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SOME PROPERTIES OF SOME ANALYTIC FUNCTIONS

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ABSTRACT. In the present paper we introduce a new class of analytic functions and give some results

1. Introduction

Let A denote the class of functions $f(z)$ which are analytic in $U = \{z : |z| < 1\}$ with $f(0) = 0, f'(0) = 1$. As usual

$$S^*(\alpha) = \left\{f : f \in A \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in U\right\}. \tag{1.1}$$

This is well-known class of starlike functions of order $\alpha(0 \leq \alpha < 1)$.

Let $f(z)$ and $F(z)$ be analytic in the unit disc U. The function $f(z)$ is said to be subordinate to $F(z)$, if $F(z)$ is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$. We denote this relation by the symbol $f(z) \prec F(z)$ or $f \prec F$.

In the present paper we introduce the following class by using the notion of subordination.

A function $f(z)$ belonging to A is said to be in the class $\mathcal{B}(\mu, \lambda)$ if and only if $f(z)$ satisfies the condition

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1 + \lambda z \ (z \in U). \tag{1.2}$$

for $0 < \mu$ and $0 < \lambda$. In [3], R. Singh considered the real part of left hand side of (1.2).

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In the present paper, we obtain some conditions for $f(z)$ to be starlike using method of differential subordination, and further we use differential inequalities too.

We cite, below following lemmas which we are going to use,

LEMMA A[1]. *Let w be a nonconstant and analytic in U with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at z_0, we have*

$$z_0 w'(z_0) = kw(z_0), \quad k \geq 1.$$

LEMMA B[2]. *Let $\phi(u, v)$ be a complex function, $\phi : D \longrightarrow C$ ($D \subset C \times C, C$ is the complex plane), and let $u = u_1 + iu_2, \quad v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satifies following conditions*

i) *$\phi(u, v)$ is continuous in D.*

ii) *$(1, 0) \in D$ and $Re\{\phi(1, 0)\} > 0$,*

iii) *$Re\{\phi(iu_2, v_1)\} \leq 0$ for all (iu_2, v_1) and such that $v_1 \leq -(1 + u_2^2)/2$*

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$.

If $Re\{\phi(p(z), zp'(z))\} > 0, \quad z \in U$, then $Re\{p(z)\} > 0, \quad z \in U$.

2. Main results

We, first need the following result on differential subordination.

LEMMA 1. *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in U and satisfies the condition*

$$\frac{zp'(z)}{\mu} + p(z) \prec 1 + \lambda z \tag{2.1}$$

for some $\mu > 0$, $\lambda > 0$. Then

$$p(z) \prec 1 + \lambda_1 z, \quad \text{where } \lambda_1 = \frac{\lambda\mu}{1 + \mu}. \tag{2.2}$$

Proof. We define the function $w(z)$ by

$$p(z) = 1 + \lambda_1 w(z), \text{ where } \lambda_1 = \frac{\lambda\mu}{1+\mu}.$$

Then $w(z)$ is analytic in U, with $w(0) = 0$. We want to show that $|w(z)| < 1$, $z \in U$.

If not by Lemma A, there exists $z_0 \in U$, such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Then we have

$$\begin{aligned} \left|\frac{z_0 p'(z_0)}{\mu} + p(z_0) - 1\right| &= \left|\frac{\lambda_1 z_0 w'(z_0)}{\mu} + (1 + \lambda_1 w(z_0)) - 1\right| \\ &= \lambda_1 \left|\frac{kw(z_0)}{\mu} + w(z_0)\right| \\ &= \lambda_1 \left|\frac{k}{\mu} + 1\right| \\ &\geq \lambda_1 \left(\frac{1+\mu}{\mu}\right) \\ &= \left(\frac{\lambda\mu}{1+\mu}\right)\left(\frac{1+\mu}{\mu}\right) = \lambda, \end{aligned}$$

which is contradiction to (2.1). Hence $|w(z)| < 1, z \in U$. Therefore we have the desired result.

From Lemma 1, we have the following theorem.

THEOREM 1. *If $f \in A$ and satisfies (1.2) for some $\mu > 0$, and $\lambda > 0$, then*

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec 1 + \lambda_1 z, \text{ where } \lambda_1 = \frac{\lambda\mu}{1+\mu}. \tag{2.3}$$

Proof. We define the function $p(z)$ by

$$p(z) = \left(\frac{f(z)}{z}\right)^{\mu}. \tag{2.4}$$

Then

$$p'(z) = \mu \left(\frac{f(z)}{z}\right)^{\mu-1} \left(\frac{zf'(z) - f(z)}{z^2}\right)$$
$$= \frac{\mu}{z} \left(\frac{f(z)}{z}\right)^{\mu-1} f'(z) - \frac{\mu}{z} \left(\frac{f(z)}{z}\right)^{\mu}.$$

and we see that $p(z) = 1 + p_1 z + p_2 z + \cdots$ is analytic in U. Thus we have

$$f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1} = \frac{zp'(z)}{\mu} + p(z). \tag{2.5}$$

By Lemma 1, we get

$$\left(\frac{f(z)}{z}\right)^{\mu} = p(z) \prec 1 + \lambda_1 z, \text{ where } \lambda_1 = \frac{\lambda\mu}{1+\mu}.$$

Hence the proof is complete.

For $\mu = 1$, we get

COROLLARY 1. *If $f \in A$ and satisfies the condition $f'(z) \prec 1 + \lambda z$, $\lambda > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{\lambda}{2} z, \quad z \in U.$$

In the next result, we have used the concept of differential inequalities.

THEOREM 2. *Let $f \in A$ and satisfies*

$$Re\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \beta,$$

where $0 < \mu$, $-\frac{1}{2\mu} < \beta$, then

$$Re\left(\frac{f(z)}{z}\right)^{\mu} > 0.$$

Proof. We define the function $p(z)$ by

$$p(z) = \left(\frac{f(z)}{z}\right)^{\mu}.$$

From the proof of Theorem 1, we get

$$Re\left(\frac{zp'(z)}{\mu} + p(z) - \beta\right) > 0.$$

Setting the function $\phi(u, v)$ by

$$\phi(u, v) = \frac{v}{\mu} + u - \beta,$$

where $u = p(z)$ and $v = zp'(z)$, we have that

i) $\phi(u, v)$ is continuous in $D = C \times C$.

ii) $(1, 0) \in D$, and $Re\,(\phi(1, 0)) = 1 - \beta > 0$.

iii) For all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$, we get

$$\begin{aligned} Re\,(\phi(iu_2, v_1)) &= Re\left(\frac{v_1}{\mu} + iu_2 - \beta\right) \\ &= \frac{v_1}{\mu} - \beta \\ &\leq -\frac{1+u_2^2}{2\mu} - \beta \\ &< 0, \end{aligned}$$

where $0 < \mu$, $-\frac{1}{2\mu} < \beta$. Hence we get

$$Re\left(\frac{p(z)}{z}\right)^{\mu} = Rep(z) > 0.$$

From (2.5) and Theorem 2, we get the following corollary.

COROLLARY 2. *Let $p(z) = 1 + p_1 z + p_2 z + \cdots$ be analytic in U satisfying the following condition*

$$Re\left(p(zP + \frac{1}{\mu}zp'(z)\right) > -\frac{1}{2\mu}.$$

Then we get

$$Rep(z) > 0.$$

REMARK. Nunokawa showed the case $\mu = 2$ of above Corllary 2. (See [3; Lemma 5])

Now we consider some special cases of above Theorem 2.

COROLLARY 3. *If $\mu = 1$ and $f \in A$ satisfies the following condition*

$$Re\,(f'(z)) > \beta \quad \in U,$$

then

$$f \in S^*(0).$$

COROLLARY 4. *If $\mu = 2$ and $f \in A$ satisfies the following condition*

$$Re\left(\frac{f'(z)f(z)}{z}\right) > -\beta, \quad z \in U,$$

then

$$Re\left(\frac{f(z)}{z}\right)^2 > 0.$$

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