East Asian Math J. 14 (1998), No. 2, pp 399–410

EXPERIMENTAL RESULTS OF W-CYCLE MULTIGRID FOR PLANAR LINEAR ELASTICITY

JAECHIL YOO

ABSTRACT. In [3], Franca and Stenberg developed several Galerkin least squares methods for the solution of the problem of linear elasticity. That work concerned itself only with the error estimates of the method. It did not address the related problem of finding effective methods for the solution of the associated linear systems. In this work, we present computational experiments of W-cycle multigrid method Computational experiments show that the convergence is uniform as the parameter, ν, goes to 1/2

1. Introduction

Let Ω be a bounded convex polygonal domain in R^2 and $\partial\Omega$ be the boundary of Ω. The pure displacement boundary value problem for planar linear elasticity is given in the form

$$
\begin{aligned}
(1) \qquad & 2\mu\{\nabla\cdot\varepsilon(u)+\frac{\nu}{1-2\nu}\nabla\nabla\cdot u\}+f=0 \ \text{ in } \Omega,\\
& u=0 \ \text{ on } \partial\Omega
\end{aligned}
$$

Here $u=(u_1,u_2)$ denotes the displacement, $f=(f_1,f_2)$ is the body force, ν is Poisson's ratio and μ is the shear modulus given by $\mu = E/\{2(1+\nu)\}$ where E is the Young's modulus. Instead of using Poisson's ratio ν and Young's elasticity modulus E, we can also work with the Lame constants λ and μ. These constants are related to each other by the following equations;

$$
\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \nu=\frac{\lambda}{2(\lambda+\mu)},
$$

Received July 27, 1998 Revised October 28, 1998
1991 Mathematics Subject Classification 65N22, 65N30
Key words and phrases Galerkin least squares method, W-cycle multigrid, elasticity

$$\mu = \frac{E}{2(1+\nu)}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

We restrict Poisson's ratio to $0 \leq \nu < 1/2$ where the upper limit corresponds to an incompressible material.

Throughout this paper, we use mesh parameter h_k and grid level k which may vary from occurrence to occurrence

We define various standard differential operators as follows, see [2]:

$$\nabla \cdot v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},$$

$$\nabla \cdot \tau = \begin{pmatrix} \partial \tau_{11}/\partial x + \partial \tau_{12}/\partial y \\ \partial \tau_{21}/\partial x + \partial \tau_{22}/\partial y \end{pmatrix}, \quad \nabla v = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix},$$

$$\tau : \eta = \sum_{i=1}^{2}\sum_{j=1}^{2} \tau_{ij}\eta_{ij}, \quad \text{and } \varepsilon(v) = \frac{1}{2}\left[\nabla v + (\nabla v)^t\right].$$

Let $H^m(\Omega)$ denote the usual Sobolev space of functions with $L^2(\Omega)$ derivatives up to order m. $H^m(\Omega)$ is equipped with the norm

$$||v||_{H^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}}.$$

We use the following convention for the Sobolev seminorms:

$$|v|_{H^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha| = m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}}.$$

Let $H_0^m(\Omega) = \{v \in H^m(\Omega) : v|_{\partial\Omega} = 0\}$.

It is well known that for $f \in L^2(\Omega)$, equation (1) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$, see [4].

There is a great deal of literature dealing with approximation schemes for the equations of linear elasticity. To avoid the locking phenomenon in linear elasticity problems, there are several different approaches:

nonconforming finite element methods, the methods of reduced/selected integration, first order least squares methods, and Galerkin least squares methods. For all of these approaches, mixed finite element methods involving a pair of finite element spaces are commonly used and we have to solve large linear systems arising from the finite element discretizations. With the usual mixed finite element methods, the system is indefinite and hence the problem poses difficulties.

In recent years, modern iterative methods such as multigrid and domain decomposition methods have been applied to mixed finite element methods. Among those iterative methods, the multigrid method has been one of the most popular and fastest methods. So we study the multigrid method to solve the large sparse linear systems derived from the Galerkin least squares method for the pure displacement boundary value problem.

It is well-known that one way of driving stabilized mixed finite element methods is to combine the classical Galerkin formulation with least-squares forms of the differential equations. (See [3] and references therein). An advantage of this method is that the class of finite element spaces that can be used is considerablely enlarged, hence the methods are easily incorporated into existing finite element codes. In this paper, we present a scheme of W-cycle multigrid method to solve the linear system arising from P-1 conforming finite element method for the mixed formulation of the pure displacement boundary value problem as in [1], [5] and [6]. We give the computational results of W-cycle multigrid method with $\alpha/4$ at the coarse grid and with the constant α, where α is the stabilization parameter in the Galerkin least squares method. Moreover, we show that the number of iterations for the W-cycle multigrid methods is reduced by a half when we take twice as many smoothings in the algorithm and also reduced by a half when we cut the mesh size by a half. S. Brenner [1] reports very similar results for the pure displacement boundary value problem with the nonconforming finite element method.

This paper is organized as follows. We explain the conforming finite element method in section 2. We discuss the W-cycle multigrid algorithm in section 3. The computational results are presented in section 4.

2. The finite element method

For simplicity, we assume that $2\mu = 1$. Let $p = -\frac{1}{\epsilon}\nabla \cdot u$, where $\epsilon = (1-2\nu)/\nu$. Then (1) is equivalent to

$$
\begin{aligned}
&-\nabla \cdot \varepsilon(u) + \nabla p = f \ \text{ in } \Omega, \\
&\epsilon p + \nabla \cdot u = 0 \ \text{ in } \Omega, \\
&u = 0 \ \text{ on } \partial\Omega.
\end{aligned} \tag{2}
$$

Hence, we have the following weak formulation:

Find $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega \varepsilon(u) : \varepsilon(v)\, dxdy - \int_\Omega \Big(\nabla \cdot v\Big) p\, dxdy = \int_\Omega f \cdot v\, dxdy, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H_0^1(\Omega), \\
&\epsilon \int_\Omega pq\, dxdy + \int_\Omega \Big(\nabla \cdot u\Big) q\, dxdy = 0, \qquad \forall q \in L^2(\Omega).
\end{aligned} \tag{3}
$$

Let $\mathcal{T}^k$ be a family of triangulations of Ω, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let $h_T = \text{diam}(T)$ for each $T \in \mathcal{T}^k$ and $h_k = \max_{T \in \mathcal{T}^k} h_T$, then $h_k = 2h_{k+1}$. Now let's define the conforming finite element spaces for our multigrid method.

$$
\begin{aligned}
V_k &:= \{v \in C^0(\Omega)\ ;\ v|_T \text{ is linear for all } T \in \mathcal{T}^k \text{ and } v|_{\partial\Omega} = 0\}, \\
P_k &:= \{q \in C^0(\Omega)\ ;\ q|_T \text{ is linear for all } T \in \mathcal{T}^k\} \text{ and} \\
\tilde{P}_k &:= \{q \in C^0(\Omega)\ ;\ q|_T \text{ is linear for all } T \in \mathcal{T}^k \text{ and } \int_\Omega q\, dx = 0\}.
\end{aligned}
$$

Then the discretized problem for (3) is the following:

Find $(u_k, p_k) \in V_k \times \tilde{P}_k$ such that

$$
\mathcal{B}_k\Big((u_k,p_k),(v_k,q_k)\Big) = \mathcal{F}_f(v_k,q_k) \qquad \forall (v_k,q_k) \in V_k \times \tilde{P}_k \tag{4}
$$

where

$$\begin{aligned}
&\mathcal{B}_k\Big((u_k,p_k),(v_k,q_k)\Big)\\
&=\int_\Omega \varepsilon(u_k):\varepsilon(v_k)\,dxdy-\int_\Omega\Big(\nabla\cdot u_k\Big)q_k\,dxdy-\int_\Omega\Big(\nabla\cdot v_k\Big)p_k\,dxdy\\
&\quad-\alpha\sum_{T\in\mathcal{T}^k}h_T^2\int_T\Big(-\nabla\cdot\varepsilon(u_k)+\nabla p_k\Big)\cdot\Big(-\nabla\cdot\varepsilon(v_k)+\nabla q_k\Big)\,dxdy\\
&\quad-\epsilon\int_\Omega p_kq_k\,dxdy
\end{aligned}$$

and

$$\mathcal{F}_f(v_k,q_k)=\int_\Omega f\cdot v_k\,dxdy-\alpha\sum_{T\in\mathcal{T}^k}h_T^2\int_T f\cdot\Big(-\nabla\cdot\varepsilon(v_k)+\nabla q_k\Big)\,dxdy.$$

Note that the bilinear form $\mathcal{B}_k$ is symmetric and indefinite.

3. Multigird algorithm

In this section, we discuss the W-cycle multigrid algorithm.

In order to define the fine-to-coarse operator I_k^{k-1}, we introduce the following mesh-dependent inner product:

$$\Big((u,p),(v,q)\Big)_k:=(u,v)_{L^2(\Omega)}+h_k^2(p,q)_{L^2(\Omega)}.$$

Then $I_k^{k-1}:V_k\times P_k\to V_{k-1}\times P_{k-1}$ is defined by

$$\Big(I_k^{k-1}(u,p),(v,q)\Big)_{k-1}=\Big((u,p),(v,q)\Big)_k$$

for all $(u,p)\in V_k\times P_k$ and $(v,q)\in V_{k-1}\times P_{k-1}$.

Define $B_k:V_k\times P_k\to V_k\times P_k$ by

$$\Big(B_k(u,p),(v,q)\Big)_k=\mathcal{B}_k\Big((u,p),(v,q)\Big),$$

for all $(u,p),\ (v,q)\in V_k\times P_k$.

THEOREM 1. *(i) Given* $(u,p) \in V_k \times P_k$,

$$(u,p) \in V_k \times \tilde{P}_k \;\Leftrightarrow\; \Big((u,p),(0,1)\Big)_k = 0.$$

$$(ii)\ I_k^{k-1} : V_k \times \tilde{P}_k \to V_{k-1} \times \tilde{P}_{k-1}.$$

Proof. See [1].

THEOREM 2. *The subspace* $V_k \times \tilde{P}_k$ *is invariant under* B_k

Proof. See [1].

THEOREM 3. *The spectral radius of* B_k *is at most* Ch_k^{-2}.

Proof. See [1]

Because of the result of Theorem 3 and indefiniteness of the system, the usual iterative methods are not appropriated to solve our lineat system.

The mesh-dependent norms on $V_k \times \tilde{P}_k$ are defined as follows

$$\|\!|(u,p)|\!\|_{s,k} := \sqrt{\Big((B_k^2)^{s/2}(u,p),(u,p)\Big)_k} \quad \text{for all } (u,p) \in V_k \times \tilde{P}_k.$$

Note that B_k is nonsingular and symmetric, hence ${B_k}^2$ is positive definite with respect to $(\cdot,\cdot)_k$. Therefore, this norm is well-defined for each $s \in R$. Moreover,

$$\|\!|(u,p)|\!\|_{0,k} := \sqrt{\|u\|^2_{L^2(\Omega)} + h_k^2\|p\|^2_{L^2(\Omega)}} \quad \text{for all } (u,p) \in V_k \times \tilde{P}_k,$$

$$\Big|\mathcal{B}_k\Big((u,p),(v,q)\Big)\Big| \le \|\!|(u,p)|\!\|_{2,k}\|\!|(v,q)|\!\|_{0,k} \quad \text{for all } (u,p),(v,q) \in V_k \times \tilde{P}_k,$$

and

$$\|\!|(u,p)|\!\|_{2,k} = \sup_{(v,q)\in V_k\times\tilde{P}_k\setminus\{(0,0)\}} \frac{\Big|\mathcal{B}_k\Big((u,p),(v,q)\Big)\Big|}{\|\!|(v,q)|\!\|_{0,k}} \text{ for all } (u,p) \in V_k\times\tilde{P}_k.$$

Let

$$\begin{aligned}
&\mathcal{B}^*_{k-1}\Big((u,p),(v,q)\Big)\\
&= \int_\Omega \varepsilon(u) : \varepsilon(v)\ dxdy - \int_\Omega \Big(\nabla\cdot u\Big) q\ dxdy - \int_\Omega \Big(\nabla\cdot v\Big) p\ dxdy\\
&\quad - \frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}} h_T^2(-\nabla\cdot\varepsilon(u)+\nabla p, -\nabla\cdot\varepsilon(v)+\nabla q)_{L^2(T)}\\
&\quad - \epsilon\int_\Omega pq\ dxdy
\end{aligned}$$

and

$$\mathcal{F}^*_f(v,q) = \int_\Omega f\cdot v\ dxdy - \frac{\alpha}{4}\sum_{T\in\mathcal{T}^{k-1}} h_T^2\Big(f, -\nabla\cdot\varepsilon(v)+\nabla q\Big)_{L^2(T)}$$

Note that $\mathcal{B}^*_{k-1}$ and $\mathcal{F}^*_f$ are different from $\mathcal{B}_{k-1}$ and $\mathcal{F}_f$. The difference is in the least squares term. We divide the stabilization parameter α by 4 to define $\mathcal{B}^*_{k-1}$ and $\mathcal{F}^*_f$.

Define $\tilde{P}_k^{k-1} : V_k \times \tilde{P}_k \to V_{k-1}\times\tilde{P}_{k-1}$ by

$$\mathcal{B}^*_{k-1}\Big(\tilde{P}_k^{k-1}(u,p),(v,q)\Big) = \mathcal{B}_k\Big((u,p),(v,q)\Big)$$

for all $(u,p)\in V_k\times\tilde{P}_k$ and $(v,q)\in V_{k-1}\times\tilde{P}_{k-1}$.

Now we describe the k-th level iteration scheme of the conforming W-cycle multigrid algorithm. The k-th level iteration with initial iterate (y_0,z_0) yields $CMG\Big(k,(y_0,z_0),(w,r)\Big)$ as a conforming approximate solution to the following problem.

Find $(y,z)\in V_k\times\tilde{P}_k$ such that

$$B_k(y,z) = (w,r), \quad \text{where } (w,r)\in V_k\times\tilde{P}_k.$$

For $k=1$, $CMG\Big(1,(y_0,z_0),(w,r)\Big)$ is the solution obtained from a direct
method. In other words,

$$CMG\Big(1,(y_0,z_0),(w,r)\Big) = (B_1)^{-1}(w,r).$$

For $k > 1$, there are two steps.
Smoothing step : Let $(y_m, z_m) \in V_k \times \tilde{P}_k$ be defined recursively by the initial iterate (y_0, z_0) and the equations

$$(y_l, z_l) = (y_{l-1}, z_{l-1}) + \frac{1}{\Lambda_k^2} B_k\Big((w, r) - B_k(y_{l-1}, z_{l-1})\Big), \quad 1 \le l \le m,$$

where $\Lambda_k := Ch_k^{-2}$ is greater than or equal to the spectral radius of B_k, and m is the number of smoothings.
Correction step : The coarser-grid correction in $V_k \times \tilde{P}_k$ is obtained by applying the $(k-1)$-th level conforming iteration. More precisely,

$$(v_0, q_0) = (0, 0) \text{ and}$$
$$(v_i, q_i) = CMG\Big(k-1, (v_0, q_0), (\bar{w}, \bar{r})\Big), i = 1, 2$$

where $(\bar{w}, \bar{r}) \in V_{k-1} \times \tilde{P}_{k-1}$ is defined by $(\bar{w}, \bar{r}) := I_k^{k-1}\Big((w, r) - B_k(y_m, z_m)\Big)$.
Then $CMG\Big(k, (y_0, z_0), (w, r)\Big) = (y_m, z_m) + I_{k-1}^k(v_2, q_2)$.

REMARK 1. In the smoothing step, we use B_k instead of the restriction of B_k. Because the space $V_k \times P_k$ has a natural coordinate system which consists of the values of piecewise linear functions at mesh points on the triangles. In view of Theorem 1 and Theorem 2, the result of the smoothing step and the correction step belongs to $V_k \times \tilde{P}_k$. Therefore, in the actual implementation of the multigrid method, we use only the natural coordinate system of $V_k \times P_k$. Note that B_k is represented by a sparse banded matrix and B_k is not invertible.

4. Experimental results

We apply the W-cycle multigrid algorithm to the pure displacement boundary value problem (2) studied in [1]. The domain Ω is the unit square, and the body force $f = (f_1, f_2)$ is taken to be as follows :

$$f_1 = \pi^2[2\sin 2\pi y(-1 + 2\cos 2\pi x) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y],$$
$$f_2 = \pi^2[2\sin 2\pi x(1 - 2\cos 2\pi y) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x \sin\pi y].$$

The exact solution $u = (u_1, u_2)$ is

$$u_1 = \sin 2\pi y(-1 + \cos 2\pi x) + \frac{\epsilon}{\epsilon + 2} \sin \pi x \sin \pi y,$$
$$u_2 = \sin 2\pi x(1 - \cos 2\pi y) + \frac{\epsilon}{\epsilon + 2} \sin \pi x \sin \pi y.$$

The programs execute until the discrete L^2 relative error is less than 5% of the initial error. We use the initial iterates, $u^0 = (u_1^0, u_2^0) = (0, 0)$ and $p^0 = 0$. The computations were done in double-precision arithmetic for various α's, smoothing steps and Poisson's ratio ν's. The numbers in the columns represent the number of iterations to achieve an L^2 relative error of less than 5% in the displacement.

We know that the number of iterations for the W-cycle multigrid is reduced in half when we take twice as many smoothings and cut in half when we have the mesh size by a half We also observe that our multigrid is robust for the moderate α's in that the convergence is uniform as the parameter, Poisson's ratio ν, goes to 1/2

Also, we give the numerical experiments with the fixed α for all levels and with $\alpha/4$ at the coarse grid for W-cycle multigrid methods. A very attractive feature of using the fixed α for all levels in our *CMG* algorithm is its inherent simplicity, the bilinear form at the coarse grid is the same form at the fine grid. In other word, the structure of the linear system at the coarse grid is same as that of the linear system at the fine grid. The numerical experiments show that the number of iterations of W-cycle multigrid method is nearly same in both cases with α fixed and α modified.

Note that the size of our linear system is 12675 by 12675 for the case of N = 64 and 3267 by 3267 for the case of N=32.

	$N = 32$				$N = 64$			
smoo	α=1	α=0 3	α=0 1	α=0 01	α=1	α=0 3	α=0 1	α=0 01
1	1096	1084	1081	1080	554	552	551	551
2	548	542	541	540	277	276	276	276
3	366	362	361	360	185	184	184	184
4	274	271	271	270	139	138	138	138

Table 1: $\alpha/4$ at the coarse grid and $\nu = 0.3$

	$N = 32$				$N = 64$			
smoo	α=1	α=0 3	α=0 1	α=0.01	α=1	α=0.3	α=0 1	α=0 01
1	1084	1079	1082	1084	551	552	552	553
2	542	540	541	542	276	276	276	277
3	362	360	361	362	184	184	184	185
4	271	270	271	271	138	138	138	139

Table 2: $\alpha/4$ at the coarse grid and $\nu = 0.45$

	$N = 32$				$N = 64$			
smoo	α=1	α=0 3	α=0 1	α=0 01	α=1	α=0 3	α=0 1	α=0 01
1	1091	1102	1113	1119	564	568	570	570
2	546	551	557	560	282	284	285	285
3	371	368	371	372	188	190	191	190
4	273	276	279	280	141	142	143	143

Table 3: $\alpha/4$ at the coarse grid and $\nu = 0.495$

	$N = 32$				$N = 64$			
smoo	α=1	α=0.3	α=0 1	α=0 01	α=1	α=0.3	α=0 1	α=0 01
1	div	div	1118	1125	566	571	573	574
2	div	div	559	563	284	286	287	287
3	div	div	373	376	div	191	191	192
4	div	div	280	282	div	143	144	144

Table 4: $\alpha/4$ at the coarse grid and $\nu = 0.4995$

	$N = 32$				$N = 64$			
smoo	α=1	α=0.3	α=0 1	α=0.01	α=1	α=0 3	α=0 1	α=0 01
1	1180	1097	1088	1081	609	563	556	552
2	590	549	544	541	305	282	278	276
3	394	366	363	361	203	188	186	184
4	295	275	272	271	153	141	139	138

Table 5: Fixed α for all levels and $\nu = 0.3$

	$N = 32$				$N = 64$			
smoo	α=1	α=0.3	α=0 1	α=0 01	α=1	α=0.3	α=0 1	α=0 01
1	1177	1101	1092	1084	611	568	559	553
2	589	551	546	542	306	284	280	277
3	393	367	364	362	204	190	187	185
4	295	276	273	271	153	142	140	139

Table 6: Fixed α for all levels and $\nu = 0.45$

	$N = 32$				$N = 64$			
smoo	α=1	α=0.3	α=0 1	α=0 01	α=1	α=0 3	α=0 1	α=0 01
1	1174	1100	1094	1109	610	569	561	565
2	587	550	547	555	305	285	281	283
3	391	367	365	370	204	190	187	189
4	294	275	274	278	153	143	143	142

Table 7: Fixed α for all levelsand $\nu = 0.495$

	$N = 32$				$N = 64$			
smoo	α=1	α=0 3	α=0 1	α=0 01	α=1	α=0 3	α=0 1	α=0 01
1	1173	1100	1094	1114	610	569	561	568
2	587	550	547	557	305	285	281	284
3	391	367	365	372	204	190	187	190
4	294	275	274	279	153	143	141	142

Table 8: Fixed α for all levels and $\nu = 0.4995$

References

[1] S C Brenner, *A nonconforming mixed multigrid method for the pure displacement problem in planar linear elasticity*, SIAM J Numer. Anal , **30** (1993), 116-135

[2] P. Cialet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978

[3] L P Franca and R Stenberg, *Error analysis of some Galerkin least squares methods for the elasticity equations*, SIAM J Numer Anal. **28** (1991), 1680-1697.

[4] P. Grisvard, *Singularités en elasticité*, Arch Rational Mech Anal. **107** (1989), 157-180.

[5] C.-O. Lee, *Multigrid methods for the pure traction problem of linear elasticity: mixed formulation*, SIAM J Numer. Anal. **35** (1998), 121-145.

[6] J. Yoo, *Multigrid method for the Galerkin least squares method in linear elasticity: The pure displacement problem*, Proceedings of the 9th Copper Mountain Conference on iterative methods, Cray Research, Inc. 1996.

Department of Mathematics
Dongeui University
Pusan 614-714, Korea