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# A GEOMETRIC CRITERION FOR MEMBERSHIP IN NEW CLASSES $\mathbb{A}_{1,1}{}^2(r)$

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## 1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $T$ and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}_T$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^{\perp}\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by $Q_T$. One knows that $\mathcal{A}_T$ is the dual space of $Q_T$ and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, \quad [L] \in Q_T.$$

The Banach space $Q_T$ is called a *predual* of $\mathcal{A}_T$. For $x$ and $y$ in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x, \quad {}^{\forall}u \in \mathcal{H}$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbb{A}_{m,n}$ (to be defined in Section 2) were defined by Bercovici-Foias-Pearcy in [3]. Also these classes are closely related to the study of the theory of dual algebras.

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Especially, B. Chevreau and C. Pearcy [7] defined the properties $E^r_{\theta,\gamma}$ (to be defined in Section 2), and B. Chevreau, G. Exner and C. Pearcy [6] obtained some new sufficient conditions for membership in the class $\mathbb{A}_{1,\aleph_0}$ (to be defined in Section 2) concerning the properties $E^r_{\theta,\gamma}$. In this paper, we construct new classes and obtain a geometric criterion for membership in the classes $\mathbb{A}^l_{m,n}$ (to be defined in Section 3).

## 2. Notation and preliminaries

The notation and terminology employed herein agree with those in [4], [5], [7], [12]. We shall denote by $D$ the open unit disc in the complex plane $C$, and we write $\mathbb{T}$ for the boundary of $D$. The space $L^p = L^p(\mathbb{T}), 1 \leq p \leq \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure $m$ on $\mathbb{T}$. The space $H^p = H^p(\mathbb{T}), 1 \leq p \leq \infty$, is the usual Hardy space. It is well-known that the space $H^\infty$ is the dual space of $L^1/H^1_0$, where

$$(3) \qquad H^1_0 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \quad for \quad n = 0, 1, 2, \cdots\},$$

and the duality is given by the pairing

$$(4) \qquad \langle f, [g] \rangle = \int_T fgdm \quad \text{for} \quad f \in H^\infty, \quad [g] \in L^1/H^1_0.$$

Recall that any contraction $T$ can be written as a direct sum $T = T_1 \oplus T_2$, where $T_1$ is a completely nonunitary contraction and $T_2$ is a unitary operator. If $T_2$ is absolutely continuous or acts on the space $(0)$, $T$ will be called an *absolutely continuous contraction.* The following Foias-Sz.Nagy functional calculus provides a good relationship between the function space $H^\infty$ and a dual algebra $\mathcal{A}_T$.

THEOREM 2.1 [4, Theorem 4.1]. *Let $T$ be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*

(a) $\Phi_T(1) = 1_{\mathcal{H}}, \quad \Phi_T(\xi) = T,$

(b) $\|\Phi_T(f)\| \leq \|f\|_\infty, \quad f \in H^\infty,$

(c) *$\Phi_T$ is continuous if both $H^\infty$ and $\mathcal{A}_T$ are given their weak* topologies,* (*d*) *the range of* $\Phi_T$ *is weak* dense in* $\mathcal{A}_T$,

(*e*) *there exists a bounded, linear, one-to-one map* $\phi_T : Q_T \to L^1/H_0^1$ *such that* $\phi_T{}^* = \Phi_T$, *and*

(*f*) *if* $\Phi_T$ *is an isometry, then* $\Phi_T$ *is a weak* homeomorphism of* $H^\infty$ *onto* $\mathcal{A}_T$ *and* $\phi_T$ *is an isometry of* $Q_T$ *onto* $L^1/H_0^1$.

DEFINITION 2.2 [3]. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let $m$ and $n$ be any cardinal numbers such that $1 \le m, n \le \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{i,j}], \quad 0 \le i < m, 0 \le j < n, \tag{5}$$

where $\{[L_{i,j}]\}_{\substack{0\le i<m\\ 0\le j<n}}$ is an arbitrary $m\times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0\le i<m}, \{y_j\}_{0\le j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m$ and $n$ are positive integers and $r$ is a fixed real number satisfying $r \ge 1$, a dual algebra $\mathcal{A}$ (with property $(\mathbb{A}_{m,n})$) is said to have property $(\mathbb{A}_{m,n}(r))$ if for every $s > r$ and every $m\times n$ array $\{[L_{i,j}]\}_{\substack{0\le i<m\\ 0\le j<n}}$ from $Q_{\mathcal{A}}$ such that the rows and columns of the matrix $([L_{i,j}])$ are summable, there exist sequences $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ from $\mathcal{H}$ that satisfy (5) and also satisfy the following conditions:

$$\|x_i\|^2 \le s \sum_{0\le j<n} \|[L_{ij}]\|, \ 0 \le i < m, \tag{6a}$$

and

$$\|y_j\|^2 \le s \sum_{0\le i<m} \|[L_{ij}]\|, \ 0 \le j < n. \tag{6b}$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathbb{A}_{m,\aleph_0}(r))$ ( for some real number $r \ge 1$) if for every $s > r$ and every array $\{[L_{i,j}]\}_{\substack{0\le i<m\\ 0\le j<\infty}}$ from $Q_{\mathcal{A}}$ with summable rows, there exist sequences $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<\infty}$ from $\mathcal{H}$ that satisfy (5) and $(6a, b)$ with the replacement of $n$ by $\aleph_0$. Properties $(\mathbb{A}_{\aleph_0,n}(r))$ and $(\mathbb{A}_{\aleph_0,\aleph_0}(r))$ are defined similary. For brief notation, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. Furthermore, if $m$ and $n$ are cardinal numbers such that $1 \le m, n \le \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all $T$ in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$.

DEFINITION 2.3 [7]. Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $0 \leq \theta < \gamma \leq 1$. We denote by $\mathcal{E}_\theta^r(\mathcal{A})$ (resp. $\mathcal{E}_\theta^l(\mathcal{A})$) the set of all $[L]$ in $Q_\mathcal{A}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from $\mathcal{H}$ satisfying

($a$) $\limsup_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \leq \theta$,

($b$) $\|x_i\| \leq 1,\ \|y_i\| \leq 1,\ 1 \leq i < \infty$,

($c^r$) $\|[x_i \otimes z]\| \to 0$ for all $z$ in $\mathcal{H}$(*resp.*($c^l$)$\|[z \otimes y_i]\| \to 0$ for all $z$ in $\mathcal{H}$), and

($d^r$) $\{y_i\}$ converges weakly to zero (resp. ($d^l$) $\{x_i\}$ converges weakly to zero).

For $0 \leq \theta < \gamma \leq 1$, the dual algebra $\mathcal{A}$ is said to have property $E_{\theta,\gamma}^r$ (resp. $E_{\theta,\gamma}^l$) if the closed absolutely convex hull of the set $\mathcal{E}_\theta^r(\mathcal{A})$ (resp. $\mathcal{E}_\theta^r(\mathcal{A})$) contains the closed ball $B_{0,\gamma}$ of radius $\gamma$ centered at the origin in $Q_\mathcal{A}$:

$$\overline{aco}(\mathcal{E}_\theta^r(\mathcal{A})) \supset \{[L] \in Q_\mathcal{A} : \|[L]\| \leq \gamma\} = B_{0,\gamma}. \tag{7}$$

$$(resp.\quad \overline{aco}(\mathcal{E}_\theta^l(\mathcal{A})) \supset B_{0,\gamma})$$

To establish our results, it will be convenient to use the minimal coisometric extension theorem [12]: every contraction $T$ in $\mathcal{L}(\mathcal{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence. Given such $T$ and $B$, one knows that there exists a canonical decomposition of the isometry $B^*$ as

$$B^* = S \oplus R^* \tag{8}$$

corresponding to a decomposition of the space

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R}, \tag{9}$$

where, if $\mathcal{S} \neq (0)$, $S$ is a unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \neq (0)$, $R$ is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be (0). ([7])

LEMMA 2.4 [7, Lemma 3.2]. *If $T$ is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ with minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, and the subspace $\mathcal{R}$ of $\mathcal{K}$ in (9) is nonzero, then the unitary operator $R$ in (8) is absolutely continuous.*

LEMMA 2.5 [7, Lemma 3.5]. *Suppose $T \in \mathbb{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$. Then $B \in \mathbb{A}(\mathcal{K}), \Phi_T \circ \Phi_B^{-1}$ is an isometry and weak* homeomorphism from $\mathcal{A}_B$ onto $\mathcal{A}_T$, and $j = \varphi_B^{-1} \circ \varphi_T$ is a linear isometry of $Q_T$ onto $Q_B$. Moreover,*

$$j([C_\lambda]_T) = [C_\lambda]_B, \quad \lambda \in D, \tag{10}$$

*and*

$$j([x \otimes y]_T) = [x \otimes y]_B, \quad x, y \in \mathcal{H}. \tag{11}$$

LEMMA 2.6 [7, Lemma 3.6]. *If $T$ belongs to $\mathbb{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K}), x, y \in \mathcal{H}$, and $w, z \in \mathcal{K}$, then*

$$\|[x \otimes y]_T\| = \|[x \otimes y]_B\|, \tag{12}$$

$$[x \otimes z]_B = [x \otimes Pz]_B, \tag{13}$$

*and*

$$[w \otimes z]_B = [Qw \otimes Qz]_B + [Aw \otimes Az]_B. \tag{14}$$

LEMMA 2.7 [7, Lemma 3.7]. *Suppose $T \in \mathbb{A}(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, and $\{x_n\}_{n=1}^\infty$ is a sequence from $\mathcal{H}$ such that*

$$\|[x_n \otimes y]_T\| \to 0, \quad {}^\forall y \in \mathcal{H}, \tag{15}$$

*then we have*

$$\|[x_n \otimes z]_B\| \to 0, \quad {}^\forall z \in \mathcal{K}, \tag{16}$$

$$\|[Qx_n \otimes z]_B\| \to 0, \quad {}^\forall z \in \mathcal{K}, \tag{17}$$

*and*

$$\|[Ax_n \otimes z]_B\| \to 0, \quad {}^\forall z \in \mathcal{K} \tag{18}$$

LEMMA 2.8 [7, Lemma 3.8]. *Suppose $T \in \mathbb{A}(\mathcal{H})$ and has $B$ in $\mathcal{L}(\mathcal{K})$ for its minimal coisometric extension. If $\{z_n\}$ is any sequence in $\mathcal{K}$ that converges weakly to zero, then*

$$\|[w \otimes z_n]_B\| \to 0, \quad {}^{\forall} w \in \mathcal{S}. \tag{19}$$

Suppose $U$ is an absolutely continuous unitary operator in $\mathcal{L}(\mathcal{N})$ with spectral measure $E_U$, and let $\mu$ be a scalar spcetral measure for $U$. Then one knows, via the absolute continuity, that there exists a Borel set $\Sigma \subset \mathbb{T}$ such that $\mu$ is equivalent to Lebesgue measure $m|_\Sigma$ (where this measure is defined to be zero on Borel subsets of $\mathbb{T}\backslash\Sigma$). For any vectors $x$ and $y$ in $\mathcal{N}$, let us denote by $\mu_{x,y}$ the complex measure on $\mathbb{T}$ defined by

$$\mu_{x,y}(\mathcal{B}) = (E_U(\mathcal{B})x, y) \tag{20}$$

for every Borel subset $\mathcal{B}$ of $\mathbb{T}$. Obviously all of these complex measures $\mu_{x,y}$ are absolutely continuous with respect to the measure $m|_\Sigma$. Therefore, for each pair $x, y \in \mathcal{N}$, there is a function in $L^1(\Sigma)$, which is denote by $x \overset{U}{\cdot} y$ or $x \cdot y$, that is the Radon- Nikodym derivatives of $\mu_{x,y}$ with respect to $m|_\Sigma$. We thus have, of course,

$$(l(U)x, y) = \int_{\mathbb{T}} l \quad d\mu_{x,y} = \int_{\Sigma} l\{x \cdot y\} dm, \quad l \in L^\infty(\Sigma). \tag{21}$$

LEMMA 2.9 [7, Lemma 3.9]. *Suppose $T \in \mathbb{A}(\mathcal{H})$ and has $B = S^* \oplus R$ as its minimal coisometric extension, with $\mathcal{R} \neq (0)$. Then, for every pair of vectors $w, z \in \mathcal{R}$, we have*

$$[w \overset{R}{\cdot} z] = \varphi_B([w \otimes z]_B). \tag{22}$$

LEMMA 2.10. *For $k = 1, 2$, suppose $T_k$ belongs to $\mathbb{A}(\mathcal{H})$ and has minimal coisometric extension $B_k = S_k^* \oplus R_k$ in $\mathcal{L}(\mathcal{K})$ with $\mathcal{R}_k \neq (0)$. Let $\Sigma_k \subset \mathbb{T}$ and $\mathcal{R}_{0,k} \subset \mathcal{R}_k$ be as in* [7, Proposition 3.10], *and denote the projection of $\mathcal{K}$ onto $\mathcal{R}_{0,k}$ by $A_{0,k}$. Let also $\epsilon$ and $\rho$ be arbitrary* *real numbers such that $\epsilon > 0$ and $0 < \rho < 1$. If $a_1 \in \mathcal{H}, b_k \in \mathcal{R}_k$ and $h_k \in L^1(\Sigma_k)$ are given, and we write $h_{1,k} = (A_k a_1 \overset{R}{\cdot} b_k) + h_k$, then there exist, for each $k = 1, 2,$ $u_k \in \mathcal{H}$ and $c_k \in \mathcal{R}_k$ such that*

$$\|h_{1,k} - A_k(a_1 + u_k) \overset{R}{\cdot} c_k\|_1 < \epsilon, \tag{23}$$

$$\|Qu_k\| < \epsilon, \tag{24}$$

$$\|(A_k - A_{0,k})u_k\| < \epsilon, \tag{25}$$

$$\|u_k\| \leq 2\|h_k\|_1^{1/2}, \tag{26}$$

$$\|c_k\| \leq (1/\rho)\{\|b_k\| + \|h_k\|_1^{1/2}\}, \tag{27}$$

*and*

$$c_k - b_k \in \mathcal{R}_{0,k}, \tag{28}$$

*where the notation $\| \quad \|_1$ indicates the norm on $L^1(\Sigma)$.*

*Proof.* It is clear from [7, Theorem 3.11].

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions $T$ in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$, and $\mathbb{N}$ is denoted by the set of all natural numbers.

LEMMA 2.11 [8, Theorem 2.1]. *Suppose $\{T_k\}_{k=1}^{\infty}$ is any sequence of operators contained in the class $\mathbb{A}_{\aleph_0} \cap C_{\cdot 0}$, $\{[L_k]_{T_k}\}_{k=1}^{\infty}$ is an arbitrary sequence (where $[L_k]_{T_k} \in Q_{T_k}$), and $\{\epsilon_k\}_{k=1}^{\infty}$ is any sequence of positive numbers. Then there exists a dense set $\mathcal{D} \subset \mathcal{H}$ such that for every $x$ in $\mathcal{D}$, there exists a sequence $\{y_k^x\}_{k=1}^{\infty} \subset \mathcal{H}$ satisfying*

$$[x \otimes y_k^x]_{T_k} = [L_k]_{T_k}, \quad k \in \mathbb{N}, \tag{29}$$

*and*

$$\|y_k^x\| > \epsilon_k, \quad k \in \mathbb{N}. \tag{30}$$

## 3. Main results

From the idea of lemma 2.11, we construct new classes as following:

DEFINITION 3.1. Let $m, n$ and $l$ be any cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$. We denote by $\mathbb{A}^{l}_{m,n}(\mathcal{H})$ the class of all sets $\{T_k\}_{k=1}^{l}$ such that $T_k$ belong to $\mathbb{A}(\mathcal{H})$ for all $k = 1, 2, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}, \tag{31}$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $Q_{T_k}$ for each $1 \leq k \leq l$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m$ and $n$ are positive integers and $r$ is a fixed real number satisfying $r \geq 1$, then we denoted by $(\mathbb{A}^{l}_{m,n}(r))$ the class of all sets $\{T_k\}_{k=1}^{l}$ such that $T_k$ belong to $\mathbb{A}(\mathcal{H})$ for all $k = 1, 2, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form (31) has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ consisting of a pair of sequences of vectors from $\mathcal{H}$ and also satisfy the following conditions:

$$\|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}^{(k)}]_{T_k}\|, \ 0 \leq i < m, \ 1 \leq k \leq l \tag{32}$$

and

$$\|y_j^{(k)}\|^2 \leq s \sum_{0 \leq i < m} \|[L_{ij}^{(k)}]_{T_k}\|, \ 0 \leq j < n, \ 1 \leq k \leq l. \tag{33}$$

REMARK 3.2. If $\{T_k\}_{k=1}^{\infty}$ are in the class $\mathbb{A}_{\aleph_0} \cap C_0$, then $\{T_k\}_{k=1}^{\infty} \in \mathbb{A}^{\aleph_0}_{1,1}$, by lemma 2.11.

LEMMA 3.3. *Suppose* $m, n$ *and* $l$ *are cardinal numbers such that* $1 \le m, n, l \le \aleph_0$ *and* $T_k \in \mathbb{A}(\mathcal{H})$ *has minimal coisometric extension* $B_k$ *in* $\mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)$ *for* $k$, $1 \le k \le l$. *Then* $\{T_k\}_{k=1}^l \in \mathbb{A}_{m,n}^l$ *if and only if for* $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m\\0\le j<n}} \subset Q_{T_k}, 1 \le k \le l$, *there exists a Cauchy sequence* $\{x_{i,p}\}_{p=1}^\infty$ *in* $\mathcal{H}$ *and sequences* $\{w_{j,p}^{(k)}\}_{p=1}^\infty$ *in* $\mathcal{S}_k$ *and* $\{b_{j,p}^{(k)}\}_{p=1}^\infty$ *in* $\mathcal{R}_k$ *such that* $\{w_{j,p}^{(k)} + b_{j,p}^{(k)}\}$ *is bounded and* $\|(\varphi_{B_k}^{-1} \circ \varphi_{T_k})([L_{ij}^{(k)}]_{T_k}) - [x_{i,p} \otimes (w_{j,p}^{(k)} + b_{j,p}^{(k)})]_{B_k}\| \to 0$.

*Proof.* It is clear from [7, Proposition 4.7].

CONVENTION. *In the following theorems we assume that* $\mathcal{R}_k$ *are either simultaneously* $(0)$ *or not* $(0)$.

THEOREM 3.4. *For* $k = 1, 2$, *suppose that* $T_k (\in \mathbb{A}(\mathcal{H}))$ *has minimal coisometric extension* $B_k$ *in* $C_0(\mathcal{K})$, *and* $\mathcal{A}_{T_k}$ *has property* $E_{\theta,\gamma}^r$ *for some* $0 < \theta < \gamma \le 1$. *Suppose also that, for each* $k = 1, 2$, $0 < \rho < 1$, $[L_k] \in Q_{B_k}$, $a \in \mathcal{H}$, $w_k \in \mathcal{S}_k$, $b_k \in \mathcal{R}_k$, *and* $\delta > 0$ *are given such that*

$$\max_k\{\|[L_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}\|\} < \delta. \tag{34}$$

*Then there exist* $\hat{a} \in \mathcal{H}$, $\hat{w_k} \in \mathcal{S}_k$, $\hat{b_k} \in \mathcal{R}_k$, $k = 1, 2$, *such that*

$$\max_{k=1,2}\{\|[L_k]_{B_k} - [\hat{a} \otimes (\hat{w_k} + \hat{b_k})]_{B_k}\|\} < (\theta/\gamma)\delta, \tag{35}$$

*and*

$$\begin{aligned} &\|\hat{a} - a\| < 6(\delta/\gamma)^{1/2}, \quad \|\hat{w_k} - w_k\| < (\delta/\gamma)^{1/2} \\ &\|\hat{b_k}\| < 1/\rho\{\|b_k\| + (\delta/\gamma)^{1/2}\}, \qquad k = 1, 2. \end{aligned} \tag{36}$$

*Proof.* Of course, either of the spaces $\mathcal{S}_k$ or $\mathcal{R}_k$ may be zero, for all $k$, but the proof is unchanged in these special cases. Let

$$[D_k]_{B_k} = [L_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k} \tag{37}$$

and set $d = \max_k\{\|[D_k]_{B_k}\|\}$, so, $0 \leq d < \delta$. We may assume that $d > 0$, since otherwise we can simply take $\hat{a} = a, \hat{w_k} = w_k$ and $\hat{b_k} = b_k$ for each $k = 1, 2$. And, we choose $\epsilon > 0$ such that

$$(\theta/\gamma)d + \epsilon < (\theta/\gamma)\delta. \tag{38}$$

With $j$ as in Lemma 2.5, note that $\|(\gamma/d)j^{-1}([D_k]_{B_k})\| < \gamma$, and thus, by hypothesis, for each $k = 1, 2$, there exist $N \in \mathbb{N}$, elements $[P_{1,k}], \cdots, [P_{N,k}]$ from $\mathcal{E}_\theta^r(\mathcal{A}_{T_k})$, and scalars $\tilde{\alpha}_{1,k}, \cdots, \tilde{\alpha}_{N,k}$ such that

$$\|(\gamma/d)j^{-1}([D_k]_{B_k}) - \sum_{i=1}^{N}\tilde{\alpha}_{i,k}[P_{i,k}]_{T_k}\| < (\epsilon/2)(\gamma/d), \tag{39}$$

and $\sum_{i=1}^{N}|\tilde{\alpha}_{i,k}| < 1, k = 1, 2$. Upon setting $\alpha_{i,k} = (d/\gamma)\tilde{\alpha}_{i,k}$, for each $i, k$, we obtain, by multiplying (39) by $d/\gamma$,

$$\|j^{-1}([D_k]_{B_k}) - \sum_{i=1}^{N}\alpha_{i,k}[P_{i,k}]_{T_k}\| < (\epsilon/2), \quad k = 1, 2, \tag{40}$$

and

$$\sum_{i=1}^{N}|\alpha_{i,k}| < d/\gamma, \quad k = 1, 2. \tag{41}$$

For each $i = 1, \cdots, N$, by definition of $\mathcal{E}_\theta^r(T_k)$, there exist sequences $\{x_{n_i}^{(i,k)}\}_{n_i=1,k=1}^{\infty,2}$ and $\{y_{n_i}^{(i,k)}\}_{n_i=1,k=1}^{\infty,2}$ in the unit ball of $\mathcal{H}$ such that

$$\|[P_{i,k}]_{T_k} - [x_{n_i}^{(i,k)} \otimes y_{n_i}^{(i,k)}]_{T_k}\| < \theta + (\epsilon/2)(\gamma/d), \quad n_i \in \mathbb{N}, \tag{42}$$

$$\lim_{n_i \to \infty}\|[x_{n_i}^{(i,k)} \otimes z]_{T_k}\| = 0, \quad {}^\forall z \in \mathcal{H}, \tag{43}$$

and

$$\{y_{n_i}^{(i,k)}\}_{n_i=1}^{\infty} \tag{44}$$

converges weakly to zero for each $k = 1, 2$. By (40) and (42), we get, for any choice of the $N$-tuple $\nu = (n_1, \cdots, n_N)$,

$$\begin{aligned}(45)\qquad &\|j^{-1}([D_k]_{B_k}) - \sum_{i=1}^{N} \alpha_{i,k}[x_{n_i}^{(i,k)} \otimes y_{n_i}^{(i,k)}]_{T_k}\| \\ &< \epsilon/2 + (d/\gamma)\{\theta + (\epsilon/2)(\gamma/d)\} = \epsilon + (d\theta/\gamma)\end{aligned}$$

and, we obtain, using (11),

$$(46)\qquad \|[D_k]_{B_k} - \sum_{i=1}^{N} \alpha_{i,k}[x_{n_i}^{(i,k)} \otimes y_{n_i}^{(i,k)}]_{B_k}\| < \epsilon + (d\theta/\gamma)$$

for every choice of $\nu$. Take $\tau > 0$ such that

$$(47)\qquad (\theta\delta/\gamma) - \{(d\theta/\gamma) + \epsilon\} = 5\tau.$$

Using (14) and (37) we may combine (46) and (47) to yield

$$\begin{aligned}(48)\qquad &\|[L_k]_{B_k} - [Qa \otimes w_k]_{B_k} - (A)\| \\ &< (\theta\delta/\gamma) - 5\tau,\end{aligned}$$

where

$$(A) = \sum_{i=1}^{N} \alpha_{i,k}[Qx_{n_i}^{(i,k)} \otimes Qy_{n_i}^{(i,k)}]_{B_k} - [M_k(\nu)]_{B_k}\|$$

and

$$(49)\qquad [M_k(\nu)]_{B_k} = [Aa \otimes b_k]_{B_k} + \sum_{i=1}^{N} \alpha_{i,k}[Ax_{n_i}^{(i,k)} \otimes Ay_{n_i}^{(i,k)}]_{B_k}$$

for every choice of $\nu$. Let us define, for arbitrary $\nu = (n_1, \cdots, n_N)$,

$$(50)\qquad u_\nu = \sum_{k=1}^{2} \sum_{i=1}^{N} \beta_i^{(k)} x_{n_i}^{(i,k)}, \qquad v_\nu^{(k)} = \sum_{i=1}^{N} \overline{\beta_i^{(k)}} y_{n_i}^{(i,k)},$$

where $(\beta_i^{(k)})^2 = \alpha_{i,k}$ for $i = 1, \cdots, N,\ k = 1, 2$. Then, for every choice of $\nu$,

$$\begin{aligned} &[Q(a+u_\nu)\otimes(w_k+Qv_\nu^{(k)})]_{B_k} \\ &= [Qa\otimes w_k]_{B_k} + [Qu_\nu\otimes w_k]_{B_k} + [Qa\otimes Qv_\nu^{(k)}]_{B_k} \\ &\qquad + [Qu_\nu\otimes Qv_\nu^{(k)}]_{B_k}, \quad k=1,2, \end{aligned} \tag{51}$$

and

$$\begin{aligned} &\|[Qu_\nu\otimes Qv_\nu^{(k)}]_{B_k}\| \\ &\le \sum_{i=1}^{N}|\alpha_{i,k}|\|[Qx_{n_i}^{(i,k)}\otimes Qy_{n_i}^{(i,k)}]_{B_k}\| \\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^{N}|\beta_i^{(k)}\beta_j^{(k)}|\|[Qx_{n_i}^{(i,k)}\otimes Qy_{n_j}^{(j,k)}]_{B_k}\| \\ &+ \sum_{k_1=1}^{2}\sum_{i,j=1}^{N}|\beta_i^{(k_1)}\beta_j^{(k)}|\|[Qx_{n_i}^{(i,k_1)}\otimes Qy_{n_j}^{(j,k)}]_{B_k}\|, \quad k=1,2. \end{aligned} \tag{52}$$

Thus we see from (51), (52), and $B_k \in C_0,\ k = 1, 2$, it suffices to choose the indices $n_1^o, \cdots, n_N^o$ ( one at a time, in the indicated order ) sufficiently large that for $\nu_o = (n_1^o, \cdots, n_N^o)$ the following properties are valid:

$$\|[Qa\otimes Qv_{\nu_o}^{(k)}]_{B_k}\| < \tau/4, \tag{53}$$

$$\|[Qu_{\nu_o}\otimes w_k]_{B_k}\| < \tau/4, \tag{54}$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{N}|\beta_i^{(k)}\beta_j^{(k)}|\|[Qx_{n_i^o}^{(i,k)}\otimes Qy_{n_j^o}^{(j,k)}]_{B_k}\| < \tau/4, \tag{55}$$

$$\sum_{k_1=1}^{2}\sum_{i,j=1}^{N}|\beta_i^{(k_1)}\beta_j^{(k)}|\|[Qx_{n_i^o}^{(i,k_1)}\otimes Qy_{n_j^o}^{(j,k)}]_{B_k}\| < \tau/4, \tag{56}$$

$$\|[Au_{\nu_o} \otimes b_k]_{B_k}\| < \tau, \tag{57}$$

and

$$\|u_{\nu_o}\|^2 < 2\delta/\gamma, \quad \|v_{\nu_o}^{(k)}\|^2 < \delta/\gamma, \quad k = 1, 2. \tag{58}$$

Therefore, by combining (51)-(55), we obtain for each $k = 1, 2$,

$$\begin{aligned} \|[Qa \otimes w_k]_{B_k} + \sum_{i=1}^{N} \alpha_{i,k} [Qx_{n_{i,o}}^{(i,k)} \otimes Qy_{n_{i,o}}^{(i,k)}]_{B_k} \\ - [Q(a + u_{\nu_o}) \otimes (w_k + Qv_{\nu_o}^{(k)})]_{B_k}\| < \tau. \end{aligned} \tag{59}$$

We next define

$$a_1 = a + u_{\nu_o}, \quad \tilde{w}_k = w_k + Qv_{\nu_o}^{(k)}, \quad k = 1, 2, \tag{60}$$

and conclude from (60), (59) and (48) that

$$\|[L_k]_{B_k} - [Qa_1 \otimes \tilde{w}_k]_{B_k} - [M_k(\nu_o)]_{B_k}\| < (\theta\delta/\gamma) - 4\tau, \quad k = 1, 2. \tag{61}$$

Moreover, if in $[M_k(\nu_o)]_{B_k}$ we replace $a$ by $a_1$, and so define, for $k = 1, 2$,

$$[M_k^{(1)}(\nu_o)]_{B_k} = [Aa_1 \otimes b_k]_{B_k} + \sum_{i=1}^{N} [Ax_{n_{i,o}}^{(i,k)} \otimes Ay_{n_{i,o}}^{(i,k)}]_{B_k}, \tag{62}$$

then by (49), (57), (60), and (61) we have

$$\|[L_k]_{B_k} - [Qa_1 \otimes \tilde{w}_k]_{B_k} - [M_k^{(1)}(\nu_o)]_{B_k}\| < (\theta\delta/\gamma) - 3\tau, \quad k = 1, 2. \tag{63}$$

Now suppose that $\mathcal{R}_k = (0)$, for all $k = 1, 2$. Then $b_k = 0$, $[M_k^{(1)}(\nu_o)]_{B_k} = 0, Qa_1 = a_1$, and

$$\|[L_k]_{B_k} - [a_1 \otimes \tilde{w}_k]_{B_k}\| < (\theta\delta/\gamma) - 3\tau, \quad k = 1, 2.$$

Then, by (60) and (58), we have

$$\|a - a_1\| < (2\delta/\gamma)^{1/2} \quad and \quad \|w_k - \tilde{w}_k\| < (\delta/\gamma)^{1/2}, \quad k = 1, 2,$$

so (with $\tilde{b}_k = 0$) the proof in this case is complete.
Hence we may suppose that $\mathcal{R}_k \neq (0)$, $k = 1, 2$, we let $\Sigma_k \subset \mathbb{T}$ be as in Lemma 2.10, and we prepare to apply Lemma 2.10 to deal with the term $[M_k^{(1)}(\nu_o)]_{B_k}$ in (63). By (62) and Lemma 2.9 we have

$$(64) \qquad \varphi_{B_k}([M_k^{(1)}(\nu_o)]_{B_k}) = [Aa_1 \overset{R}{\cdot} b_k] + \sum_{i=1}^{N} \alpha_{i,k}[Ax_{n_i o}^{(i,k)} \overset{R}{\cdot} Ay_{n_i o}^{(i,k)}].$$

Thus we define the function $h_k$, $k = 1, 2$, in $L^1(\Sigma_k)$ to be

$$h_k = \sum_{i=1}^{N} \alpha_{i,k}(Ax_{n_i o}^{(i,k)} \overset{R}{\cdot} Ay_{n_i o}^{(i,k)}).$$

We note from (21) and (41) that $\|h_k\|_1 \leq \delta/\gamma$, and we set $\epsilon' = \{\tau/(2(\|w'\| + 1))\}(< \tau)$ where $\|w'\| = \max_{k=1,2} \|\tilde{w}_k\|$. With $a_1$ and $b_k, k = 1, 2$, as in (64), an application of Lemma 2.10 yields the existence of $\tilde{u}_k \in \mathcal{H}$ and $c_k \in \mathcal{R}_k$, $k = 1, 2$, such that

$$(65) \qquad \|Aa_1 \overset{R}{\cdot} b_k + \sum_{i=1}^{N} \alpha_{i,k}(Ax_{n_i o}^{(i,k)} \overset{R}{\cdot} Ay_{n_i o}^{(i,k)}) - A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \overset{R}{\cdot} c_k\|_1 < \epsilon' + \tau < 2\tau, \quad k = 1, 2,$$

$$(66) \qquad \|Q(\sum_{k=1}^{2} \tilde{u}_k)\| < \tau/(\|w'\| + 1),$$

$$(67) \qquad \|\sum_{k=1}^{2} \tilde{u}_k\| \leq 4(\delta/\gamma)^{1/2},$$

$$(68) \quad \|c_k\| \leq (1/\rho)\{\|b_k\| + \|h_k\|_1^{1/2}\} < \{\|b_k\| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2.$$

Since $L^1(\Sigma_k) \subset L^1(\mathbb{T})$ and the norm in $L^1(\mathbb{T})$ dominates the norm in $(L^1/H_0^1)(\mathbb{T})$, we obtain using (62), (22), and (65),

$$\|[M_k^{(1)}(\nu_o)]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]_{B_k}\| < 2\tau, \quad k = 1, 2. \tag{69}$$

Thus from (63) and (69) we get

$$\begin{aligned} &\|[L_k]_{B_k} - [Qa_1 + \tilde{w}_k]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]_{B_k}\| \\ &\leq \|[L_k]_{B_k} - [Qa_1 + \tilde{w}_k]_{B_k} - [M_k^{(1)}(\nu_o)]_{B_k}\| \\ &\qquad + \|[M_k^{(1)}(\nu_o)]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]_{B_k}\| \\ &< (\theta\delta/\gamma) - 3\tau + 2\tau = (\theta\delta/\gamma) - \tau, \quad k = 1, 2, \end{aligned} \tag{70}$$

and since, by (66), we have

$$\begin{aligned} \|[Q(\sum_{k=1}^{2} \tilde{u}_k) \otimes \tilde{w}_k]_{B_k}\| &\leq \|Q(\sum_{k=1}^{2} \tilde{u}_k)\| \cdot \|\tilde{w}_k\| \\ &< (\tau/(\|w'\| + 1))\|w'\| < \tau, \quad k = 1, 2, \end{aligned} \tag{71}$$

the inequality (70) yields

$$\begin{aligned} &\|[L_k]_{B_k} - [Q(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes \tilde{w}_k]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]_{B_k}\| \\ &< (\theta\delta/\gamma), \quad k = 1, 2. \end{aligned} \tag{72}$$

Since $\tilde{w}_k \in \mathcal{S}_k$ and $c_k \in \mathcal{R}_k$, by using (14) one can rewrite (72) as

$$\|[L_k]_{B_k} - [(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes (\tilde{w}_k + c_k)]_{B_k}\| < (\theta\delta/\gamma), \quad k = 1, 2.$$

So if we define

$$\hat{a} = a_1 + \sum_{k=1}^{2} \tilde{u}_k = a + u_{\nu_o} + \tilde{u}_1 + \tilde{u}_2$$

$$\hat{b}_k \doteq c_k, \quad \hat{w}_k = \tilde{w}_k, \quad k = 1, 2,$$

then (35) is satisfied. Moreover,

$$\|\hat{a} - a\| \le \|u_{\nu_o}\| + \|\sum_{k=1}^{2} \tilde{u}_k\| < (2\delta/\gamma)^{1/2} + 4(\delta/\gamma)^{1/2} = 6(\delta/\gamma)^{1/2},$$

from (58) and (67), so the first inequality in (35) is satisfied. Furthermore, from (60) and (58) we have

$$\|\hat{w}_k - w_k\| \le \|Qv_{\nu_o}^{(k)}\| < (\delta/\gamma)^{1/2}, \quad k = 1, 2.$$

Finally,

$$\|\hat{b}_k\| = \|c_k\| < (1/\rho)\{\|b_k\| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2.$$

We are ready to prove main theorems.

THEOREM 3.5. *For $k = 1, 2$, suppose that $T_k(\in \mathbb{A}(\mathcal{H}))$ has minimal coisometric extension $B_k$ in $C_0(\mathcal{K})$, and $\mathcal{A}_{T_k}$ has property $E^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$. Suppose also that $\delta > 0$, $[L_k] \in Q_{T_k}$, $a \in \mathcal{H}$, $w_k \in \mathcal{S}_k$ and $b_k \in \mathcal{R}_k$ are given such that*

$$\max_{k=1,2}\{\|[L_k]_{T_k} - [a \otimes \mathbb{P}(w_k + b_k)]_{T_k}\|\} < \delta, \tag{73}$$

*where $\mathbb{P}$ is the projection of $\mathcal{K}$ onto the subspace $\mathcal{H}$. Then there exist $\hat{a} \in \mathcal{H}$, $\hat{w}_k \in \mathcal{S}_k$ and $\hat{b}_k \in \mathcal{R}_k$ such that*

$$[L_k]_{T_k} = [\hat{a} \otimes \mathbb{P}(\hat{w}_k + \hat{b}_k)]_{T_k}, \quad k = 1, 2, \tag{74}$$

$$\|\hat{a} - a\| < 6(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \tag{75}$$

$$\|\hat{w}_k - w_k\| < (\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad k = 1, 2, \tag{76}$$

*and*

$$\|\hat{b}_k\| < 2\|b_k\| + 2(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad k = 1, 2 \tag{77}$$

*Proof.* Since if $\mathcal{A}_{T_k}$ has property $E^{\tau}_{0,\gamma}$ for each $k$, it also has property $E^{\tau}_{\theta,\gamma}$ for all $0 < \theta < \gamma$, the right-hand side of (75), (76), and (77) are continuous functions of $\theta$ and $\delta$, it suffices to treat the case $0 < \theta < \gamma$. Suppose now that (73) holds, let $\{s_n\}$ be a sequence of positive numbers strictly decreasing to 3/4 such that $s_1 = 1$, and define $\rho_n = (s_{n+1}/s_n)$, $n \in \mathbb{N}$. Set

$$[\hat{L}_k]_{B_k} = \varphi_{B_k}^{-1} \circ \varphi_{T_k}([L_k]), \quad k = 1, 2.$$

Then we have, by (73), (11), (12), and (13),

$$\max_{k=1,2}\{\|[\hat{L}_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}\|\} < \delta. \tag{78}$$

We now set

$$a = a_1, \; w_k = w_{1,k}, \; b_k = b_{1,k}, \quad k = 1, 2,$$

and apply Theorem 3.4 to obtain $a_2 \in \mathcal{H}$, $w_{2,k} \in \mathcal{S}_k$ and $b_{2,k} \in \mathcal{R}_k$, $\quad k = 1, 2$, such that

$$\max_{k=1,2}\{\|[\hat{L}_k]_{B_k} - [a_2 \otimes (w_{2,k} + b_{2,k})]_{B_k}\|\} < (\theta/\gamma)\delta, \tag{79}$$

$$\begin{aligned} &\|a_2 - a_1\| < 6(\delta/\gamma)^{1/2}, \quad \|w_{2,k} - w_{1,k}\| < (\delta/\gamma)^{1/2}, \\ &\|b_{2,k}\| < (1/\rho_1)\{\|b_{1,k}\| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2. \end{aligned} \tag{80}$$

Suppose now that vectors $\{a_p\}_{p=1}^n$ in $\mathcal{H}$, $\{w_{p,k}\}_{p=1}^n$ in $\mathcal{S}_k$, and $\{b_{p,k}\}_{p=1}^n$ in $\mathcal{R}_k$, have been chosen so that for $p = 2, \cdots, n$, $\quad k = 1, 2$,

$$\max_{k=1,2}\{\|[\hat{L}_k]_{B_k} - [a_p \otimes (w_{p,k} + b_{p,k})]_{B_k}\|\} < (\theta/\gamma)^{p-1}\delta, \tag{81$_p$}$$

$$\|a_p - a_{p-1}\| < 6(\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2}, \tag{82$_p$}$$

$$\|w_{p,k} - w_{p-1,k}\| < (\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2}, \tag{83$_p$}$$

and

$$(84_p) \qquad \|b_{p,k}\| < (1/\rho_{(p-1)}\{\|b_{p-1,k}\| + (\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2}\}.$$

Then, applying Theorem 3.4, we deduce the existence of vectors $a_{n+1}$ in $\mathcal{H}, w_{n+1,k}$ in $\mathcal{S}_k$, and $b_{n+1,k}$ in $\mathcal{R}_k$ such that the inequalities $(81)_{n+1}$, $(82)_{n+1}, (83)_{n+1}$, and $(84)_{n+1}$ are valid. Therefore, by induction, there exist sequences $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{H}, \{w_{n,k}\}_{n=1}^{\infty}$ in $\mathcal{S}_k$, $k = 1, 2$, and $\{b_{n,k}\}_{n=1}^{\infty}$ in $\mathcal{R}_k$, $k = 1, 2$, satisfying the appropriate inequalities for all $n$ in $\mathbb{N}$, and it is clear from $(82)_p$ and $(83)_p$ that $\{a_n\}$ and $\{w_{n,k}\}$ are Cauchy, for each $k = 1, 2$. Define

$$\hat{a} = \lim_{n\to\infty} a_n,$$
$$\hat{w}_k = \lim_{n\to\infty} w_{n,k}, \quad k = 1, 2,$$

and observe that since

$$\begin{aligned}\|\hat{a} - a\| &= \|\sum_{p=2}^{\infty}(a_p - a_{p-1})\| \\ &\le \sum_{p=2}^{\infty}\|a_p - a_{p-1}\| \\ &= 6(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}),\end{aligned}$$

and

$$\|\hat{w}_k - w_k\| < (\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}),$$

inequalities (75)and (76) are satisfied.
Furthermore, by iterating $(84)_p$, we see that

$$\begin{aligned}\frac{1}{2}\quad \|b_{n,k}\| &\le s_n\|b_{n,k}\| \\ &\le \|b_k\| + (\delta/\gamma)^{1/2}\sum_{p=1}^{n-1} s_p(\theta/\gamma)^{(p-1)/2},\end{aligned}$$

and therefore that

$$\|b_{n,k}\| \le 2\|b_k\| + 2(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad n \in \mathbb{N},\ k = 1, 2.$$

Thus the sequence $\{b_{n,k}\}$ is bounded and w.l.o.g., we may suppose that $\{b_{n,k}\}$ converges weakly to $\hat{b}_k$. Hence

$$\|\hat{b}_k\| \le 2\|b_k\| + 2(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad k = 1, 2,$$

which establishes (77). That (74) is valid now follows from $(81)_p$ as in the proof of Lemma 3.3.

THEOREM 3.6. *Under the hypotheses of Theorem 3.5, suppose that* $\|[L_1]_{T_1}\|$ *and* $\|[L_2]_{T_2}\|$ *are equal. Then, we have* $\{T_1, T_2\} \in \mathbb{A}^2_{1,1}(r(\theta,\gamma))$ *,where*

$$r(\theta,\gamma) = (18/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2. \tag{85}$$

*Proof.* By theorem 3.5, the set $\{T_1, T_2\}$ certainly belongs to some $\mathbb{A}^2_{1,1}(r)$. To see that $r$ may be taken to be as in (85), let $\epsilon > 0$ and set $a = 0$, $w_k = 0$, $b_k = 0$ and $\delta = \max_{k=1,2}\{\|[L_k]_{T_k}\|\} + \epsilon$ in (73). Then from (75), (76) and (77), we see that

$$\begin{aligned}
&\|\hat{a}\| \, \|\mathbb{P}(\hat{w}_k + \hat{b}_k)\| \\
&\le \|\hat{a}\|(\|\hat{w}_k\| + \|\hat{b}_k\|) \\
&< 6(\delta/\gamma)^{1/2}(1/\{1-(\theta/\gamma)^{1/2}\})\cdot \\
&\qquad [(\delta/\gamma)^{1/2}(1/\{1-(\theta/\gamma)^{1/2}\}) + 2(\delta/\gamma)^{1/2}(1/\{1-(\theta/\gamma)^{1/2}\})] \\
&= 18(\delta/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2 \\
&= (18/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2(\|[L_k]_{T_k}\| + \epsilon),
\end{aligned}$$

by the hypothesis. Therefore, we have

$$\{T_1, T_2\} \in \mathbb{A}^2_{1,1}(r(\theta,\gamma)),$$

where

$$r(\theta,\gamma) = (18/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2.$$

## References


[1] C. Apostol, H Bercovici, C Foias and C. Pearcy, *Invariant subspaces, dilation theory, and structure of the predual of a dual algebra. I*, Funct. Anal. **63** (1985), 369–404

[2] H Bercovici, B Chevreau, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. II*, Math Z **187** (1984), 97–103

[3] H. Bercovici, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I*, Michigan Math. J. **30** (1983), 335–354.
[4] ———, *Dual algebras with applications to invariant subspaces and dilation theory*, C.B.M.S Regional Conference Series, 56, A M.S., Providence, R. I (1985).
[5] A. Brown and C. Pearcy, *Introduction to operator theory. I*, Elements of functional analysis, Springer-Verlag, New York, 1977.
[6] B. Chevreau, G. Exner, and C Pearcy, *On the structure of contraction operators, III*, Michigan Math. J. **36** (1989), 29–62.
[7] B. Chevreau, and C Pearcy, *On the structure of contraction operators, I*, J. Funct. Anal **76** (1988), 1–29
[8] ———, *On common noncyclic vectors for families of operators*, Houston J. of Math **17** (1991), 637-650
[9] G. Exner and P. Sullivan, *Normal operators and the classes* $\mathbb{A}_n$, J. Operator Theory **19** (1988), 81-94.
[10] I. Jung, *Dual operator algebras and the classes* $\mathbb{A}_{m,n}I$, J Operator Theory (to appear).
[11] M. Marsalli, *A classification of operator algebras*, J Operator Theory **24** (1990), 155-163.
[12] B. Sz-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, North Holland Akademiai Kiado, Amsterdam/Budapest, 1970.

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