \in & K, \\ x_{n+1} = & W(Sy_n, Ty_n, x_n, \alpha_n, \beta_n), \\ y_n = & W(Sx_n, Tx_n, \gamma_n, \rho_n), n = 0, 1, 2, \cdots, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\rho_n\}$ satisfy that $0 \leq \alpha_n, \beta_n, \gamma_n, \rho_n \leq 1$, $0 \leq \alpha_n + \beta_n < 1$, $0 \leq \gamma_n + \rho_n < 1$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n)$ diverges. Then $\{x_n\}$ converges to a unique common fixed point of S and T in K.

Proof. By (3.4) $\sim$ (3.8), we have

$$\begin{aligned} \delta(D_{0,m}) = & \max\{p(x_0, Sx_j), p(x_0, Tx_j), p(x_0, Sy_j), p(x_0, Ty_j), \\ & p(Sx_j, x_0), p(Tx_j, x_0), p(Sy_j, x_0), p(Ty_j, x_0) : n \leq j \leq m\} \\ \leq & \frac{1}{1-q}\{p(x_0, Sx_0) + p(Sx_0, x_0)\}, \end{aligned}$$

and

$$\begin{aligned} \delta(D_{n,m}) = & \max\{p(x_n, Sx_j), p(x_n, Tx_j), p(x_n, Sy_j), p(x_n, Ty_j), \\ & p(Sx_j, x_n), p(Tx_j, x_n), p(Sy_j, x_n), p(Ty_j, x_n) : n \leq j \leq m\} \\ \leq & [1 - (1-q) \cdot (\alpha_{n-1} + \beta_{n-1})] \cdot \delta(D_{n-1,m}). \end{aligned}$$

Continuing this process, we obtain

$$\delta(D_{n,m}) \le \prod_{j=0}^{n-1}[1-(1-q)\cdot(\alpha_j+\beta_j)]\cdot\delta(D_{0,m}).$$

Since $1-q>0$ and $\sum_{j=0}^{\infty}(\alpha_j+\beta_j)$ diverges, we have $\prod_{j=0}^{\infty}[1-(1-q)\cdot(\alpha_j+\beta_j)]=0$. Thus $\lim_{n,m\to\infty}\delta(D_{n,m})=0$. The remaining proof is the same as in the proof of Theorem 3.1. □

THEOREM 3.3. *Let K be a nonempty closed convex subset of a complete convex metric space X and let $S,T:K\to K$ are two mappings such that*

$$\begin{aligned}\max\{&d(Sx,Sy),d(Tx,Ty),d(Sx,Ty),d(Tx,Sy)\}\\ \le& q\cdot\max\{d(x,y),d(x,Sx),d(x,Sy),d(x,Tx),d(x,Ty),(y,Sx),\\ &d(y,Sy),d(y,Tx),d(y,Ty),d(Sx,Tx),d(Sy,Ty)\}\end{aligned}$$

for all $x,y\in K$ and some $q\in[0,1)$. Then for every $y\in K$ with $y\ne Sy$ or $y\ne Ty$,

$$\inf\{d(x,y)+d(x,Sx)+d(x,Tx):x\in K\}>0.$$

Proof. Assume that there exists $z\in X$ with $z\ne Sz$ or $z\ne Tz$ and

$$\inf\{d(x,z)+d(x,Sx)+d(x,Tx):x\in K\}=0.$$

Then there exists a sequence $\{z_n\}$ in X such that

$$\lim_{n\to\infty}\{d(z_n,z)+d(z_n,Sz_n)+d(z_n,Tz_n)\}=0.$$

Since $d(z_n,z)\to 0, d(z_n,Sz_n)\to 0$ and $d(z_n,Tz_n)\to 0$, we have $\{Sz_n\}$ and $\{Tz_n\}$ converge to z. By hypothesis

$$\begin{aligned}d(Sz,Tz_n)\le q\cdot\max\{&d(z,z_n),d(z,Sz),d(z,Sz_n),d(z,Tz),\\ &d(z,Tz_n),d(z_n,Sz),d(z_n,Sz_n),d(z_n,Tz),\\ &d(z_n,Tz_n),d(Sz,Tz),d(Sz_n,Tz_n)\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$d(Sz, z) \leq q \cdot \max\{d(z, Sz), d(z, Tz), d(Sz, Tz)\}$$

and similarly,

$$\begin{aligned} d(Sz_n, Tz) \leq & q \cdot \max\{d(z_n, z), d(z_n, Sz_n), d(z_n, Sz), d(z_n, Tz_n), \\ & d(z_n, Tz), d(z, Sz_n), d(z, Sz), d(z, Tz_n), \\ & d(z, Tz), d(Sz_n, Tz_n), d(Sz, Tz)\}. \end{aligned}$$

Hence, we obtain $d(z, Tz) \leq q \cdot \max\{d(z, Sz), d(z, Tz), d(Sz, Tz)\}$ and similarly

$$\begin{aligned} d(Sz, Tz) \leq & q \cdot \max\{d(z, z), d(z, Sz), d(z, Sz), d(z, Tz), d(z, Tz), \\ & d(z, Sz), d(z, Sz), d(z, Tz), d(z, Tz), d(Sz, Tz), d(Sz, Tz)\} \\ = & q \cdot \max\{d(z, Sz), d(z, Tz), d(Sz, Tz)\}. \end{aligned}$$

Therefore, we have $\max\{d(z, Sz), d(z, Tz), d(Sz, Tz)\} \leq q \cdot \max\{d(z, Sz), d(z, Tz), d(Sz, Tz)\}$. This is a contradiction. Therefore, for every $y \in K$ with $y \neq Sy$ or $y \neq Ty$,

$$\inf\{d(x, y) + d(x, Sx) + d(x, Tx) : x \in K\} > 0. \quad \square$$

Using Theorem 3.2 and Theorem 3.3, we have the following corollaries.

COROLLARY 3.4. *Let K be a nonempty closed convex subset of a complete convex metric space X and $S, T : K \to K$ be a quasi-contractive mappings, i.e., there exists a constant $q \in [0, 1)$ such that for all $x, y \in K$,*

$$\begin{aligned} & \max\{d(Sx, Sy), d(Tx, Ty), p(Sx, Ty), p(Tx, Sy)\} \\ & \leq q \cdot \max\{d(x, y), d(x, Sx), d(x, Sy), d(x, Tx), d(x, Ty), d(y, Sx), \\ & \qquad d(y, Sy), d(y, Tx), d(y, Ty), d(Sx, Tx), d(Sy, Ty)\}. \end{aligned}$$

Suppose that $\{x_n\}$ *is an Ishikawa type iterative scheme defined by*

$$\begin{aligned} x_0 &\in K, \\ x_{n+1} &= W(Sy_n, Ty_n, x_n, \alpha_n, \beta_n), \\ y_n &= W(Sx_n, Tx_n, x_n, \gamma_n, \rho_n), n = 0, 1, 2, \cdots, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\rho_n\}$ *satisfy that* $0 \le \alpha_n, \beta_n, \gamma_n, \rho_n \le 1$, $0 \le \alpha_n + \beta_n < 1$, $0 \le \gamma_n + \rho_n < 1$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n)$ diverges. *Then* $\{x_n\}$ *converges to the unique common fixed point of* S and T *in* K.

COROLLARY 3.5 [1]. *Let* K *be a nonempty closed convex subset of a complete convex metrice space* X *and let* $T : K \to K$ *be a quasi-contractive mapping i.e., there exists a constant* $q \in [0, 1)$*such that for all* $x, y \in K$

$$d(Tx, Ty) \le q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Suppose that $\{x_n\}$ *is a sequence defined by*

$$\begin{aligned} x_0 &\in K, \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n), \\ y_n &= W(Tx_n, x_n, \beta_n), n = 0, 1, 2, \cdots, \end{aligned}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy that* $0 \le \alpha_n, \beta_n < 1$ *and* $\sum_{n=0}^{\infty} \alpha_n$ *diverges. Then* $\{x_n\}$ *converges to a unique fixed point of* T *in* K.

THEOREM 3.6. *Let* K *be a nonempty closed* p*-convex subset of a complete* p*-convex metric space with a* w*-distance* p*, and let* $S, T : K \to K$ *are two mappings. Suppose that there exists a nondecreasing upper semi-continuous function* $\varphi : \mathbb{R}_+^9 \to \mathbb{R}_+$*(the set of all nonnegative real numbers) in such that for all* $t > 0$, $\varphi(2t, t, t, 0, 0, 2t, 2t, t, t) < t$ *and such that for all* $x, y \in K$,

(3.9)

$$\begin{aligned} &\max\{p(Sx, Sy), p(Tx, Ty), p(Sx, Ty), p(Tx, Sy), \\ &\quad p(Sy, Sx), p(Ty, Tx), p(Ty, Sx), p(Sy, Tx)\} \\ &\le \varphi(p(x, y), p(x, Sx), p(x, Tx), p(x, Sy), p(x, Ty), \\ &\quad p(y, Sy), p(y, Ty), p(y, Sx), p(y, Tx)). \end{aligned}$$

If $\{x_n\}$ is the iterative sequence defined by (3.2) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\rho_n\}$ are bounded away from zero, and if $p(x_n, z) \to 0, p(z, x_n) \to 0$, $p(Sx_n, Sx_n) \to 0, p(Tx_n, Tx_n) \to 0, p(Sy_n, Sy_n) \to 0, p(Ty_n, Ty_n) \to 0$, $p(Sx_n, Tx_n) \to 0, p(Tx_n, Sx_n) \to 0, p(Sy_n, Ty_n) \to 0, p(Ty_n, Sy_n) \to 0$, then z is the common fixed point of S and T.

Proof. By (2.1) and (3.2), we write

$$\begin{aligned} p(x_n, Sy_n) \leq & p(x_n, x_{n+1}) + p(x_{n+1}, Sy_n) \\ = & p(x_n, x_{n+1}) + p(W(Sy_n, Ty_n, x_n, \alpha_n, \beta_n), Sy_n) \\ \leq & p(x_n, x_{n+1}) + \alpha_n p(Sy_n, Sy_n) + \beta_n p(Ty_n, Sy_n) \\ & + (1 - \alpha_n - \beta_n) p(x_n, Sy_n). \end{aligned}$$

Thus, we obtain

$$(\alpha_n + \beta_n) \cdot p(x_n, Sy_n) \leq p(x_n, x_{n+1}) + \alpha_n p(Sy_n, Sy_n) + \beta_n p(Ty_n, Sy_n).$$

Similarly, we have

$$(\alpha_n + \beta_n) \cdot p(x_n, Ty_n) \leq p(x_n, x_{n+1}) + \alpha_n p(Sy_n, Ty_n) + \beta_n p(Ty_n, Ty_n).$$

From $p(x_n, z) \to 0$ and $p(z, x_n) \to 0$, we obtain $p(x_n, x_{n+1}) \to 0$ and since $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded away from zero, we have $p(x_n, Sy_n) \to 0$ and $p(x_n, Ty_n) \to 0$.

By (3.9) we have

$$\begin{aligned} \max\{ & p(Sx_n, Sy_n), p(Tx_n, Ty_n), p(Sx_n, Ty_n), p(Tx_n, Sy_n), \\ & p(Sy_n, Sx_n), p(Ty_n, Tx_n), p(Ty_n, Sx_n), p(Sy_n, Tx_n)\} \\ \leq & \varphi(p(x_n, y_n), p(x_n, Sx_n), p(x_n, Tx_n), p(x_n, Sy_n), p(x_n, Ty_n), \\ & p(y_n, Sy_n), p(y_n, Ty_n), p(y_n, Sx_n), p(y_n, Tx_n)), \end{aligned}$$

where

$$\begin{aligned}
p(x_n, y_n) =& p(x_n, W(Sx_n, Tx_n, x_n, \gamma_n, \rho_n)) \\
\leq& \gamma_n p(x_n, Sx_n) + \rho_n p(x_n, Tx_n) + (1 - \gamma_n - \rho_n) p(x_n, x_n) \\
\leq& \gamma_n [p(x_n, Sy_n) + p(Sy_n, Sx_n)] + \rho_n [p(x_n, Ty_n) \\
&+ p(Ty_n, Tx_n)] + (1 - \gamma_n - \rho_n) p(x_n, x_n) \\
\leq& p(x_n, Sy_n) + p(Sy_n, Sx_n) + p(x_n, Ty_n) \\
&+ p(Ty_n, Tx_n) + p(x_n, x_n), \\
p(x_n, Sx_n) \leq& p(x_n, Sy_n) + p(Sy_n, Sx_n), \\
p(x_n, Tx_n) \leq& p(x_n, Ty_n) + p(Ty_n, Tx_n), \\
p(y_n, Sy_n) =& p(W(Sx_n, Tx_n, x_n, \gamma_n, \rho_n), Sy_n) \\
\leq& \gamma_n p(Sx_n, Sy_n) + \rho_n p(Tx_n, Sy_n) \\
&+ (1 - \gamma_n - \rho_n) p(x_n, Sy_n) \\
\leq& p(Sx_n, Sy_n) + p(Tx_n, Sy_n) + p(x_n, Sy_n),
\end{aligned}$$

$$\begin{aligned}
p(y_n, Ty_n) \leq& p(Sx_n, Ty_n) + p(Tx_n, Ty_n) + p(x_n, Ty_n), \\
p(y_n, Sx_n) =& p(W(Sx_n, Tx_n, x_n, \gamma_n, \rho_n), Sx_n) \\
\leq& \gamma_n p(Sx_n, Sx_n) + \rho_n p(Tx_n, Sx_n) \\
&+ (1 - \gamma_n - \rho_n) p(x_n, Sx_n) \\
\leq& p(Sx_n, Sx_n) + p(Tx_n, Sx_n) \\
&+ p(x_n, Sy_n) + p(Sy_n, Sx_n), \\
p(y_n, Tx_n) \leq& p(Sx_n, Tx_n) + p(Tx_n, Tx_n) \\
&+ p(x_n, Ty_n) + p(Ty_n, Tx_n).
\end{aligned}$$

If $\max\{\limsup p(Sx_n, Sy_n), \limsup p(Tx_n, Ty_n), \limsup p(Sx_n, Ty_n),$ $\limsup p(Tx_n, Sy_n), \limsup p(Sy_n, Sx_n), \limsup p(Ty_n, Tx_n), \limsup$ $p(Ty_n, Sx_n), \limsup p(Sy_n, Tx_n)\} = r > 0$, then from the above inequalites and the assumptions of φ, it follows that

$$\begin{aligned} r =& \max\{\limsup p(Sx_n, Sy_n), \limsup p(Tx_n, Ty_n), \\ & \limsup p(Sx_n, Ty_n), \limsup p(Tx_n, Sy_n), \\ & \limsup p(Sy_n, Sx_n), \limsup p(Ty_n, Tx_n), \\ & \limsup p(Ty_n, Sx_n), \limsup p(Sy_n, Tx_n)\} \\ \leq& \varphi(2r, r, r, 0, 0, 2r, 2r, r, r) < r. \end{aligned}$$

This is a contradiction. Hence

$$\begin{aligned} \max\{ & \limsup p(Sx_n, Sy_n), \limsup p(Tx_n, Ty_n), \\ & \limsup p(Sx_n, Ty_n), \limsup p(Tx_n, Sy_n), \\ & \limsup p(Sy_n, Sx_n), \limsup p(Ty_n, Tx_n), \\ & \limsup p(Ty_n, Sx_n), \limsup p(Sy_n, Tx_n)\} \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows from $p(x_n, Sx_n) \leq p(x_n, Sy_n) + p(Sy_n, Sx_n)$ and $p(z, Sx_n) \leq p(z, x_n) + p(x_n, Sx_n)$ that $p(z, Sx_n) \to 0$ as $n \to \infty$. By similar methods, we obtain $p(z, Tx_n) \to 0$ as $n \to \infty$. Again from (3.9), we have

$$\begin{aligned} \max\{ & p(Sz, Sx_n), p(Tz, Tx_n), p(Sz, Tx_n), p(Tz, Sx_n), p(Sx_n, Sz), \\ & p(Tx_n, Tz), p(Tx_n, Sz), p(Sx_n, Tz)\} \\ \leq& \varphi(p(z, x_n), p(z, Sz), p(z, Tz), p(z, Sx_n), p(z, Tx_n), \\ & p(x_n, Sx_n), p(x_n, Tx_n), p(x_n, Sz), p(x_n, Tz)). \end{aligned}$$

Since

$$\begin{aligned} p(z, Sz) \leq& p(z, x_n) + p(x_n, Sx_n) + p(Sx_n, Sz), \\ p(z, Tz) \leq& p(z, x_n) + p(x_n, Tx_n) + p(Tx_n, Tz), \\ p(x_n, Sz) \leq& p(x_n, Sx_n) + p(Sx_n, Sz), \end{aligned}$$

and

$$p(x_n, Tz) \leq p(x_n, Tx_n) + p(Tx_n, Tz),$$

if $\max\{\limsup p(Sz, Sx_n)$, $\limsup p(Tz, Tx_n)$, $\limsup p(Sz, Tx_n)$, $\limsup p(Tz, Sx_n)$, $\limsup p(Sx_n, Sz)$, $\limsup p(Tx_n, Tz)$, $\limsup p(Tx_n$

$, Sz), \limsup p(Sx_n, Tz)\} = r > 0$, then from the above inequalities and these assumptions of φ, it follows that

$$r \leq (2r, r, r, 0, 0, 2r, 2r, r, r) < r.$$

This is a contradiction and so $\lim_{n\to\infty} p(Sz, Sx_n) = \lim_{n\to\infty} p(Tz, Tx_n) = \lim_{n\to\infty} p(Tz, Sx_n) = \lim_{n\to\infty} p(Sx_n, Sz) = \lim_{n\to\infty} p(Tx_n, Tz) = \lim_{n\to\infty} p(Tx_n, Sz) = \lim_{n\to\infty} p(Sx_n, Tz) = \lim_{n\to\infty} p(Sz, Tx_n) = 0$. If follows that $p(z, Sz) \leq p(z, Sx_n) + p(Sx_n, Sz) \longrightarrow 0, p(z, Tz) \leq p(z, Tx_n) + p(Tx_n, Tz) \longrightarrow 0$, and $p(z, z) \leq p(z, x_n) + p(x_n, z) \longrightarrow 0$, as $n \longrightarrow \infty$. Thus we obtain $p(z, Sz) = 0$, $p(z, Tz) = 0$ and $p(z, z) = 0$. By Lemma 2.5, we have $Sz = z = Tz$. □

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Department of Applied Mathematics
Changwon National University
Changwon 641-773, Korea
E-mail : jsume@sarim.changwon.ac.kr